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Stochastic Processes and their Applications 125 (2015) 2010–2025

stochastic processes and their applications

www.elsevier.com/locate/spa

Laws relating runs and steps in gambler's ruin

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Received 20 June 2014; received in revised form 21 November 2014; accepted 12 December 2014
Available online 19 December 2014

Abstract

Let $\mathbf{X}_j$ denote a fair gambler's ruin process on $\mathbb{Z} \cap [-N, N]$ started from $\mathbf{X}_0 = 0$, and denote by $\mathcal{R}_N$ the number of runs of the absolute value, $|\mathbf{X}_j|$, until the last visit $j = \mathcal{L}_N$ by $\mathbf{X}_j$ to 0. Then, as $N \to \infty$, $N^{-2}\mathcal{R}_N$ converges in distribution to a density with Laplace transform: $\tanh(\sqrt{\lambda})/\sqrt{\lambda}$. In law, we find: $2\left(\lim_{N\to\infty} N^{-2}\mathcal{R}_N\right) = \lim_{N\to\infty} N^{-2}\mathcal{L}_N$. Denote by $\mathcal{R}'_N$ and $\mathcal{L}'_N$ the number of runs and steps respectively in the meander portion of the gambler's ruin process. Then, $N^{-1}\left(2\mathcal{R}'_N - \mathcal{L}'_N\right)$ converges in law as $N \to \infty$ to the density $(\pi/4)\text{sech}^2(\pi x/2), \ -\infty < x < \infty$.

MSC: 60F05; 05A15

Keywords: Gambler's ruin; Runs; Last visit; Meander; Generalized Fibonacci polynomial

1. Introduction

Define a simple random walk $\{\mathbf{S}_j, \ j \geq 0\}$ on $\mathbb{Z}$ such that $\mathbf{S}_0 = 0$, and the increments $\varepsilon_j := \mathbf{S}_j - \mathbf{S}_{j-1}, \ j = 1, 2, \ldots,$ are independent random variables with $P(\varepsilon_j = 1) = p$ and $P(\varepsilon_j = -1) = 1 - p$ for all $j \geq 1$. A simple, symmetric random walk is the case $p = \frac{1}{2}$, and in general we utilize the parameter $\xi := p(1-p)$. Here $\xi = \text{Var}(\varepsilon)$ and the case $\xi = \frac{1}{4}$ corresponds to the symmetric case which is recurrent, and the $\xi < \frac{1}{4}$ corresponds to the asymmetric case which is transient, [4]. We call $\Gamma_{m,n} := \{(j, \mathbf{S}_j), \ j = m, \ldots, n\}$ a path of the random walk. An excursion is a path $\Gamma_{m,n}$ such that $\mathbf{S}_m = \mathbf{S}_n = 0$, but $\mathbf{S}_j \neq 0$ for $m < j < n$. A positive excursion

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http://dx.doi.org/10.1016/j.spa.2014.12.005

is an excursion that lies above the x-axis save for its endpoints. We define the number of runs in a path $\Gamma_{m,n}$ as one plus the number of indices j, $m < j < n$, such that $\varepsilon_{j+1} = -\varepsilon_j$. For a positive excursion path, the number of runs is just twice the number of peaks, where a peak corresponds to $\varepsilon_j = 1$ and $\varepsilon_{j+1} = -1$. Define the index j, or step, of first return of a random walk to the origin by $\mathbf{L} := \inf\{j \geq 1 : \mathbf{S}_j = 0\}$; $\mathbf{L}$ is finite a.s. for a simple random walk by recurrence. Define the excursion sequence from the origin by $\Gamma_0 := \{(j, \mathbf{S}_j), j = 0, \ldots, \mathbf{L}\}$; $\mathbf{L}$ is called the number of steps of Γ_0. Finally, define $\mathbf{R}$ as the number of runs along Γ_0, where $\mathbf{R} = \infty$ on $\{\mathbf{L} = \infty\}$ in the case of the asymmetric walk, $p \neq \frac{1}{2}$.

We now introduce the motivation and results of this paper. Define the height $\mathbf{H}$ of the excursion Γ_0 as the maximum absolute value of the walk over this excursion:

$$\mathbf{H} := \max\{|\mathbf{S}_j| : j = 1, \ldots, \mathbf{L}\}. \tag{1.1}$$

In the transient case $\xi < \frac{1}{4}$, $\mathbf{H}$ takes the value $+\infty$ with positive probability $\sqrt{1-4\xi} = |2p-1|$. Our primary goal is to establish in Theorem 1 an explicit formula for the conditional joint generating function of $\mathbf{R}$ and $\mathbf{L}$ given $\{\mathbf{H} \leq N\}$:

$$K_N(z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} \leq N\}. \tag{1.2}$$

The case of steps alone for simple random walk, that is $y = 1$ and $p = \frac{1}{2}$ in (1.2), was solved by [3] as noted in [5, p. 327]. Our motivation for solving (1.2) is to apply the result to the symmetric gambler's ruin problem, especially for the case $p = \frac{1}{2}$. The symmetric gambler's ruin process on $[-N, N]$ for general p is a Markov chain $\{\mathbf{X}_j, j \geq 0\}$ on fortunes $f \in \mathbb{Z} \cap [-N, N]$, started from fortune $\mathbf{X}_0 = 0$, with transition from fortune f to $f + 1$ with probability p, and from f to $f - 1$ with probability $1 - p$, independent of f. The process terminates at the first step $j = \tau_N$ (stopping time) such that $|\mathbf{X}_j| = N$, that is when the gambler has either won or lost a total of N units on successive independent games with unit bets and probability p of winning each game. We may think of repeated independent attempts for an excursion of the random walk $\{\mathbf{S}_j\}$ to reach at least height N and thus terminate the gambler's ruin process. Hence if we denote by $\mathcal{M}_N$ the number of excursions of the random walk until the step $\mathcal{L}_N$ of last visit to the fortune $f = 0$ by the gambler's ruin process, then $\mathcal{M}_N$ is a geometric random variable. Define $\mathcal{R}_N$ as the total number of runs of the path $\{(j, |\mathbf{X}_j|), j = 0, \ldots, \mathcal{L}_N\}$. Equivalently, $\mathcal{R}_N$ is the sum of the number of runs over each of the $\mathcal{M}_N$ excursions that make up the gambler's ruin path until the last visit. Thus by applying Theorem 1 we may explicitly write the joint generating function of $\mathcal{R}_N$ and $\mathcal{L}_N$; see (3.6). As a means to prove Theorem 1, we establish Proposition 2. This proposition yields the joint generating function for the number of runs, $\mathcal{R}'_N$, and steps, $\mathcal{L}'_N$, over the so-called meander, that is the remainder of the gambler's ruin process following the last visit; see (1.5)–(1.6). We apply Proposition 2 and Theorem 1 to find limit distributions relating runs and steps as $N \to \infty$ in Corollary 2.

