Large Deviations for Certain Integer Valued Statistics in Gambler's Ruin

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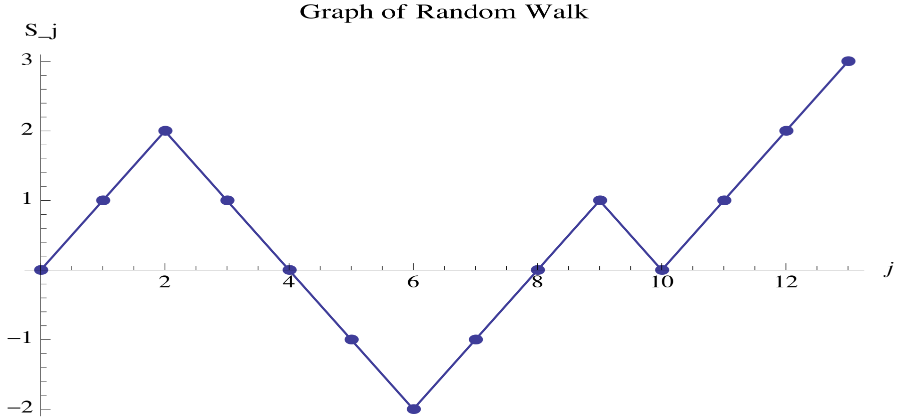
Random Walk

- Walk on Integers: $\mathbf{S}_j$, $j = 0, 1, 2, \ldots$
- Increments: $\varepsilon_j := \mathbf{S}_j - \mathbf{S}_{j-1}$
- Transitions: $P(\varepsilon_{j+1} = 1) = P(\varepsilon_{j+1} = -1) = \frac{1}{2}$



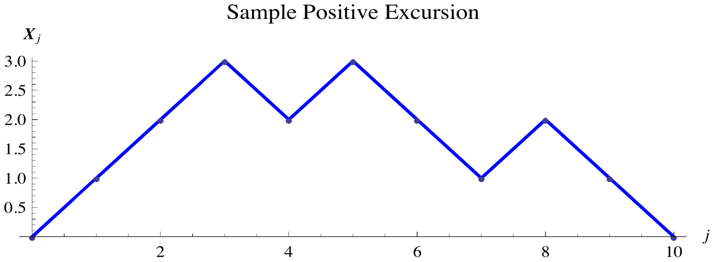
Excursion:

A Path of First Return to Zero by Random Walk

Sample :

#Steps: $\mathbf{L} = 10$, Height: $\mathbf{H} = 3$

#Runs: $\mathbf{R} = 6$, #Short Runs: $\mathbf{V} = 3$



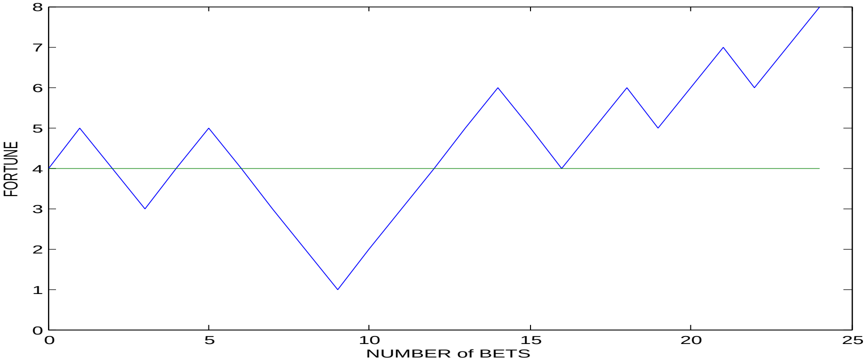
Gambler's Ruin, $\{\mathbf{X}_j\}$

Initial Fortune $N \in (0, 2N)$. Unit Bets. Independent Fair Games.

Terminate Game at Step $\mathcal{T}$ when **Fortune** First Equals $2N$ or 0.

Equivalently, $\mathbf{X}_0 = 0 \in (-N, N)$, $\mathcal{T} := \inf\{j : |\mathbf{X}_j| = N\}$.

