Laws Relating Runs, Long Runs and Steps in Gambler's Ruin with Persisitence in Two Strata

Gregory J. Morrow

Department of Mathematics
University of Colorado Colorado Springs

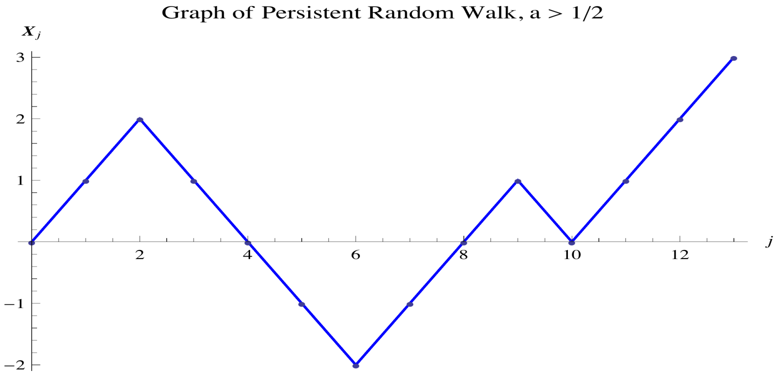
Aug 20, 2015/ 8th Intl Conf
Lattice Path Combinatorics and Applications, Cal Poly Pomona

Outline

1. Introduction
   - Persistent Random Walk and Excursion
   - Gambler's Ruin
   - Persistence in Strata
2. Conditional Joint G. F. of Excursion Statistics given Height
   - Future Maxima Decomposition
   - Recurrence Relation for One-sided G.F. , $g_{m,n}$
   - Extend 2-Parameter Fibonacci Polys over Stratum Boundary
3. Applications
   - Joint Distribution of Excursion Statistics: Homogeneous Case
   - Meander Limit, with Order N Scaling
   - Last Visit Limit, with Order N scaling

Persistent Random Walk

- Walk on Integers: $\mathbf{X}_j$, $j = 0, 1, 2, \ldots$
- Increments: $\varepsilon_j := \mathbf{X}_j - \mathbf{X}_{j-1}$
- Transitions: $P(\varepsilon_{j+1} = 1|\varepsilon_j = 1) = P(\varepsilon_{j+1} = -1|\varepsilon_j = -1) = a$



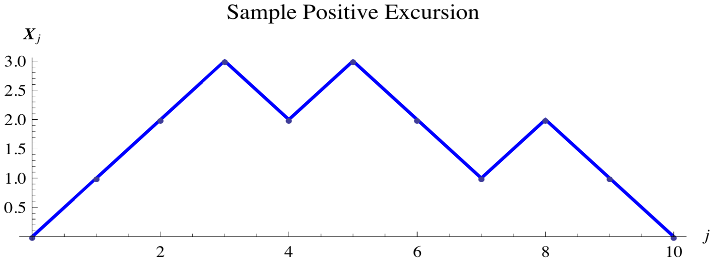
Excursion:

A Path of First Return to Zero by Random Walk

Sample :

#Steps: $\mathbf{L} = 10$, Height: $\mathbf{H} = 3$

#Runs: $\mathbf{R} = 6$, #Short Runs: $\mathbf{V} = 3$



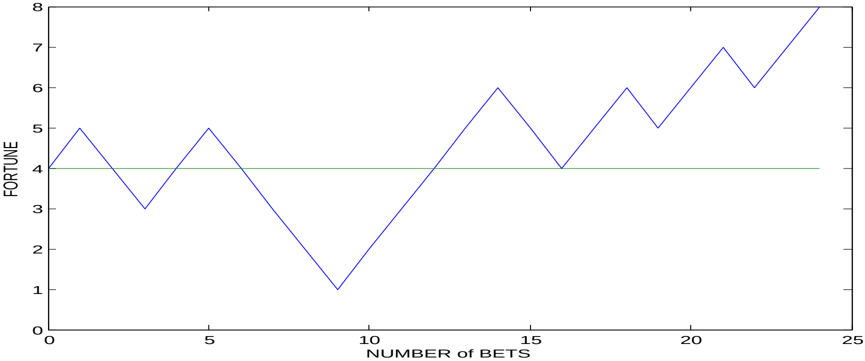
Gambler's Ruin

Initial Fortune $\in (0, 2N)$. Unit Bets. Correlated Run of Fortune.

Terminate Game at Step $\mathcal{T}$ when **Fortune** First Equals $2N$ or 0.

Equivalently, $\mathbf{X}_0 \in (-N, N)$, $\mathcal{T} := \inf\{j : |\mathbf{X}_j| = N\}$.