A key feature of our proof of an explicit formula for (1.2) involves certain bivariate polynomials $\{w_n(x, u), n \geq 1\}$, see (1.10), (2.14), that generalize the univariate Fibonacci polynomials $F_n(x) = w_n(x, 0)$. We now sketch our approach to see how these bivariate polynomials arise. First, instead of directly attacking the condition $\{\mathbf{H} \leq N\}$ in (1.2), we condition on the event $\{\mathbf{H} = n\}$ as follows. Let $G_n(y, z)$ denote the conditional joint probability generating function of the number of runs and steps along Γ_0 given that the height is n:

$$G_n(y, z) := E(y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} = n), \quad n \geq 1. \tag{1.3}$$

Define for $n \geq 1$, the stopping time

$$\mathbf{L}'_n := \inf\{j \geq 1 : |\mathbf{S}_j| = n\}. \tag{1.4}$$

Let $\Gamma'_n := \{(j, \mathbf{S}_j), j = 0, \ldots, \mathbf{L}'_n\}$ denote the first passage path to height n. By definition, $\mathbf{L}'_n$ denotes the number of steps along this first passage path; denote by $\mathbf{R}'_n$ the number of runs along Γ'_n. Define $g_n(y, z)$ as the conditional joint probability generating function of $\mathbf{R}'_n$ and $\mathbf{L}'_n$ given a non-negative path:

$$g_n(y, z) := E(y^{\mathbf{R}'_n} z^{\mathbf{L}'_n} | \mathbf{S}_j \geq 0, \ j = 1, \ldots, \mathbf{L}'_n). \tag{1.5}$$

Since any non-negative path in Γ'_n has probability given by the constant factor $(p/(1-p))^n$ times the probability of the corresponding (reflected) non-positive path, if we condition instead on a non-positive path in (1.5) then by one to one correspondence between paths we arrive at the same function $g_n(y, z)$. Observe therefore, by the definition of the meander in gambler's ruin and (1.5), that

$$E\{y^{\mathcal{R}'_N} z^{\mathcal{L}'_N}\} = z g_{N-1}(y, z). \tag{1.6}$$

Notice that for $n \geq 1$ the event $\{\mathbf{H} = n + 1\}$ implies that $\varepsilon_1 = \varepsilon_2$ (since if $\varepsilon_2 = -\varepsilon_1$ then $\mathbf{H} = 1$), and also that a run must end at step $\mathbf{L}'_{n+1}$. Thus it follows directly from (1.1)–(1.5), by regarding the first passage part Γ'_{n+1} of Γ_0 followed by the remaining part of Γ_0, that

$$G_{n+1}(y, z) = z g_n(y, z) g_{n+1}(y, z), \quad n \geq 1. \tag{1.7}$$

Our first result is a recurrence relation for the sequence $\{g_n\}$. To accomplish this we combine (1.7) with a second formula for $G_{n+1}(y, z)$ as an explicit product formula in terms of $g_m, m = 1, \ldots, n$, that we obtain by a path decomposition in Section 2; see (2.4). In the formula (2.4), we must take account of the probability that a first passage to height n remains non-negative, as defined by ρ_n:

$$\rho_n := P(\mathbf{S}_j \geq 0, \ j = 1, \ldots, \mathbf{L}'_n), \qquad \rho_{-n} := P(\mathbf{S}_j \leq 0, \ j = 1, \ldots, \mathbf{L}'_n). \tag{1.8}$$

The probability ρ_n is determined by the classical solution of the gambler's ruin problem on $[0, n+1]$ started from fortune $f = 1$; see Lemma 1 in Section 2.2.

Proposition 1.

$$g_{n+1} = C_n \frac{g_n^2}{g_{n-1}} \frac{1}{1 - [\rho_n \rho_{-n} g_n^2]}, \quad n \geq 2, \tag{1.9}$$

where $g_1 = yz, \ g_2 = (1-\xi) y z^2 / [1 - \xi y^2 z^2]$, *and* $C_n = 1 - \rho_n \rho_{-n}$.

To unravel the recursion of Proposition 1, we introduce a sequence of polynomial functions, $\{w_n(x, u)\}$, with variables $x := \xi z^2$ and $u := y^2 - 1$, such that $w_n(x, u)$ serves as a denominator of the rational expression for $g_n(y, z)$. The variable u is chosen such that $u = 0$ corresponds to eliminating the role of runs in the calculation.

Definition 1. Define $w_n(x, u)$ by the following recurrence relation:

$$w_{n+1} = (1 - xu) w_n - x w_{n-1}, \qquad w_0 = w_1 = 1, \quad n \geq 1. \tag{1.10}$$

In case $u = 0$, the recursion (1.10) becomes the definition (2.5) of the Fibonacci polynomials $\{F_n(x) = w_n(x, 0)\}$ in one variable (see [5, p. 327]). An identity for $\{w_n\}$ that arises naturally in the solution of the recurrence of Proposition 1 is as follows.

$$w_n^2 - x^n(u+1) = w_{n+1}w_{n-1}, \quad \text{for all } n = 1, 2, \ldots, \; w_0 = w_1 = 1. \tag{1.11}$$

In fact by (2.7)–(2.8), any two term recurrence $v_{n+1} = \beta v_n - x v_{n-1}, n \geq 1$, with coefficients β and x independent of n, satisfies

$$v_{n+1}v_{n-1} - v_n^2 = \beta^{-1}x^{n-1}(v_3v_0 - v_2v_1), \quad \beta \neq 0. \tag{1.12}$$

Hence we obtain (1.11) as a consequence of (1.10), (1.12). In particular, when $v_0 = 0,\ v_1 = 1$, the polynomials $v_n = v_n(\beta, -x)$ are called the generalized Fibonacci polynomials in β and $-x$, [12].

Proposition 2.

$$g_n(y,z) = \frac{F_n(\xi)yz^n}{w_n(x,u)}, \quad n \geq 1.$$

The proof of the formula in Proposition 2 follows from Proposition 1 and (1.11) by induction. The solution g_n provided by Proposition 2 obviously also yields a formula for G_n via (1.7).

Armed with an explicit formula for G_n, by (1.3) we aim to calculate $K_N(y,z)$ in (1.2) by summation:

$$K_N(y,z) = \frac{1}{P(\mathbf{H} \leq N)} \sum_{n=1}^{N} G_n(y,z)P(\mathbf{H} = n). \tag{1.13}$$

In the case of steps alone, with $p = \frac{1}{2}$, we have: $K_N(1,z) = \frac{F_N(\xi)}{F_{N-1}(\xi)} \frac{z^2F_{N-1}(x)}{F_N(x)}$ [5, p. 327]. In particular the numerator of the rational function expression is closely related to the denominator. For our two variable case we must introduce a numerator polynomial as well, that we make explicit in (2.11) of Definition 2. In fact by Remark 1 in Section 2.5, the numerator polynomials $\{q_n\}$ of Definition 2 satisfy the recurrence (2.12), that is the same two term recurrence that defines $\{w_n\}$, but with a different initial condition, namely $q_0 = 0, q_1 = 1$. So in fact $q_n(x,u)$ is the generalized Fibonacci polynomial $v_n(\beta, -x)$, evaluated at $\beta = 1 - xu$.

Theorem 1. *The generating function (1.2) has the following formula.*

$$K_N(y,z) = \frac{F_N(\xi)}{F_{N-1}(\xi)} \frac{y^2z^2q_N(x,u)}{w_N(x,u)}, \quad N \geq 1,$$

where, if $\xi = \frac{1}{4}$ the normalization constant is written $\frac{F_N(\xi)}{F_{N-1}(\xi)} = \frac{N+1}{2N}$.

To prove Theorem 1 we use an induction argument that relies on the fact that the sequence of numerators $\{q_n\}$ will satisfy the following identity.

$$q_nw_{n+1} - w_nq_{n+1} = -x^n, \quad \text{for all } n = 1, 2, \ldots. \tag{1.14}$$

In the one-variable case, we have $q_n(x,0) = F_{n-1}(x) = w_{n-1}(x,0), n \geq 1$. Thus in case $u = 0$, (1.14) follows from (1.11). We observe that w_n and q_n are related in general by

$$w_n = q_n - xq_{n-1}, \quad n \geq 1. \tag{1.15}$$

The recursion (1.15) follows by (1.10) and (2.12) because $\{d_n := w_n - q_n\}$ also satisfies the basic two term recurrence $d_{n+1} = \beta d_n - x d_{n-1}$, for $\beta = 1 - xu$, but with the following initial conditions: $d_1 = 0 = -xq_0$, and $d_2 = -x = -xq_1$.