Picture: "Last Visit" $\mathcal{L} := \inf\{j : |\mathbf{X}_j| = 0\}$, "Meander" $\mathcal{L} \leq j \leq \mathcal{T}$.



Laws Relating Runs and Steps [M,2015]

2 Parameter Fibonacci Recurrence: $\quad (\mathbf{F}) \quad v_{n+1} = \beta v_n - x v_{n-1}.$

- **Define** $\{q_n(\beta, -x)\}$, and $\{w_n(\beta, -x)\}$ by

  ($\mathbf{F}$), with: $q_0 = 0,\ q_1 = 1;\ \ w_0 = 1,\ w_1 = 1;$

- $\beta = 1$ gives Fibonacci Polynomials:

  $q_1 = 1,\ q_2 = 1,\quad q_3 = 1 - x,\quad q_4 = 1 - 2x,\quad q_5 = 1 - 3x + x^2$

  Obtain Fibonacci numbers with $x = -1$.

Theorem

$$K_{N+1} := E\{r^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} \le N\} = c \cdot r^2 z^2 \, (q_N / w_N);\ \ x := \tfrac{1}{4} z^2,\ \beta := 1 - x(r^2 - 1)$$

Steps alone ($r = 1$, $\beta = 1$) proved by [de Bruijn, Knuth, Rice ('72)]

Corollary

Define $\alpha := \sqrt{\beta^2 - 4x}$. *Let* $N \to \infty$ *in Theorem.*
Then, $K(r, z) := E\{r^{\mathbf{R}} z^{\mathbf{L}}\} = (2 - \beta - \alpha)$. *(Narayana G.F.)*

Corollary: Steps and Runs of order N^2 [M,2015]

- Denote $\mathcal{R} := \#\text{Runs}(|\mathbf{X}_j|,\ 0 \le j \le \mathcal{L})$

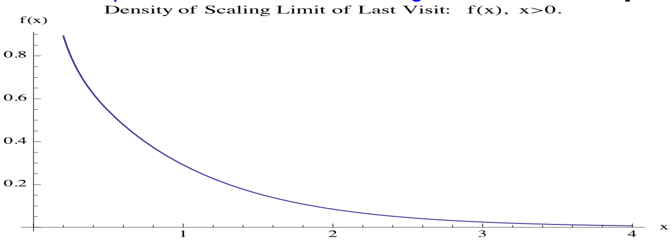
  All Excursions become Positive under Absolute Value sign

  So, Count #Runs in each Excursion and Add Up until Last Visit

Corollary

Both $\frac{1}{2}\mathcal{L}/N^2 \to f,\ \mathcal{R}/N^2 \to f;\ \ f(x) := \frac{1}{\sqrt{\pi x}} \sum_{\nu=-\infty}^{\infty} (-1)^{\nu} e^{-\nu^2/x},\ x > 0.$

Case of Steps $\mathcal{L}$ Alone Stands as Analogue of ArcSine Law [F.Knight ('69)]



Upward and Downward Conditional G.F. g_n

Focus on Construction for Meander

$$\Gamma'_n := \text{Conditional Upward First Passage Path to Level } n,$$

$$\text{given } \mathbf{X}_0 = 0, \ \& \text{ Path "One-Sided"} : \mathbf{X}_j \in [0, n]$$

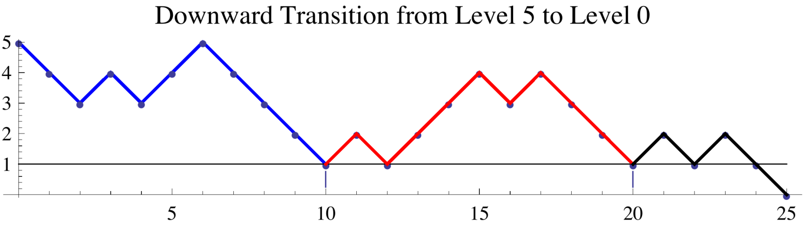
- $\mathbf{L}'_n := \#(\textbf{Steps} \text{ along } \Gamma'_n)$. $\mathbf{R}'_n := \#(\textbf{Runs} \text{ along } \Gamma'_n)$,

  If instead $\mathbf{X}_0 = n$, Obtain Statistics for Downward First Passage to 0.

