A directed polymer approach to the once-oriented last passage site percolation time constant in high dimensions

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Conference in Memory of Walter V. Philipp
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Directed Last Passage Site Model and Directed Polymer Model

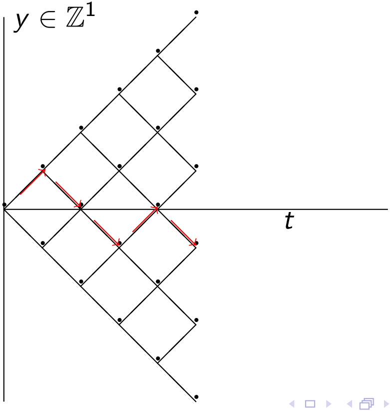
Relation between $f(\beta)$ and ν_d

Asymptotic Evaluation of ν_d for Exponential Weights

Asymptotics of ν_d for a Class of Distributions

Directed path for $d = 1$.

There are $(2d)^n$ directed paths of length n, with height of path $y \in \mathbb{Z}^d$.



"Point to Line" Last Passage Times

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- Superadditivity of "Point to Point" Last Passage Times:

$$\widetilde{T}_{0,n+m} \geq \widetilde{T}_{0,m} + \widetilde{T}_{m,m+n}, \text{ so, by Kingman,}$$

$$\lim_{n\to\infty} \widetilde{T}_{0,n}/n = \mu_d, \quad \mathbf{P}\text{-a.s. and in } \mathbf{L}^1.$$

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$$\widetilde{H}_{0,n} := \max_{\gamma_0=\mathbf{0}} \sum_{t=1}^{n} \eta(t,\gamma_t).$$

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- By Hammersley and Welsh: $\nu_d = \mu_d$.

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- Also in these special cases $(\widetilde{T}_{0,n} - n\nu_1)/n^{1/3}$ converges in distribution to the Tracy-Widom law F_2 [ref. K. Johansson CMP (2000), relations to random matrix theory]

Directed Polymer: Partition Function and Free Energy

- Directed paths are polymers. Higher total weights are favored. Partition function:

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- By Jensen's inequality, $\{\mathbf{E} \ln Z_n\}$ is superadditive; so define the Free Energy:

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- By Jensen's inequality w.r.t. $\mathbf{E}$ again, $f(\beta) \leq \lambda(\beta)$.

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- Further, since $\mathbf{E}(\eta) \geq 0$, $f_n(\beta)$ is non-decreasing. Indeed,

$$\text{for } \beta_1 > \beta_0, \quad f_n(\beta_1) = f_n(\beta_0) +$$

$$(1/n)\mathbf{E} \ln \mathbb{E} \exp(\Delta\beta T) \left[\exp(\beta_0 T)/\mathbb{E} \exp(\beta_0 T)\right]$$

$$= f_n(\beta_0) + (1/n)\mathbf{E} \ln \widetilde{\mathbb{E}} \exp(\Delta\beta T) \geq f_n(\beta_0) + \mathbf{E}\widetilde{\mathbb{E}}(\Delta\beta) T/n \geq f_n(\beta_0),$$

where at the last step Jensen's inequality was applied w.r.t. $\widetilde{\mathbb{E}}$. Here $\mathbb{E}$ is the uniform measure on directed paths and $\widetilde{\mathbb{E}}$ is simply the Gibbs measure expectation.

Proposition 1

- Estimate $\sum_\gamma \exp(\beta T(\overrightarrow{\gamma_n}))$ from below by the maximum term:

$$\mathbf{E}(\ln Z_n) = \mathbf{E}\ln[(2d)^{-n}\sum_\gamma \exp(\beta T(\overrightarrow{\gamma_n}))] \geq \beta \mathbf{E}\widetilde{H}_{0,n} - n\ln(2d)$$

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- By dividing by n and taking $n \to \infty$, we have thus proved: **Proposition 1:**

$$f(\beta) \geq \beta \nu_d - \ln(2d), \text{ and } \lim_{\beta \to \infty} f(\beta)/\beta = \nu_d.$$

Prototypical examples

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- When $\lambda(\beta)$ exists for all β we know by the martingale theory first developed by Bolthausen CMP(1989), and later by Comets-Yoshida AP (2006) that for high d, $\lambda(\beta) = f(\beta)$ for small $\beta > 0$. We essentially determine in a generic case for high d where the departure of the two curves $\lambda(\beta)$ and $f(\beta)$ occurs in terms of d.