We remark that the joint generating function $K(y, z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < \infty\}$ follows by taking the limit as $N \to \infty$ in Theorem 1. In fact $K(y, z)$ is already known for $p = \frac{1}{2}$. To see this, define the ℓth Narayana polynomial $\mathcal{N}_\ell(s) := \sum_{k=1}^{\ell} \mathcal{N}_{\ell,k} s^{k-1}$; $\mathcal{N}_1(s) = 1$, where $\mathcal{N}_{\ell,k}$ are the Narayana numbers that count the number of ℓ-Dyck paths with k peaks, [OEIS A001269]. Here a ℓ-Dyck path is a non-negative path of 2ℓ steps which is zero at the end points. Now, because a positive excursion of 2ℓ steps is obtained simply by raising a $(\ell - 1)$-Dyck path to level 1, we have the following formula in case $\xi = \frac{1}{4}$:

$$P(\mathbf{R} = 2k, \mathbf{L} = 2\ell) = 2 \times 4^{-\ell} \mathcal{N}_{\ell-1,k}, \quad \ell \geq k + 1 \geq 2, \qquad \xi = \frac{1}{4}. \tag{1.16}$$

Define the generating function of the sequence of Narayana polynomials by $\mathcal{N}(s, t) := 1 + \sum_{\ell=1}^{\infty} \mathcal{N}_\ell(s) t^\ell$. By (1.16) we find: $E\{y^{\mathbf{R}} z^{\mathbf{L}}\} = \frac{1}{2} y^2 z^2 \mathcal{N}(y^2, z^2/4)$. Yet $\mathcal{N} = \mathcal{N}(s, t)$ satisfies the identity $\mathcal{N} = 1 + t\mathcal{N} + st(\mathcal{N}^2 - \mathcal{N})$, which can be seen by decomposing Dyck paths into positive excursions plus Dyck paths with at least one zero between their endpoints. So we solve for $\mathcal{N}$ and thus obtain (see [9]):

$$K(y, z) = 1 + xu - \sqrt{(xu - 1)^2 - 4x}, \qquad \xi = \frac{1}{4}. \tag{1.17}$$

We shall recover (1.17) as the special case $\xi = \frac{1}{4}$ of Corollary 3 in Section 3.

The following corollaries constitute our main applications of Theorem 1.

Corollary 1. *Let $p = \frac{1}{2}$. Then, as $N \to \infty$, we have that $N^{-2}\mathcal{R}_N$ converges in distribution to the probability density*

$$f(x) = 2 \sum_{\nu=1}^{\infty} e^{-\pi^2 (2\nu-1)^2 x/4} = \frac{1}{\sqrt{\pi x}} \sum_{\nu=-\infty}^{\infty} (-1)^\nu e^{-\nu^2/x}, \quad x > 0.$$

The Laplace transform of $f(x)$ is given by $\int_0^\infty e^{-\lambda x} f(x) \mathrm{d}x = \tanh(\sqrt{\lambda})/\sqrt{\lambda}$. In law, we have: $\lim_{N\to\infty} N^{-2}\mathcal{L}_N = 2 \lim_{N\to\infty} N^{-2}\mathcal{R}_N$. *Further,*

$$\lim_{N\to\infty} E\{\exp(-\lambda \mathcal{R}'_N / N^2)\} = \sqrt{\lambda} / \sinh \sqrt{\lambda}, \tag{1.18}$$

and the same limiting Laplace transform (1.18) holds with $\frac{1}{2}\mathcal{L}'_N$ in place of $\mathcal{R}'_N$.

The second formula for $f(x)$ in Corollary 1 may be checked directly to have the indicated Laplace transform by: $\int_0^\infty x^{-1/2} e^{-\nu^2/x} e^{-\lambda x} \mathrm{d}x = \sqrt{\pi/\lambda} e^{-2\sqrt{\lambda}|\nu|}$, [6, 17.13–31]. The equality of the two formulae for $f(x)$ in Corollary 1 may also be obtained as a consequence of a theta function identity (for example apply [2, formula (3.12)], with a reciprocal transformation in x). The density $g(x) := \frac{1}{2} f(x/2)$, $x > 0$, for the scaling limit of the number of steps $\mathcal{L}_N$ until the last visit was first established by [7] as the density for the last time that a standard Brownian motion hits zero before exiting the interval $[-1, 1]$. By independence and Corollary 1 we also have that $(\mathcal{R}_N + \mathcal{R}'_N)/N^2$ converges in law to a distribution with Laplace transform $1/\cosh \sqrt{\lambda}$. The probability distributions determined by the Laplace transforms $\sqrt{\lambda}/\sinh \sqrt{\lambda}$ and $1/\cosh \sqrt{\lambda}$ may be found by [2, Table 1, pp. 442–443].

Let $p = \frac{1}{2}$. By Corollary 1 we expect $2\mathcal{R}_N - \mathcal{L}_N$ to have order N instead of N^2. Denote $\Delta_N := 2\mathcal{R}_N - \mathcal{L}_N - \mathcal{M}_N$ and $\Delta'_N := 2\mathcal{R}'_N - \mathcal{L}'_N$. We obtain the following absolutely continuous scaling limits with densities shown under the characteristic function transforms. Here we remark that $\mathcal{M}_N$ is a centering term in the definition of Δ_N since it may be seen by (1.17) that $2\mathbf{R} - \mathbf{L}$ is the mixture of a symmetric distribution with an atom located at integer $d = 2 : K(e^{2it}, e^{-it}) - \frac{1}{2}e^{2it} = 1 - \frac{1}{2}\cos(2t) - \sqrt{(1 - \frac{1}{2}\cos(2t))^2 - \frac{1}{4}}$, $t \in \mathbb{R}$.

Corollary 2. *Let $p = \frac{1}{2}$. Then, for $t \in \mathbb{R}$, there holds:*

$$\text{(i)} \lim_{N\to\infty} E\{e^{it\Delta_N/N}\} = \frac{\tanh(t)}{t} = \int_{-\infty}^{\infty} e^{itx}\left[\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{2\nu+1}e^{-(2\nu+1)\pi|x|/2}\right]\mathrm{d}x,$$

$$\text{(ii)} \lim_{N\to\infty} E\{e^{it\Delta'_N/N}\} = \frac{t}{\sinh(t)} = \int_{-\infty}^{\infty} e^{itx}\left[\frac{\pi}{4\cosh^2(\pi x/2)}\right]\mathrm{d}x,$$

$$\text{(iii)} \lim_{N\to\infty} E\{e^{it(\Delta_N+\Delta'_N)/N}\} = \frac{1}{\cosh(t)} = \int_{-\infty}^{\infty} e^{itx}\left[\frac{1}{2\cosh(\pi x/2)}\right]\mathrm{d}x.$$

Corollaries 1 and 2 are proved in Section 3.1.

In Section 4 we write representations for the distribution of $\mathbf{R}$ in terms of Legendre functions. This leads to Remark 4, which relates certain Catalan rook paths of Section 3 to the distribution of $\mathbf{R}$ for the case of simple, symmetric random walk.

2. Path decomposition

We construct a path decomposition that will enable us to establish the formula (2.4), and which leads, when combined with (1.3), to the recurrence (1.9). Therefore the decomposition gives rise to the denominator polynomials (1.10) which appear as the solution to (1.9) in Proposition 2.