- **Define**:

$$g_n(r, z) := E\{r^{\mathbf{R}'_n} z^{\mathbf{L}'_n}\}.$$

- A Downward Path for g_5



Recurrence Relation for g_n

Decompose Downward Transition for g_n by:

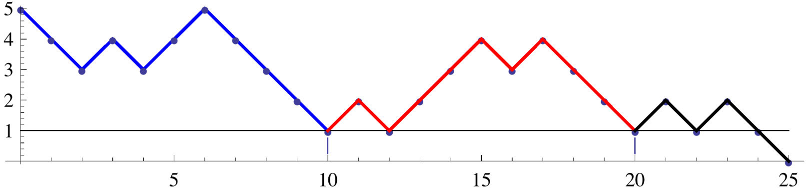
Return to Level 1 After Each Future Maximum.

- $\rho_n :=$ Probability of One-sided Transition
- $z \cdot h^+ :=$ G.F. over Termination Seq. $(UD)^\ell D$, from Level 1
- $\lambda_m := \sum_{\ell=0}^{\infty} (\rho_m g_m \cdot \rho_m g_m)^\ell = \frac{1}{1-\rho_m^2 g_m^2}$.
- **Recurrence**

$$g_n = c_n \cdot g_{n-1} \cdot \lambda_{n-1} \cdot \lambda_{n-2} \cdots \lambda_2 \cdot z \cdot h^+$$

- Therefore, **(g)** $g_{n+1} = c \cdot \lambda_n \cdot g_n^2 / g_{n-1} = c \cdot \frac{g_n^2}{g_{n-1}} \frac{1}{1-\rho_n^2 g_n^2}$.

Downward Transition with First Two Steps the Same



Denominators of g_n

Recall:

(**F**) $w_{n+1} = \beta w_n - x w_{n-1}, \; w_0 = 1, \;\; w_1 = 1; \; \beta = 1 - x(r^2 - 1), \; x = \frac{1}{4} z^2$

Lemma

We have the Identity: $w_n^2 - r^2 x^n = w_{n-1} w_{n+1}$

Proof. $v_{n+1} = \beta v_n - x v_{n-1} \implies v_{n+1} v_{n-1} - v_n^2 = \beta^{-1} x^{n-1} (v_3 v_0 - v_2 v_1)$. □

Lemma

*(**g**) has solution* $g_n = F_n \, r \cdot z^n / w_n, \;\; F_n := \frac{n+1}{2^n}; \;\; \& \;\; \lambda_n = \frac{(w_n)^2}{w_{n-1} w_{n+1}}; n \geq 1.$

Proof. By Induction. $g_1 = \frac{2}{2} r \cdot z / 1 = r \cdot z$.

$$g_{n+1} = \frac{c F_n^2 \; r^2 \; z^{2n} w_{n-1}}{F_{n-1} \; r \cdot z^{n-1} w_n^2} \times \frac{w_n^2}{w_n^2 - r^2 2^{-2n} z^{2n}} = \frac{c F_n^2 \; r \cdot z^{n+1} w_{n-1}}{F_{n-1}} \times \frac{1}{w_{n-1} w_{n+1}}$$

$$= F_{n+1} \; r \cdot z^{n+1} / w_{n+1},$$

by cancellation and previous Lemma ($\& \; \rho_n F_n = 2^{-n}, \;\; 2^{-2n} z^{2n} = x^n$).