The Gaussian case

- Carmona and Hu PTRF (2002) have shown in the Gaussian case ($\lambda(\beta) = \beta^2/2$) that :

$$f(\beta) \leq \min\left(\beta^2/2, \beta\sqrt{2\ln(2d)}\right), \quad \text{so, } \nu_d \leq \sqrt{2\ln(2d)}$$

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- Obviously, putting $m_d := \mathbf{E}\max(\eta_1, \ldots, \eta_{2d})$,

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- But one can show directly in the Gaussian case that

$$m_d = \sqrt{2\ln(2d)} + o(1)$$

Hence in this case: $\nu_d = \sqrt{2\ln(2d)} + o(1)$, as $d \to \infty$.

Exponential case: $\nu_d \gtrsim \ln(2d)$, as $d \to \infty$

- We have $\nu_d \geq m_d$. We first estimate $m_d := \mathbf{E} \max(\eta_1, \ldots, \eta_{2d})$ from below.

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- Write $G(t) := t^{2d}$ for $t = t(x) := \mathbf{P}(\eta \leq x) = 1 - \exp(-x)$.

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- By the change of variables $t = t(x)$ we have

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$$m_d = \int_0^\infty x \mathrm{d}t^{2d}(x) = \int_0^1 -\ln(1-t)\mathrm{d}G(t).$$

- But, $-\ln$ is a convex function. So by Jensen's inequality,

$$m_d \geq -\ln(\int_0^1 (1-t)\mathrm{d}G(t)) = -\ln(1 - 2d/(2d+1))$$

$$= \ln(2d) + o(1), \text{ as } d \to \infty.$$

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- By the previous page we therefore have

$$\nu_d \sim \ln(2d), \text{ as } d \to \infty.$$

Assumptions to Construct a Lower Bound for m_d

- Write $F(x) := \mathbf{P}(\eta \leq x)$. Assume

  $$F(x) = 1 - \exp(-u(x)), \text{ for } u(x) \text{ non-decreasing to } \infty, \text{ as } x \to \infty.$$

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- Also, because $u(x)$ need not be continuous, assume there are exponents

  $$u_0(x) \leq u(x) \leq u_1(x), \text{ with } u_i'(x) > 0, \text{ and } u_1(x) = (1+o(1))u_0(x)$$

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- Assume that $F_i(x) := 1 - \exp(-u_i(x))$ is concave for large x, each $i = 0, 1$.
- Finally, assume the exponential tail condition

  $$\liminf_{x\to\infty} u_i'(x) > 0, \quad i = 0, 1.$$

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- Proof uses that $F_i^{-1}(t)$ is convex, $i = 0, 1$ and "brackets" $F^{-1}(t)$. In particular, by Jensen's inequality

$$m_d = \int_0^1 F^{-1}(t) \mathrm{d}(t^{2d}) \gtrsim \int_{t_0}^1 F_1^{-1}(t) \mathrm{d}\left(t^{2d}/(1 - t_0^{2d})\right)$$
$$\gtrsim F_1^{-1}(1 - 1/(2d) + O(1/d^2)).$$

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$$\gtrsim F_1^{-1}(1 - 1/(2d) + O(1/d^2)).$$

- We obtain, using that $U_1 := u_1^{-1}$ grows at most linearly,

$$F_1^{-1}(1-1/(2d)+O(1/d^2)) = U_1(\ln(2d)+o(1)) \sim U_1(\ln(2d)).\square$$

A Class of Distributions

- Now assume $u(x) = xv(x) + \delta(x), \quad x \geq x_0$ [cf. Ben-Ari (2007)], where $\delta(x) = O(x)$, and $v(x)$ is twice continuously differentiable and satisfies the following shape, growth, and regularity conditions:

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- growth: $\liminf_{x\to\infty} xv'(x) > 0.$

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- Now assume $u(x) = xv(x) + \delta(x), \quad x \geq x_0$ [cf. Ben-Ari (2007)], where $\delta(x) = O(x)$, and $v(x)$ is twice continuously differentiable and satisfies the following shape, growth, and regularity conditions:

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- **Theorem 1**. Under these conditions,

$$\nu_d \sim U(\ln(2d)), \text{ as } d \to \infty.$$

Proof of Thm 1 by Log MGF and Proposition 1

- Define a random variable η_+ taking values in $[x_0, \infty)$ to have the distribution function $F_+(x) := \mathbf{P}(\eta \leq x | \eta \geq x_0)$, so

$$\lambda_+(\beta) \geq f_+(\beta) \geq \nu_d(\eta_+)\beta - \ln(2d) \geq \nu_d(\eta)\beta - \ln(2d)$$

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- **Lemma 2**. Define $w(x) := (xv(x))' = xv'(x) + v(x)$. Put $U = U_d = U(\ln(2d))$ for $U =$ the inverse of $xv(x)$, and $\beta = \beta_d := w(U) - M$ for a constant $M > 0$.Then

$$\lambda_+(\beta_d) \lesssim U^2 v'(U), \text{ as } d \to \infty.$$

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- But $u(x) + x^2 v'(x) = xw(x)$, so $U \gtrsim \nu_d$.

Comments on the Proof

- Since by the lower bound of Lemma 1, $\nu_d \gtrsim U(\ln(2d))$, and as a consequence of Proposition 1 and Lemma 2, we have the upper bound $\nu_d \lesssim U(\ln(2d))$, the Theorem is proved.

Comments on the Proof

- Since by the lower bound of Lemma 1, $\nu_d \gtrsim U(\ln(2d))$, and as a consequence of Proposition 1 and Lemma 2, we have the upper bound $\nu_d \lesssim U(\ln(2d))$, the Theorem is proved.
- **Comments on the proof of Lemma 2.** After integration by parts, we have the formula:

$$\exp(\lambda_+(\beta)) = C_0\beta \int_{x_0}^{\infty} \exp(\beta x - xv(x) + \delta(x))dx.$$

Split the integral by $\int_{x_0}^{V(2\beta)} + \int_{V(2\beta)}^{\infty}$, where $V := v^{-1}$. The second integral is negligible since for $x \geq V(2\beta)$, we have

$$x(\beta - v(x)) + \delta(x) \leq (M - \beta)x \leq -(\beta/2)x$$

.

Comments on the Proof

- The main contribution of $\exp(\lambda_+(\beta))$ is thus bounded above by

$$C_0\beta \exp\left(\max_{x_0\le x\le V(2\beta)}\{x(\beta+M)-xv(x)\}\right)[V(2\beta)-x_0]$$

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- The exponent has a unique maximum by strict convexity of $xv(x)$, and we choose β implicitly by choosing the critical point to satisfy $x=x_\beta=U=U(\ln(2d))$. This yields

$$w(x)=\beta+M \text{ for } w(x)=(xv(x))',$$

  so the maximum exponent is written $U(w(U)-v(U))=U^2v'(U)$. Lemma 2 follows after some work to show that under the stated growth condition $\ln(\beta V(2\beta))=o(U^2v'(U))$.

The Poisson case

- By Stirling's formula $\ln(x!) = c + o(1) + (x + 1/2)\ln(x) - x$, so we find that the Poisson case is represented by $v(x) = \ln(x) - 1$ with $\delta(x) = O(\ln(x))$. The growth condition is just satisfied: $xv'(x) = 1$, and the regularity condition $(xv(x))'^2/(xv(x))'' \sim x\ln(x)^2$ is easily satisfied.

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- We have U as the inverse of $xv(x)$, or $U(u) \sim u/\ln(u)$. Therefore

$$\nu_d \sim \ln(2d)/\ln\ln(2d).$$