Fix an integer $a \geq 1$. We decompose the event $\{\mathbf{H} = a + 1\}$ pathwise. For the sake of argument we assume a positive excursion, so Γ_0 stays above the x-axis save for its endpoints. To simplify notation we write a path $\gamma \in \{\mathbf{H} = a + 1\}$ as a sequence of integer ordinates y. Thus $\gamma = \gamma_0, \gamma_1, \gamma_2, \ldots, 1, 0$, where $\gamma_0 = 0$ and $\gamma_1 = 1$, and where of course we have the random walk constraint: if $\gamma_j = y$ then $\gamma_{j+1} = y \pm 1$. The other constraints for γ are that γ reaches a maximum ordinate $a + 1$ and does not reach ordinate $y = 0$ again after the initial ordinate until the end of the sequence. After the first step, γ must eventually return to the ordinate $y = 1$ at least once. Denote $j_1 = 1$. We shall define inductively a certain subset J of steps j for which $\gamma_j = 1$. We shall denote $J = \{j_i : 1 \leq i \leq r + 1\}$ where $1 = j_1 < j_2 < \cdots < j_{r+1}$, for some $r \geq 1$, where the sequence $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{j_*}$ ends at step $j_* = 1 + j_{r+1}$, and where each step $j_{i+1} \in J$, $1 \leq i \leq r$, will be defined in terms of j_i and the path γ. To set up this approach, define $M_1 := a + 1$ as the absolute maximum of the path: $M_1 := \max\{\gamma_j : j_1 + 1 \leq j \leq j_*\}$. There is a first step $m(1)$ with $m(1) > j_1$ at which $\gamma_{m(1)} = M_1$. Define j_2 as the first step $j > m(1)$ at which $\gamma_j = 1$. Thus $j_1 < m(1) < j_2$ and $\gamma_j > 1$ for all $m(1) \leq j < j_2$ and $\gamma_{j_2} = 1$ Thus we have a first non-negative passage from zero to $M_1 - 1$ by the "upside-down" path $\tilde{\gamma}$ defined by $\tilde{\gamma}_j := M_1 - \gamma_j, m(1) \leq j \leq j_2$. That is $\tilde{\gamma}_j$ may take the value zero many times before reaching the value $M_1 - 1 = a$, and the zeros of $\tilde{\gamma}$ correspond to maximum values M_1 achieved by the original path γ at steps j with $m(1) \leq j < j_2$. Next define the "future maximum" M_2 by $M_2 := \max\{\gamma_j : j_2 + 1 \leq j \leq j_*\}$, and define $m(2)$ as the first step $j > j_2$ such that $\gamma_j = M_2$.

So, if $j_2 < j_*$, then starting from j_2 the path γ makes a first passage to height M_2 at step $m(2)$, while staying at or above level 1 for steps $j_2 \le j \le m(2)$. We must have that $M_2 \le M_1$.

We proceed inductively to define the epochs j_i, corresponding future maxima M_i, and indices $m(i)$, such that each future maximum M_i will satisfy $M_i = \gamma_{m(i)}$. Given that $j_i \in J$ has been constructed, define

$$M_i := \max\{\gamma_j : j_i + 1 \le j \le j_*\}, \quad \text{and} \quad m(i) := \inf\{j \ge j_i + 1 : \gamma_j = M_i\}.$$

Now if $M_i \ge 2$, define also

$$j_{i+1} := \inf\{j > m(i) : \gamma_j = 1\}.$$

If instead $M_i = 0$ then put $r := i$, so that the path γ terminates at epoch $m(r) = 1 + j_{r+1} = j_*$. Thus in summary, for $i \ge 1$, M_{i+1} is the maximum of the height of the remainder of the path after the path has returned to level 1 for the first time after the first step $m(i) \ge j_i + 1$ such that $\gamma_{m(i)} = M_i$.

We work with the non-decreasing sequence $M_1, M_2, \dots, M_r$, where by construction $M_1 = a + 1 \ge M_2 \ge \cdots \ge M_r \ge 2 > M_{r+1} = 0$. Write $\ell_b := |\{i : M_i = b + 1\}|, a \ge b \ge 1$, for the number of future maxima that take the value $b + 1$, where the absolute value signs denote cardinality of the given set of indices. Here by necessity $\ell_a \ge 1$ since the path γ must achieve its absolute maximum value $a + 1$. But the only condition in general is that $\ell_b \ge 0$ for $1 \le b < a$. Denote $\mathcal{E}^+ := \{\mathbf{S}_j \ge 0, j = 1, \dots, \mathbf{L}\}$ for the event we have a positive excursion. Denote by $H_a(\ell_a, \ell_{a-1}, \dots, \ell_1)$ the collection of non-negative paths $\gamma \in \{\mathbf{H} = a + 1\} \cap \mathcal{E}^+$ such that the future maximum value $a + 1$ is repeated ℓ_a times, the future maximum a is repeated ℓ_{a-1} times, and in general, the future maximum value $b + 1$ is repeated ℓ_b times for $1 \le b \le a$. Then we have the decomposition:

$$\{\mathbf{H} = a + 1\} \cap \mathcal{E}^+ = \bigcup_{\ell_a \ge 1, \ell_{a-1} \ge 0, \dots, \ell_1 \ge 0} H_a(\ell_a, \ell_{a-1}, \dots, \ell_1). \tag{2.1}$$

For $a = 1$ we have $\{\mathbf{H} = 2\} \cap \mathcal{E}^+ = \cup_{\ell_1 \ge 1} H_1(\ell_1)$, where $H_1(\ell_1)$ contains just one path, namely $\gamma = (0, 1, 2, 1, 2, 1, \dots, 2, 1, 0)$, wherein ℓ_1 is the number of repetitions of the ordinate $y = 2$. An example of a path $\gamma \in H_4(1, 0, 2, 3)$ is as follows:

$$\gamma = (0, 1, \overbrace{2, 1, 2, 3, 4, 5, 4, 5, 4, 3, 2, 1}^{\delta(1)}, \overbrace{2, 1, 2, 3, 2, 3, 2, 1}^{\delta(2)}, \overbrace{2, 3, 2, 1}^{\delta(3)} \overbrace{2, 1}^{\delta(4)} \overbrace{2, 1}^{\delta(5)}, \overbrace{2, 1}^{\delta(6)}, 0).$$

Note that $\sum_{b=1}^{a} \ell_b = r$, so in our example $r = 1 + 0 + 2 + 3 = 6$, and we have marked as $\delta(i),\ i = 1, \dots, 6$, the successive subpaths of γ defined by splitting up γ at the epochs $j_i \in J$. Thus in general define

$$\delta(i) := (\gamma_{j_i+1}, \gamma_{j_i+2}, \dots, \gamma_{j_{i+1}}), \quad 1 \le i \le r. \tag{2.2}$$

2.1. Path decomposition formula for G_a

Fix for the moment i with $1 \le i \le r$. If we insert a first ordinate 1 at the beginning the subpath $\delta(i)$ defined by (2.2), then by viewing the modified subpath first from the level $y = 1$ looking upwards we obtain a non-negative first passage to height $M_i - 1$, and by viewing the remainder of the subpath from the ordinate $y = M_i$ looking downward we have (in the upside down coordinate system) a non-negative first passage to height $b = M_i - 1$. So really we break up each subpath $\delta(i)$ into two pieces, with the dividing line between the two pieces coming at the point that the ordinate M_i is first reached in $\delta(i)$. Thus if we denote any realization δ of $\delta(i) = 2, 2 \pm 1, \ldots, 2, 1$ by $\delta = \delta_2, \delta_3, \ldots, \delta_n$, for $n = n(\delta)$ and $\delta_2 = 2$, and if we denote $b = M_i - 1$, then we obtain the following computation:

$$\sum_{\delta} P(\mathbf{S}_2 = 2, \mathbf{S}_3 = \delta_3, \mathbf{S}_4 = \delta_4, \ldots, \mathbf{S}_{n(\delta)} = 1 | \mathbf{S}_1 = 1) = \rho_b \rho_{-b}, \tag{2.3}$$

where the sum is over all possible realizations δ of $\delta(i)$, and ρ_b is defined in (1.8). Recall that for each path $\gamma \in H_a(\ell_a, \ldots, \ell_1)$ we have that $M_i - 1$ takes the value b with frequency ℓ_b, $1 \le b \le a$, and recall the definitions (1.1)–(1.5) for G_a and g_a. Note that for the path $\gamma = 0, 1, \delta(1), \delta(2), \ldots, \delta(r), 0$, with $\delta(i)$ defined by (2.2), we have that γ has the same number of runs as $1, \delta(1), \delta(2), \ldots, \delta(r)$, where we understand the sequences displayed as concatenations of subpaths; that is we do not need to include the first step and final step of the original γ to compute its number of runs. Therefore by using the full decomposition (2.1) and by applying (2.3) and the comments immediately preceding (1.7), we obtain

$$G_{a+1} = cz^2 \left(\sum_{\ell_a=1}^{\infty} [\rho_a \rho_{-a} g_a^2]^{\ell_a} \right) \prod_{b=1}^{a-1} \sum_{\ell_b=0}^{\infty} [\rho_b \rho_{-b} g_b^2]^{\ell_b}, \tag{2.4}$$

where c is the normalization constant so that $G_{a+1}(1, 1) = 1$, and the factor z^2 corresponds to the first and last steps in a positive excursion path.