Limit Distributions for Order N scaling

Define:

- $\mathcal{R}' := \#\text{Runs}(\mathbf{X}_j,\ \mathcal{L} \le j \le \mathcal{T});$
- $\mathcal{L}' := \#\text{Steps}(\mathbf{X}_j,\ \mathcal{L} \le j \le \mathcal{T});$
- $\Delta' := 2\mathcal{R}' - \mathcal{L}';$ (Meander)
- $\Delta := 2\mathcal{R} - \mathcal{L} - \mathcal{M};\quad \mathcal{M} := \#\text{Excursions to } \mathcal{L};$ (Last Visit)

Since $E\{r^{\mathcal{R}'} z^{\mathcal{L}'}\} = z \cdot g_{N-1}(r, z)$, Obtain the Transform for (a) (Meander):

Corollary

As $N \to \infty$,

$$(a)\ \lim E\{e^{it\Delta'_N/N}\} = \frac{t}{\sinh(t)} = \int_{-\infty}^{\infty} e^{itx} \left[\frac{\pi}{4} sech^2(\pi x/2)\right] dx,$$

$$(b)\ \lim E\{e^{it\Delta_N/N}\} = \frac{\tanh(t)}{t} = \int_{-\infty}^{\infty} e^{itx} \left[\frac{2}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu+1} e^{-(2\nu+1)\pi|x|/2}\right] dx,$$

$$(c)\ \lim E\{e^{it(\Delta_N+\Delta'_N)/N}\} = \frac{1}{\cosh(t)} = \int_{-\infty}^{\infty} e^{itx} \left[\frac{1}{2} sech(\pi x/2)\right] dx.$$

Apply instead Thm (Formula for K_N) to prove part (b) of Main Corollary.

Proof of Theorem

- Conditional G.F. for Excursion Stats:

$$G_n(r,z) := E\{r^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} = n\} = z \cdot g_{n-1} g_n.$$

Lemma

Identity: $q_n w_{n+1} - w_n q_{n+1} = -x^n.$

Proof. We have: $w_n = q_n - x q_{n-1}$, $n \geq 1$, (both sides satisfy the same recurrence and initial conditions). Plug in for w_n and w_{n+1}.

$$q_n w_{n+1} - w_n q_{n+1} = x(q_{n+1} q_{n-1} - q_n^2)$$
$$= x \cdot x^{n-1} \beta^{-1} (q_3 q_0 - q_2 q_1) = -x^n.$$

Theorem

$$C_N \, E\{r^{\mathbf{R}} z^{\mathbf{L}} \mid \mathbf{H} \leq N\} = \sum_{n=1}^{N} G_n \, P(\mathbf{H} = n) = \sum_{n=1}^{N} \frac{r^2 z^2 \, x^{n-1}}{w_{n-1} w_n} = r^2 z^2 \, \frac{q_N}{w_N}.$$

Induction Proof. $\frac{1}{w_0 w_1} = \frac{q_1}{w_1} = 1.$ Lemma $\Longrightarrow$ $\frac{q_N}{w_N} + \frac{x^N}{w_N w_{N+1}} = \frac{q_{N+1}}{w_{N+1}}.$
uses: $P(\mathbf{H} = n) = \frac{1}{n(n+1)}$. $\square$

Formula for $K_N \Longrightarrow$ **Last Visit Statistics G. F.**

- $\mathcal{M} := \#$ Excursions Until Last Visit $\mathcal{L}$.
- $\mathcal{M}$ is a **Geometric** random variable:

$$p_m := P(\mathcal{M} = m) = P(\mathbf{H} < N)^m P(\mathbf{H} \geq N), \; m = 0, 1, 2, \ldots .$$

- Sum Independent Copies of Excursion Stats, Obtain Last Visit Stats:

$$\mathcal{R} := \sum_{m=0}^{\mathcal{M}} \mathbf{R}^{(m)}, \;\; \mathcal{L} := \sum_{m=0}^{\mathcal{M}} \mathbf{L}^{(m)}$$

- **Last Visit Statistics G. F.**

$$(\dagger) \quad E\{r^{\mathcal{R}} z^{\mathcal{L}} u^{\mathcal{M}}\} = \sum_{m=0}^{\infty} p_m \, [u \cdot K_N]^m = \frac{P(\mathbf{H} \geq N)}{1 - u \cdot K_N \cdot P(\mathbf{H} < N)}$$

From $(\dagger)$ we obtain part (b) of Main Corollary.

Trigonometric Substitution Method.