2.2. *Fibonacci polynomial solution to gambler's ruin*

To cast a form for working with the quantities $\rho_n \rho_{-n}$ in (2.4) (see definition (1.8)), we recall the Fibonacci polynomials $\{F_n(x)\}$, defined by:

$$F_{n+1} = F_n - xF_{n-1}, \qquad F_0 = F_1 = 1, \quad n = 1, 2, \ldots. \tag{2.5}$$

The classical Fibonacci polynomials are defined by $F_0 = 0$, $F_1 = 1$, so our $\{F_n\}$ are actually the classical ones shifted by one index. It is well known (see [11, pp. 75–76]) that, with $\alpha_0 := \sqrt{1 - 4x}$,

$$F_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-x)^k = \frac{2^{-n-1}}{\alpha_0} \left[(1 + \alpha_0)^{n+1} - (1 - \alpha_0)^{n+1} \right]. \tag{2.6}$$

We pause to establish the identity (1.12) that arises from any constant coefficient two term recurrence $v_{n+1} = \beta v_n - x v_{n-1}$, $n \ge 1$.

$$\begin{aligned} v_{n+2}v_{n-1} - v_{n+1}v_n &= (\beta v_{n+1} - x v_n) v_{n-1} - (\beta v_n - x v_{n-1}) v_n \\ &= \beta(v_{n+1}v_{n-1} - v_n^2) = \beta\beta^{-1}[v_{n+1}(v_n + x v_{n-2}) \\ &\quad - v_n(v_{n+1} + x v_{n-1})] \\ &= x(v_{n+1}v_{n-2} - v_n v_{n-1}). \end{aligned} \tag{2.7}$$

Since obviously (2.7) may be iterated, we obtain

$$v_{n+2}v_{n-1} - v_{n+1}v_n = \beta(v_{n+1}v_{n-1} - v_n^2) = x^{n-1}(v_3 v_0 - v_2 v_1). \tag{2.8}$$

Therefore (1.12) follows from (2.8).

Lemma 1. *If $\xi < \frac{1}{4}$, then the following formulae hold for $n \ge 1$:*

(i) $\rho_n = p^n / F_n(\xi), \qquad \rho_n \rho_{-n} = \xi^n / F_n(\xi)^2;$
(ii) $P(\mathbf{H} = n) = 2\xi^n / \left[F_{n-1}(\xi) F_n(\xi)\right];$
(iii) $P(\mathbf{H} \le n) = 2\xi F_{n-1}(\xi) / F_n(\xi).$

If $\xi = \frac{1}{4}$, then $\rho_n = 1/(n+1)$ and $P(\mathbf{H} \le n) = n/(n+1)$.

Proof of Lemma 1. By the solution to the gambler's ruin problem [4, p. 345], we have, with $q := 1 - p$,

$$\rho_n = \frac{(q/p) - 1}{(q/p)^{n+1} - 1} = p^n \frac{p - q}{p^{n+1} - q^{n+1}}, \quad \text{and} \quad \rho_{-n} = (q/p)^n \rho_n. \tag{2.9}$$

But, since $\xi = pq$ yields $\sqrt{1 - 4\xi} = |p - q|$, by (2.6) we simply have $F_n(\xi) = |p^{n+1} - q^{n+1}|/|p - q|$. So statement (i) follows by (2.9). The statement (ii) then follows from (i) by the equality: $P(\mathbf{H} = n+1) = p\rho_n \rho_{-(n+1)} + q\rho_{-n}\rho_{n+1}$. Formula (iii) follows easily from (ii) and the identity $F_n^2 - x^n = F_{n+1}F_{n-1}$ (case $u = 0$ of (1.10)) by an induction that verifies the following identity:

$$\sum_{n=1}^{N} \frac{2\xi^n}{F_{n-1}(\xi) F_n(\xi)} = \frac{2\xi F_{N-1}(\xi)}{F_N(\xi)}, \qquad N \ge 1. \tag{2.10}$$

If $p = \frac{1}{2}$, the formula $\rho_n = 1/(n+1)$ follows from the classical solution of the gambler's ruin started from 1 on the interval $[0, n+1]$, and the last formula holds since $\rho_n = P(\mathbf{H} \ge n+1)$. □

2.3. The polynomials q_n and w_n

Definition 2. Define

$$q_n(x, u) := \frac{2^{-n}}{\alpha} \left[(\beta + \alpha)^n - (\beta - \alpha)^n\right], \qquad n \ge 0, \tag{2.11}$$

where α and β are defined by $\beta := 1 - xu$ and $\alpha := \sqrt{(xu - 1)^2 - 4x}$.

Remark 1. The following recursion is equivalent to the definition (2.11):

$$q_{n+1} = \beta q_n - x q_{n-1}, \qquad q_0 = 0,\ q_1 = 1,\ n \ge 1;\ \beta = 1 - xu. \tag{2.12}$$

Proof of Remark 1. The equivalence of (2.11) and (2.12) is evident in [12]. To see this directly, note that (2.11) gives $q_0 = 0$ and $q_1 = 1$. Further by substitution of (2.11) into (2.12), and by $\beta(\beta \pm \alpha) - 2x = \frac{1}{2}(\beta \pm \alpha)^2$, the recursion (2.12) is satisfied by (2.11) for all $n \ge 0$. □

Lemma 2. *The generating function of the sequence $\{q_n,\ n \ge 1\}$ in (2.12) is:*

$$q(x, u, t) := \sum_{n=1}^{\infty} q_n(x, u) t^n = \frac{t}{1 + (xu - 1)t + xt^2}. \tag{2.13}$$

Proof of Lemma 2. The expression on the right side of (2.13) follows immediately by applying the recurrence (2.12) to the definition of $q(x, u, t)$ and solving algebraically for $q(x, u, t)$. □

Remark 2.

$$q_n\left(\frac{1}{4},u\right) = Q_n(u) := 2^{-n+1}\sum_{j=0}^{n-1}\binom{n+j}{2j+1}\left(-\frac{u}{2}\right)^j, \quad n \geq 1.$$

Proof of Remark 2. Calculate the generating function: $Q(u,t) := \sum_{n=1}^{\infty} Q_n(u)t^n$ by applying the formula $\sum_{m=0}^{\infty} s^m \binom{j+m}{j} = (1-s)^{-j-1}$, and find that $Q(u,t)$ matches (2.13) at $x = \frac{1}{4}$. □

Lemma 3. *We have the following closed form expression for* $w_n(x,u)$*:*

$$w_n(x,u) = \frac{2^{-n}}{\alpha}\left\{(\beta+\alpha)^n - (\beta-\alpha)^n - 2x[(\beta+\alpha)^{n-1} - (\beta-\alpha)^{n-1}]\right\}, \tag{2.14}$$

with α *and* β *as defined in Definition 2.*

Proof of Lemma 3. The proof follows by (1.15) and (2.11) of Definition 2. □

The closed formula in (2.6) is consistent with (2.14) when $u = 0$.

2.4. *Recurrence for* $\{g_n\}$*: Proofs of Propositions 1 and 2*

Now that we have a closed formula for the polynomials $\{w_n\}$ in hand, by the path decomposition we proceed to establish a formula for g_n of (1.5).

Proof of Proposition 1. By (2.4), after calculating the geometric sums, we find

$$G_{N+1} = cz^2 g_N^2 \prod_{n=1}^{N} \frac{1}{1-[\rho_n\rho_{-n}g_n^2]}, \tag{2.15}$$

where c is a normalization constant that depends only on N and ξ, such that $G_{N+1}(1,1) = 1$. Next by (1.7) and (2.15) we have

$$g_{N+1} = z^{-1}g_N^{-1}G_{N+1} = czg_N \prod_{n=1}^{N} \frac{1}{1-[\rho_n\rho_{-n}g_n^2]}. \tag{2.16}$$

Finally, (2.16) may be re-written as

$$g_{N+1} = c\frac{g_N}{g_{N-1}}\frac{1}{1-[\rho_N\rho_{-N}g_N^2]}\left(zg_{N-1}\prod_{n=1}^{N-1}\frac{1}{1-[\rho_n\rho_{-n}g_n^2]}\right). \tag{2.17}$$

Now the factor in parentheses at the end of the expression in (2.17) is, by (2.16), up to a normalization constant, just the factor g_N. Hence we obtain the main formula of Proposition 1. The formula for g_2 follows by direct calculation from (2.16) with $N = 1$, $\rho_1\rho_{-1} = \xi$ and $c = 1 - \xi$. The constant C_n is evaluated by setting $g_n(1,1) = 1$. □

Proof of Proposition 2. The proof proceeds by induction. We have already verified the conclusion of the proposition for $n = 1$ since $F_1(\xi) = w_1(x,u) = 1$. Now assume that

$$\text{(H)} \quad g_n(y,z) = \frac{F_n(\xi)yz^n}{w_n(x,u)}, \quad \text{for all } n \leq N, \text{ with some } N \geq 1.$$

Then by Proposition 1, (H), Lemma 1 and (1.11), we have

$$g_{N+1}(y,z) = C_{N+1}\frac{F_N(\xi)^2 y^2 z^{2N} w_{N-1}}{F_{N-1}(\xi) y z^{N-1} w_N^2}\frac{w_N^2}{w_N^2 - \xi^N y^2 z^{2N}} = \frac{C y z^{N+1} w_{N-1}}{w_{N+1} w_{N-1}},$$

for a normalization constant C. The constant is evaluated by using that $g_n(1,1) = 1$. Thus the induction step is complete. □

2.5. Proof of Theorem 1

Lemma 4. *The identity (1.14) holds.*

Proof of Lemma 4. One proof may be accomplished by manipulating the closed forms given by (2.11) and (2.14). Another proof is to show that (1.14) follows from (2.12) via (1.12) and (1.15). Indeed, apply (1.15) to substitute for w_{n+1} and w_n in the left side of (1.14), so that

$$q_n w_{n+1} - w_n q_{n+1} = x(q_{n+1} q_{n-1} - q_n^2).$$

By (1.12) this last expression is $x(-x^{n-1}) = -x^n$ because $\beta^{-1}(q_3 q_0 - q_2 q_1) = \beta^{-1}(0 - \beta) = -1$. □

Proof of Theorem 1. Theorem 1. By (1.13), (1.7), Lemma 1, and Proposition 2, if we denote $C_N = P(\mathbf{H} \le N)$, then

$$C_N K_N(y,z) = \sum_{n=1}^{N} \frac{F_{n-1}(\xi) F_n(\xi) y^2 z^{2n}}{w_{n-1} w_n}\left(\frac{2\xi^n}{F_{n-1}(\xi) F_n(\xi)}\right) = 2y^2 \sum_{n=1}^{N} \frac{x^n}{w_{n-1} w_n}. \tag{2.18}$$

Next we show by induction the identity:

$$\sum_{n=1}^{N} \frac{x^{n-1}}{w_{n-1} w_n} = \frac{q_N}{w_N}, \quad N \ge 1. \tag{2.19}$$

Indeed, recall that $w_0 = w_1 = 1$ and $q_1 = 1$. So (2.19) holds for $N = 1$. Now assume (2.19) for some $N \ge 1$, and apply Lemma 4. Since by (1.14),

$$\frac{q_N}{w_N} + \frac{x^N}{w_N w_{N+1}} = \frac{q_{N+1}}{w_{N+1}},$$

the induction is complete, so (2.19) is proved. The proof now obtains from (2.18) and (2.19) since by Lemma 1, $C_N = 2\xi F_{N-1}(\xi)/F_N(\xi)$, and since, with the extra factor of x in (2.18) as compared with (2.19), we write $x/\xi = z^2$. □

3. Applications of Theorem 1

We first calculate the joint generating function

$$K(y,z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < \infty\} \tag{3.1}$$

by passing to the limit $N \to \infty$ in Theorem 1. We introduce a substitution variable θ to simplify the expression for the quotient q_N/w_N. Recall the definitions of α and β in Definition 2, (2.11). An appropriate substitution is:

$$\beta := \sqrt{4x}\cos\theta, \quad \text{or} \quad \cos\theta = (1 - xu)/\sqrt{4x}. \tag{3.2}$$

Next, since $\alpha = \sqrt{\beta^2 - 4x}$, we have by (3.2) that

$$\beta \pm \alpha = \sqrt{4x}(\cos\theta \pm i \sin\theta) = \sqrt{4x}e^{\pm i\theta}, \tag{3.3}$$

with $\Im\theta < 0$ for $|y| < 1, z \neq 0$. The idea of the substitution (3.2)–(3.3) may be found in [4, p. 352]. We can now plug in the formulae (2.11), (2.14), into the expression for $K_N(y,z)$ provided in Theorem 1, and apply the substitution (3.3) in as well, to obtain that

$$\frac{F_{N-1}(\xi)}{F_N(\xi)} K_N(y,z) = \frac{y^2z^2(e^{iN\theta} - e^{-iN\theta})}{(e^{iN\theta} - e^{-iN\theta}) - \sqrt{x}\left(e^{i(N-1)\theta} - e^{-i(N-1)\theta}\right)}, \tag{3.4}$$

where, if $\xi = \frac{1}{4}$ then $F_{N-1}(\xi)/F_N(\xi) = \frac{N+1}{2N}$. Further, by a bit more simplification involving the complex exponential formula for the sine, we have that (3.4) is also written as

$$K_N(y,z) = \frac{F_N(\xi)}{F_{N-1}(\xi)} \left(\frac{y^2z^2 \sin N\theta}{\sin N\theta - \sqrt{x}\sin(N-1)\theta} \right). \tag{3.5}$$

Corollary 3.

$$E\{y^{\mathbf{R}}z^{\mathbf{L}} | \mathbf{H} < \infty\} = \frac{\left(1 + \sqrt{1-4\xi}\right) y^2z^2}{1 + xu + \sqrt{(xu-1)^2 - 4x}},$$

where $u = y^2 - 1,\ x = \xi z^2,\ |y| < 1, |z| < 1.$

Proof of Corollary 3. We let $N \to \infty$ in Theorem 1 for $K_N(y,z)$ as expressed in (3.4). Simply note that for $\theta = \arccos((1 + x - xy^2)/\sqrt{4x})$ and $|y| < 1, z \neq 0,\ \Im\theta < 0$, we have $|e^{i\theta}| > 1$. Further by (2.6), $\lim_{N\to\infty} 2F_N(\xi)/F_{N-1}(\xi) = c := 1 + \sqrt{1-4\xi}$. So, by (3.3)–(3.4),

$$\lim_{N\to\infty} K_N(y,z) = \frac{1}{2} \frac{cy^2z^2}{[1 - \sqrt{x}e^{-i\theta}]} = \frac{cy^2z^2}{2 - \beta + \alpha}.$$