- $q_{n+1} = \beta q_n - x q_{n-1}, \;\; q_0 = 0, \;\; q_1 = 1 \implies \sum q_n t^n = \frac{t}{1-\beta t + x t^2},$

$$\implies q_n = \frac{2^{-n}}{\alpha}\left[(\beta+\alpha)^n - (\beta-\alpha)^n\right], \quad \alpha = \sqrt{\beta^2 - 4x}$$

$$w_n = \frac{2^{-n}}{\alpha}\left[(\beta+\alpha)^n - (\beta-\alpha)^n\right] - 2x\left[(\beta+\alpha)^{n-1} - (\beta-\alpha)^{n-1}\right].$$

- Put $\beta = \sqrt{4x}\cos\theta, \implies \beta \pm \alpha = \sqrt{4x}(\cos\theta \pm i\sin\theta) = \sqrt{4x}\, e^{\pm i\theta}$.
- Thus, putting $r = e^{2it}$, $z = e^{-it}$, to conform with $\Delta' := 2\mathcal{R}' - \mathcal{L}'$; (Meander),

$$g_N = \frac{(N+1)e^{it}\sin\theta}{2\sin N\theta - e^{-it}\sin(N-1)\theta}, \quad \cos\theta = e^{it} - \frac{1}{4}e^{3it} + \frac{1}{4}e^{-it}.$$

- Replace t by t/N, so $\theta \sim it/N$, and let $N \to \infty$:

$$\lim g_N = \lim \frac{(N+1)it/N}{\sin(it)} = \frac{t}{\sinh t}.$$

Large Deviations Case.

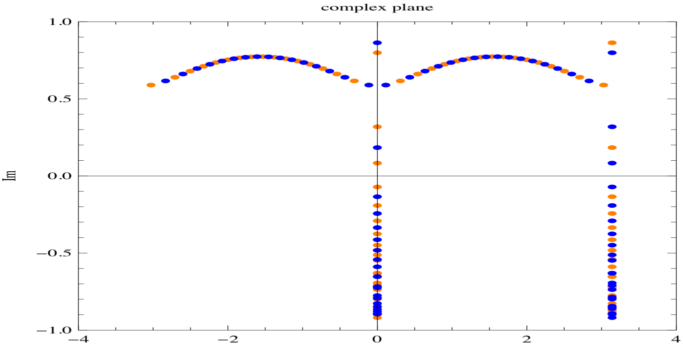
Lemma

If $r = e^{2it}$, $z = e^{-it}$, $s = e^{it}$, *then* $\quad (w_N - \frac{1}{2}sq_N)(w_N + \frac{1}{2}sq_N) = w_{2N}.$

Write: $s = e^{i(t+iy)}$, or $t + iy = -i \log s$.

Poles $t + iy$ of G.F.s: Δ_{N+1} & Δ'_{2N+1}, via zeros of $w_N - \frac{1}{2}sq_N$ &

Transf. Poles of G.F.s: Last Visit (Orange N=20); Meander (All, N=40)



Symmetry.

Large Dev. Principle and Moderate Dev. Principle

$\{Y_N\}$ satisfies a Large Deviation Principle (LDP) if $\exists$ “speed" $a_N \to \infty$ and lower-semicontinuous rate function $I : \mathbb{R} \to [0, \infty]$ such that: if $G \subseteq \mathbb{R}$ is open and $F \subseteq \mathbb{R}$ is closed, then

$$(*) \quad \begin{aligned} &\liminf_{N\to\infty} \frac{1}{a_N} \log P(Y_N \in G) \geq -\inf_{x\in G} I(x), \text{ and,} \\ &\limsup_{N\to\infty} \frac{1}{a_N} \log P(Y_N \in F) \leq -\inf_{x\in F} I(x). \end{aligned}$$

- IID Case: $Y_N := \frac{1}{b_N} \sum_{n=1}^{N} \xi_n$.