Therefore the corollary follows by the definitions of β and α in Definition 2, (2.11). □

For the case of simple random walk, the generating function of Corollary 3 may be written $K(y,1) = \frac{1}{4}[3 + y^2 - \sqrt{y^4 - 10y^2 + 9}]$. We have that $K(y,1)$ is the "twin" generating function of $P_{\text{rook}}(t) = \sum_{n=0}^{\infty} a(n)t^n$, or $P_{\text{rook}}(t) = (8t)^{-1}[1 + 3t - \sqrt{1 - 10t + 9t^2}]$, where $a(n)$ is the number of Catalan rook paths of length $2n$, [8, Theorem 6]: $K(y,1) = 2P_{\text{rook}}(1/y^2)$. Here $a(n)$ is the number of lattice paths from $(0,0)$ to $(2n,0)$ using steps in $S := \{(k,k) \text{ or } (k,-k) : k \geq 1\}$ that never cross the x-axis. As we shall see in developing the Remark 4 of Section 4, both generating functions recover the sequence $\{a(n)\}$ or 1, 5, 29, 185, 1257, ... [10, A059231].

3.1. Proofs of Corollaries 1 and 2

We first calculate the joint generating function of $\mathcal{R}_N$ and $\mathcal{L}_N$. Recall that $\mathcal{M}_N$ is the number of excursions of height at most $N-1$ until there is an excursion of height at least N for the random walk. By the fact that the simple random walk starts afresh at the end of each excursion, we have that $1 + \mathcal{M}_N$ is a standard geometric random variable (that is the values of $\mathcal{M}_N$ start from $m = 0$) with success probability $P(\mathbf{H} \geq N)$. Thus

$$P(\mathcal{M}_N = m) = [P(\mathbf{H} < N)]^m\, P(\mathbf{H} \geq N), \quad m = 0, 1, 2, \ldots.$$

Let $\mathbf{R}_N$ (resp. $\mathbf{L}_N$) be a random variable whose distribution is the conditional distribution of the number of runs (resp. number of steps) in an excursion, given that the height of the excursion is at most N. The probability generating function $K_{N-1}(y,z) = E\{y^{\mathbf{R}_N} z^{\mathbf{L}_N}\}$ is calculated in Theorem 1. Denote by $\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \ldots$ (resp. $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \ldots$), a sequence of independent copies of $\mathbf{R}_N$ (resp. $\mathbf{L}_N$). Then we have $\mathcal{R}_N = \sum_{m=0}^{\mathcal{M}_N} \mathbf{R}^{(m)}$, $\mathcal{L}_N = \sum_{m=0}^{\mathcal{M}_N} \mathbf{L}^{(m)}$, where the sums are understood to be zero if $\mathcal{M}_N = 0$. Thus by calculating a geometric sum we have:

$$E\{y^{\mathcal{R}_N} z^{\mathcal{L}_N}\} = \sum_{m=0}^{\infty} P(\mathcal{M}_N = m)\,(K_{N-1}(y,z))^m = \frac{P(\mathbf{H} \geq N)}{1 - P(\mathbf{H} < N) K_{N-1}(y,z)}. \tag{3.6}$$

Proof of Corollary 1. We have $\xi = \frac{1}{4}$. We compute the limiting Laplace transform of $N^{-2}\mathcal{R}_{N+1}$. We put $y = e^{-\lambda/N^2}$ and $z = 1$, and apply (3.6) with $N+1$ in place of N. By (3.2) with $x = \frac{1}{4}$, we have $y^2 = 5 - 4\cos\theta$, so $\theta = \arccos\left((5 - e^{-2\lambda/N^2})/4\right) \sim i\sqrt{\lambda}/N$, as $N \to \infty$. Now apply (3.4) in (3.6). Since at $\xi = \frac{1}{4}$, $P(\mathbf{H} \leq N) F_N(\xi)/F_{N-1}(\xi) = \frac{1}{2}$ by Lemma 1, we calculate that the limiting Laplace transform, $\lim_{N\to\infty} E\{\exp(-\lambda N^{-2}\mathcal{R}_{N+1})\}$, is written:

$$\lim_{N\to\infty} \frac{1}{N+1} \frac{2\sin N\theta - \sin(N-1)\theta}{2\sin N\theta - \sin(N-1)\theta - (5 - 4\cos\theta)\sin N\theta} = \frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}. \tag{3.7}$$

To see (3.7), we expand $\sin(N-1)\theta = \sin N\theta\cos\theta - \cos N\theta\sin\theta$, so the denominator of the main fraction becomes $\sin N\theta(-3 + 3\cos\theta) + \sin\theta\cos N\theta = I + II$. Here $I = \mathcal{O}(1/N^2)$ since $\theta = \mathcal{O}(1/N)$. So I is negligible since $II \sim (\cosh\sqrt{\lambda}) i\sqrt{\lambda}/N$, as $N \to \infty$. Observe further that the numerator satisfies: $2\sin N\theta - \sin(N-1)\theta \sim i\sinh\sqrt{\lambda}$, as $N \to \infty$. Thus by (3.7) we have established the Laplace transform in the statement of the corollary. By the Mittag-Leffler expansion for the hyperbolic tangent,

$$\tanh t = 8t \sum_{\nu=1}^{\infty} \frac{1}{(2\nu-1)^2\pi^2 + 4t^2}, \tag{3.8}$$

the first formula for the density $f(x)$ can be determined by inverting the Laplace transform in (3.7). The second formula for $f(x)$ was proved after the statement of Corollary 1.

Finally, we apply Proposition 2 directly to find the limiting Laplace transform of $\mathcal{R}'_{N+1}/N^2$, where in the statement of that proposition we have $F_N(\frac{1}{4}) = (N+1)2^{-N}$ by letting $\alpha_0 \to 0$ in (2.6). Hence by Proposition 2, (2.14), and (3.2)–(3.3), with $y = e^{-\lambda/N^2}$ and $z = 1$, so $\theta \sim i\sqrt{\lambda}/N$ as above, we obtain,

$$g_N(e^{-\lambda/N^2}, 1) \sim (N+1)\sin\theta \frac{e^{-\lambda/N^2}}{2\sin N\theta - \sin(N-1)\theta} \sim \frac{i\sqrt{\lambda}}{\sin(i\sqrt{\lambda})} = \frac{\sqrt{\lambda}}{\sinh\sqrt{\lambda}},$$

as $N \to \infty$. To handle the case of steps alone in the meander, we note that for $y = 1$, we have $\cos\theta = 1/z$ in (3.2), so for $z = e^{-\lambda/(2N^2)}$, we again obtain $\theta \sim i\sqrt{\lambda}/N$. Now in place of $y = e^{-\lambda/N^2} \sim 1$ we obtain $z^N = e^{-\lambda/(2N)} \sim 1$, with the result that $\lim_{N\to\infty} g_N(1, e^{-\lambda/(2N^2)}) = \sqrt{\lambda}/\sinh\sqrt{\lambda}$ as before. $\square$

Proof of Corollary 2. We first calculate the limiting characteristic function of $N^{-1}\Delta_{N+1}$. Since $\Delta_{N+1} = \sum_{m=1}^{\mathcal{M}_N}(2\mathbf{R}^{(m)} - \mathbf{L}^{(m)} - 1)$, by a geometric sum calculation analogous to (3.6), we have

$$E\{e^{is\Delta_{N+1}}\} = \frac{P(\mathbf{H} > N)}{1 - P(\mathbf{H} \le N)e^{-is}K_N(e^{2is}, e^{-is})}. \tag{3.9}$$