  MDP: If $\frac{b_N}{\sqrt{N}} \uparrow \infty$, $\frac{b_N}{N} \downarrow 0$, then under $E\{\xi_1\} = 0$ & exp. moments, $(*)$ holds with $I(x) = \frac{1}{2\sigma^2} x^2$, $a_N := b_N^2/N$. So, if $0 < d < 1$,

  $$\log P(|\textstyle\sum_{n=1}^{N} \xi_n| \geq x\sqrt{N^{1+d}}) \sim -\frac{1}{2\sigma^2} x^2 N^d, \; x > 0.$$

  If $b_N = N$, obtain instead Cramér's Thm: $(*)$ holds with

  $$I(x) = \sup_h \left[h \cdot x - \log E\{e^{h\cdot\xi_1}\} \right], \quad a_N = N.$$

LDP for Integer Valued Statistics

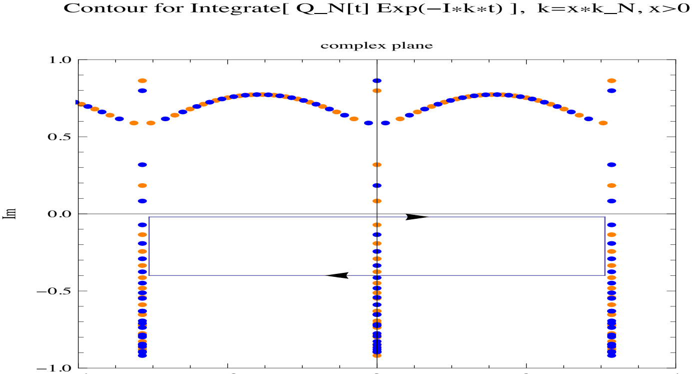
- Let $\{X_N\}$ integer valued, $a_N := k_N/N \to \infty$, $Y_N := X_N/k_N$.
  If $X_N = \Delta'_N$, LDP $(*)$ obtains. <u>Define</u>: $Q_N(z) := E\{e^{izX_N}\}$.
- Write poles of Q_N: $z_j = t_j + iy_j$, $t_j \in (-\pi, \pi]$. <u>Assume</u>:
  (1) $\exists\, \ell_- < 0, \ell_+ > 0$, s.t. $Z^* := \{t_j + iy_j : \ell_- < y_j < \ell_+\}$ satisfies
  $t_j = \epsilon\pi$, $\epsilon \in \{0, 1\}$. CASES (A) 2, or (B) 1, ϵ -values.
- Order $\{y_j\}$ by $\cdots < y(-2) < y(-1) < 0 < y(1) < y(2) < \cdots$.
  (2) $y(\pm 1) \sim \pm\mu/N$, $y(\pm 2) \sim \pm(\mu + \delta)/N$, const. $\mu > 0, \delta > 0$.
  (3) $\sup_t |Q_N(t + i\ell_-)| + |Q_N(t + i\ell_+)| = O(N^a)$.
  (4) $\sum_{z_j \in Z^*} |\mathrm{Res}(Q_N, z_j)| = O(N^b)$, $|\mathrm{Res}(Q_N, \epsilon\pi + iy(\pm 1))| \gg N^{-c}$.

Theorem

Let $k_N/(N \log N) \to \infty$. Assume (1)–(4). Under (A) assume $Res(Q_N, \pi + iy(\pm 1)) = (-1)^N Res(Q_N, iy(\pm 1))$. Then LDP $(*)$ holds with $I(x) := \mu|x|$, $x \in \mathbb{R}$.

Contour Integral

- Method of Proof:
  $Q_N(t) = \sum_{k=-\infty}^{\infty} e^{ikt} P(X_N = k) \Longrightarrow$
  $$P(X_N = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Q_N(t) e^{-ikt} dt.$$
  Let $x > 0$, $k := \lfloor x k_N \rfloor$.
- Introduce a contour in lower half-plane, one edge along t–axis, negatively oriented, avoiding poles on $t = \pm\pi$ by inward semicircles.