Now put $s := t/N$, and thus plug (3.5) into (3.9) with $y = e^{2it/N}$, $z = e^{-it/N}$, and $x = \frac{1}{4}e^{-2it/N}$. After a bit of simplification, similar as in the proof of Corollary 1, we have that

$$E\{e^{itN^{-1}\Delta_{N+1}}\} = \frac{1}{N+1}\left(\frac{\sin N\theta - \frac{1}{2}e^{-it/N}\sin(N-1)\theta}{\sin N\theta - \frac{1}{2}e^{-it/N}\sin(N-1)\theta - \frac{1}{2}e^{it/N}\sin N\theta}\right), \tag{3.10}$$

since $\sqrt{x} = \frac{1}{2}e^{-it/N}$ and $e^{-is}y^2z^2 = e^{it/N}$. There is cancelation in the denominator of (3.10) that we find by expanding $\sin(N-1)\theta$. Indeed the denominator of (3.10) is written as

$$\sin N\theta\left(1 - \frac{1}{2}e^{-it/N}\cos\theta - \frac{1}{2}e^{it/N}\right) + \frac{1}{2}e^{-it/N}\cos N\theta\sin\theta = I + II. \tag{3.11}$$

Now by (3.2), $\cos\theta = e^{it/N} - \frac{1}{4}e^{3it/N} + \frac{1}{4}e^{-it/N} \sim 1 + \frac{1}{2}t^2/N^2$. Hence $\theta \sim it/N$, as $N \to \infty$. Therefore in (3.11) we find that $I = \mathcal{O}(N^{-2})$ and $II \sim it(2N)^{-1}\cos(it)$. Thus, because the numerator of the fraction in (3.10) is asymptotically $\frac{1}{2}\sin(it)$, we find by (3.10)–(3.11) that $E\{e^{itN^{-1}\Delta_{N+1}}\} \sim \tanh(t)/t$, as $N \to \infty$. Thus statement (i) of Corollary 2 has been proved up to the form of the density. For the density, we again apply the Mittag-Leffler formula (3.8). We then invert the characteristic function to obtain a series of double exponentials using $\int_{-\infty}^{\infty} a^{-1}e^{-a|x|}e^{itx}\mathrm{d}x = 2/[a^2 + t^2]$.

To complete the proof of the corollary, we establish statement (ii). The statement (iii) will then follow by independence. We derive (ii) directly by Proposition 2, similar as in the last part of the proof of Corollary 1. By the definition of g_n in (1.5), we have:

$$E\{e^{itN^{-1}\Delta'_{N+1}}\} = zg_N(y,z), \quad \text{with } y = e^{2it/N},\ z = e^{-it/N}. \tag{3.12}$$

Hence by Proposition 2, (3.12), (2.14), and (3.2)–(3.3),

$$E\{e^{itN^{-1}\Delta'_{N+1}}\} = (N+1)(\sqrt{4x})^{-N+1}\sin\theta\frac{e^{2it/N}e^{-it(N+1)/N}}{2\sin N\theta - \sqrt{4x}\sin(N-1)\theta}. \tag{3.13}$$

Finally, $\theta \sim it/N$ as in the proof of part (i), and $\sqrt{4x} = z = e^{-it/N}$, so by (3.13), as $N \to \infty$,

$$E\{e^{itN^{-1}\Delta'_{N+1}}\} = \frac{(N+1)\sin\theta}{2\sin N\theta - e^{-it/N}\sin(N-1)\theta} \sim \frac{(N+1)it/N}{\sin(it)} \sim \frac{t}{\sinh(t)}. \tag{3.14}$$

Thus by (3.14), since $\frac{\pi}{4}\int_{-\infty}^{\infty}\mathrm{sech}^2(\pi x/2)e^{itx}\mathrm{d}x = t/\sinh(t)$, [6, 17.23–19], the proof of (ii) is complete.

Finally the proof of part (3) of the corollary is a consequence of independence of Δ_N and Δ'_N, together with the fact that $\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{sech}(\pi x/2)e^{itx}\mathrm{d}x = \mathrm{sech}(t)$, [6, 17.23–16]. □

4. Representation of $P(\mathbf{R} = 2k|\mathbf{H} < \infty)$

Define the Legendre polynomial $P_k(s)$ of degree k, by $(1 - 2st + t^2)^{-1/2} = \sum_{k=0}^{\infty} P_k(s)t^k$; see [13, 15.1]. One of its relatives in a whole family of associated Legendre functions $P_\nu^\mu(s)$

(see [1, Chapter 3]) is: $P_k^{-1}(s) = (s^2-1)^{-1/2}\int_1^s P_k(s)\mathrm{d}s$, so that

$$\pi(s,t) := \sum_{k=0}^{\infty} P_k^{-1}(s)t^k = \frac{1-\sqrt{1-2st+t^2}}{t\sqrt{s^2-1}} - \frac{1}{\sqrt{s^2-1}}. \tag{4.1}$$

We aim to match the generating function of Corollary 3 of Section 3 for $z=1$, namely

$$K(y,1) = \frac{c(u+1)}{1+\xi u+\sqrt{(\xi u-1)^2-4\xi}} = \frac{c}{4\xi}\left(1+\xi u-\sqrt{(1-\xi u)^2-4\xi}\right),$$

for $c = 1+\sqrt{1-4\xi}$, with the generating function $\pi(s,t)$ of (4.1). We accomplish this by setting $s=(1+\xi)/(1-\xi)$, and $t=\xi y^2/(1-\xi)$ in (4.1). Hence we obtain the following representation of the distribution of $\mathbf{R}$.

Corollary 4.

$$P(\mathbf{R}=2k|\mathbf{H}<\infty) = \frac{c}{\sqrt{4\xi}}\left(\frac{\xi}{1-\xi}\right)^k P_{k-1}^{-1}\left(\frac{1+\xi}{1-\xi}\right), \quad k\ge 2;$$

and $P(\mathbf{R}=2|\mathbf{H}<\infty) = c/[2(1-\xi)]$, *where* $c = 1+\sqrt{1-4\xi}$.

We have the formulae: $P_\nu^\mu(s) = \frac{2^\mu(s^2-1)^{-\mu/2}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\mu)}\int_0^\pi [s+\sqrt{s^2-1}\cos\theta]^{\nu+\mu}(\sin\theta)^{-2\mu}\mathrm{d}\theta$, and $P_{-\nu-1}^\mu(s) = P_\nu^\mu(s)$; [1, 3.4(7) and 3.7(6)]. Thus by Corollary 4 and these formula, with $\nu = k-1$ and $\mu=-1$, so that $-\nu-1+\mu = -k-1$, we obtain an integral representation as follows.

Remark 3. For all $k\ge 2$, we have

$$P(\mathbf{R}=2k|\mathbf{H}<\infty) = \left(1+\sqrt{1-4\xi}\right)\frac{\xi^k}{\pi}\int_0^\pi \frac{\sin^2(\theta)\mathrm{d}\theta}{(1+\xi+\sqrt{4\xi}\cos\theta)^{k+1}}. \tag{4.2}$$

We match the generating function $P_{\mathrm{rook}}(t)$ of Section 3 with the generating function $\pi(s,t)$ of (4.1) by setting $s=\frac{5}{3}$. Thus we see that the integer sequence $\{a(n)\}$ of [10, A059231] and discussed in Section 3, satisfies $a(n) = \frac{1}{2}3^n P_n^{-1}(\frac{5}{3})$, $n\ge 1$. Hence by Corollary 4 we obtain the following.

Remark 4. If $p=\frac{1}{2}$, then the integer sequence $\{a(n)\}$ may be written:

$$a(n) = \frac{1}{2}3^{2n+1}P(\mathbf{R}=2(n+1)), \quad n\ge 0.$$

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