Construction for Meander Case

- Meander Case: For $k > 0$,

$$iP(X_N = k) = e^{ky(-1)}\text{Res}(Q_N, iy(-1))\left[1 + (-1)^{N+k}\right]$$

$$+ \sum_{j\leq -2} e^{-ikz_j}\,\text{Res}(Q_N, z_j) + e^{k\ell_-} \times O(N^b).$$

- Goal: Remove $\log N$ from $k_N/(N\log N) \to \infty$ in case $X_N = \Delta'_N$.
- To Show: $\exists$ constant $c_0 > 0$ s.t. $\ell_- \sim -c_0$, $\ell_+ \sim +c_0$.
- Therefore, must obtain control on poles, or zeros of equation

$$(**)\quad 2\sin N\theta - e^{-iz}\sin(N-1)\theta = 0, \quad \cos\theta = e^{iz} - \tfrac{1}{4}e^{3iz} + \tfrac{1}{4}e^{-iz}.$$

- Introduce branch of θ:

$$\theta(z) = i\log w(z), \quad w(z) := \frac{1}{4}e^{-iz}\left(1 + 4e^{2iz} - e^{4iz} + (-1 + e^{2iz})\sqrt{1 - 6e^{2iz} + e^{4iz}}\right).$$

Calculation of Poles

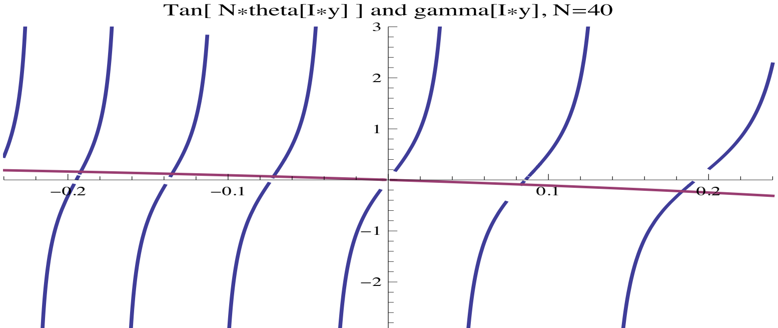
- Trig $\Longrightarrow \quad (**) \qquad \tan N\theta = \gamma(z), \quad \gamma(z) := -\frac{e^{-iz}\sin\theta}{2-e^{-iz}\cos\theta}$.

  Denote roots of $(**)$, for $z = iy$, by $y(j)$, $j = \ldots, -2, -1, 1, 2, \ldots$.

  $\theta(iy)$ is real for $|y| \le \log(1+\sqrt{2})$

Lemma

$$\left|\theta(iy(j)) - \frac{j\pi}{N}\right| \le \frac{\arctan(4|y(j)|/3)}{N}, \quad \textit{and} \quad \left|y(j) - \frac{j\pi}{N}\right| \le \frac{6j^2\pi^2+2|j|\pi}{N^2}, \quad \forall\ |j| \le c_0 N$$



Control of Poles: Meander & Last Visit

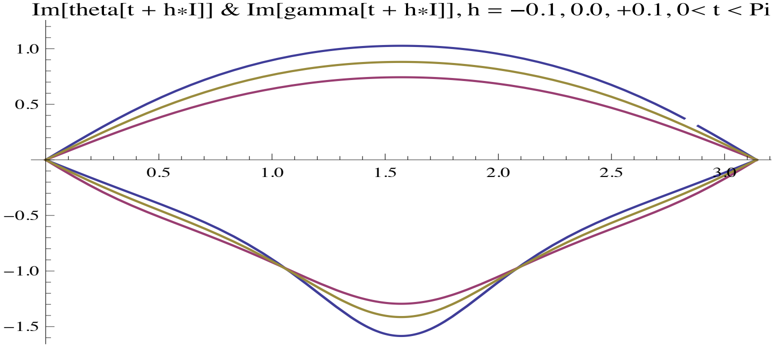
Lemma

$\exists$ constant $c_0 > 0$ s.t. $\forall$ $|y(j)| \le c_0$, the only poles are: $\epsilon\pi + iy(j)$.

Proof Sketch. $\Im(\tan(a+ib)) = \frac{\sinh 2b}{\cos 2a + \cosh 2b}$ has same sign as b,

So $\Im(\tan N\theta(t+iy))$ has same sign as $\Im(\theta(t+iy))$. We show:

$\text{sign}\,(\Im(\tan N\theta(t+iy))) > 0$ & $\text{sign}\,(\Im(\gamma(t+iy))) < 0$, $\forall\, |y| \le c_0$, $t \ne n\pi$.



Control of the magnitude of Q_{N+1}

Lemma

$\exists\ \ell_-,\ \ell_+ \sim c_0$ *such that:* $\sup_t \left(|Q_N(t+i\ell_-)| + |Q_N(t+i\ell_+)|\right) = O(N).$

Proof. Define $J_+ := \max\{j : y(j) \le c_0\}$.

Define $y = \ell_+$ such that: $\theta(iy) = (\pi J_+ - \pi/2)/N$.

Thus $|\sin N\theta(i\ell_+)| = 1, \quad \cos N\theta(i\ell_+) = 0.$

Write the denominator q_{N+1} of Q_{N+1} by

(1) $\quad q_{N+1}(t+i\ell) = \cos N\theta \left(2 - e^{-i(t+i\ell)}\cos\theta\right)\{\tan N\theta - \gamma(t+i\ell)\}.$

Consider regimes: (I) $\frac{c}{N} \le t \le \pi - \frac{c}{N}$, and (II) $0 \le t \le \frac{c}{N}$.

In regime (I) each of the 3 factors in (1) is bounded below by a constant in absolute value. In regime (II), use

$$q_{N+1} = 2\sin N\theta(t+i\ell) - e^{-i(t+i\ell)}\sin(N-1)\theta(t+i\ell) = S_1 - S_2.$$

Over (II), $|S_1| \ge \frac{7}{4}$; and $|S_2| \le \frac{5}{4}$. Finally, $Q_{N+1} = \frac{(N+1)\sin\theta}{q_{N+1}}$. $\square$

Completion of Proof: Meander Case

Corollary

If $a_N := \frac{k_N}{N} \to \infty$, then $Y_N := \frac{\Delta'_N}{k_N}$ satisfies the LDP $(*)$, with $I(x) = \pi|x|$.

Proof. Take $x > 0$, $k := \lfloor xk_N \rfloor$. $\theta(j) := \theta(iy(j))$. $m(j) := \frac{\partial}{\partial y}\theta(iy)_{y=y(j)}$.

$$\mathrm{Res}(Q_N, \epsilon\pi + y(j)) = i\,(-1)^{N\epsilon}\, N\, \sin\theta(j)/\left[D(j)(-1)^j \cos h_j\right],$$

$$D(j) := 2m(j)(N-1) - e^{y(j)}\left[m(j)(N-2)\frac{2\cos\theta(j) - e^{y(j)}}{2 - e^{y(j)}\cos\theta(j)} + \frac{-\sin\theta(j)}{2 - e^{y(j)}\cos\theta(j)}\right].$$

The sign requirement of Theorem, Case (A), holds.

$$(\dagger) \quad P(\Delta'_N = k) + O(e^{-\frac{1}{2}c_0 x k_N}) = \sum_{j=J_-}^{-1} (-1)^j \left(1 + (-1)^{N+k}\right) e^{ky(j)} \left[N\, \sin\theta(j)/\left(D(j)\cos h_j\right)\right].$$

$N\, \sin\theta(j)/\left(D(j)\cos h_j\right) \sim -\pi/N$ at $j = -1$, & $= O\left(|j|/N\right)$, uniformly for $|j| \le c_0 N$. Same estimates hold for for $y(j)$.

Introduce sum over k from $(a/\pi)k_N$ to ∞ for upper bound. Since G is open, also sum over a large region of k values for lower bound. The geometric sums on k cancel the factor $(1/N)$ coming from $(\dagger)$. $\square$

Conclusion

- The Future Maxima Decomposition is amenable to the study of the Joint Generating Function of certain lattice path statistics.
- Applications:
  - Order N scaling limit laws for certain integer valued statistics constructed from linear combinations of lattice path statistics in gambler's ruin.
  - A large deviation principle holds in this context.
- Outlook
  - The method shown extends to a local limit for the probability mass function of certain integer valued statistics in gambler's ruin, thus extending the limit laws for these statistics.
  - One wants to extend the results for limit distributions to the case of a larger “step–set" S. The step set here is $S = \{-1, 1\}$. One expects to be able to handle the case $S = \{-1, 0, 1\}$. However the case of larger steps sets in general is not clear.

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