Picture: "Last Visit" $\mathcal{L} := \inf\{j : |\mathbf{X}_j| = 0\}$, "Meander" $\mathcal{L} \leq j \leq \mathcal{T}$



Uncorrelated Case: Runs and Steps, $a = \frac{1}{2}$ [M, 2015]

2 Parameter Fibonacci Recurrence: $(\mathbf{F})\quad v_{n+1} = \beta v_n - x v_{n-1}$.

- **Define** $\{q_n\}, \{w_n\}$ by $(\mathbf{F})$: $q_0 = 0,\ q_1 = 1;\ w_0 = 1,\ w_1 = 1;$

  $\implies \sum q_n t^n = \frac{t}{1-\beta t + x t^2},\ w_n = q_n - x q_{n-1}$

Theorem

$$K_{N+1} := E\{r^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} \le N\} = c \cdot r^2 z^2 \,(q_N / w_N);\ x := \tfrac{1}{4} z^2,\ \beta := 1 - x(r^2 - 1)$$

- Denote $\mathcal{R} := \#\text{Runs}(|\mathbf{X}_j|,\ 0 \le j \le \mathcal{L})$

Corollary

Both $\frac{1}{2}\mathcal{L}/N^2 \to f,\ \mathcal{R}/N^2 \to f;\ \ f(x) := \frac{1}{\sqrt{\pi x}} \sum_{\nu=-\infty}^{\infty} (-1)^\nu e^{-\nu^2/x},\ x > 0.$

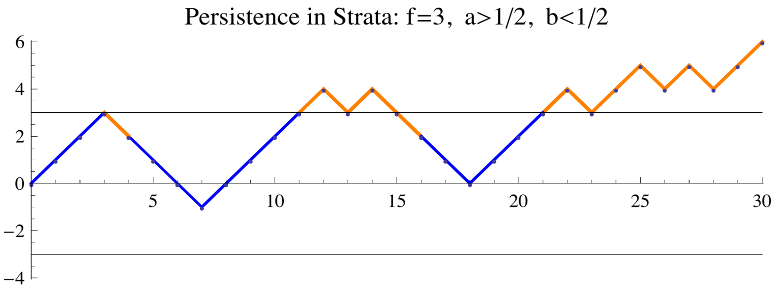
- $\mathcal{R}' := \#\text{Runs}(\mathbf{X}_j,\ \mathcal{L} \le j \le \mathcal{T});\ \mathcal{L}' := \mathcal{T} - \mathcal{L};\ \Delta' := 2\mathcal{R}' - \mathcal{L}';$ (Meander)
- $\Delta := 2\mathcal{R} - \mathcal{L} - \mathcal{M};\quad \mathcal{M} := \#\text{Excursions to } \mathcal{L};$ (Last Visit)

Corollary

$$\Delta'/N \to \tfrac{\pi}{4}\, sech^2(\pi x/2);\ \ (\Delta + \Delta')/N \to \tfrac{1}{2}\, sech(\pi x/2),\ -\infty < x < \infty.$$

Persistence in Strata

- General Persistent Symmetric Gambler's Ruin:
  $P(\varepsilon_{j+1} = 1 | \varepsilon_j = 1, |\mathbf{X}_j| = k) = P(\varepsilon_{j+1} = -1 | \varepsilon_j = -1, |\mathbf{X}_j| = k) = a_k$
- **Two Persistence Parameters:**
  $a_k = a, \ 0 \le k < f; \quad a_k = b, \quad f \le k < N; \quad \text{some } f \in (0, N)$
- **Velocity Model:** Velocities ± 1, Deterministic Change in Persistence
  Interpretation: Change in Stratum $\leftrightarrow$ Change in Medium
  cf. [Szasz and Toth, 1984] 1-D Random Environment
- **Motivation:** 3 Counting Stats, Non-trivial Scaling Limits for $b \neq a$



Key: Explicitly Calculate $K_N := E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}} | \mathbf{H} < N\}$.

Outline: Formula for $K_N = K_N(r, y, z) \Longrightarrow$

Formula for **Last Visit Statistics G. F.** (†).

Note: K_N **not needed for Meander**.

- $\mathcal{M} := \#$ Excursions Until Last Visit $\mathcal{L}$.
- $\mathcal{M}$ is a **Geometric** random variable:

$$p_m := P(\mathcal{M} = m) = P(\mathbf{H} < N)^m P(\mathbf{H} \geq N), \ m = 0, 1, 2, \ldots.$$

- Sum Independent Copies of Excursion Stats, Obtain Last Visit Stats:

$$\mathcal{R} := \sum_{m=0}^{\mathcal{M}} \mathbf{R}^{(m)}, \ \mathcal{V} := \sum_{m=0}^{\mathcal{M}} \mathbf{V}^{(m)}, \ \mathcal{L} := \sum_{m=0}^{\mathcal{M}} \mathbf{L}^{(m)}$$

- **Last Visit Statistics G. F.**

$$(\dagger) \quad E\{r^{\mathcal{R}} y^{\mathcal{V}} z^{\mathcal{L}} u^{\mathcal{M}}\} = \sum_{m=0}^{\infty} p_m \, [u \cdot K_N]^m = \frac{P(\mathbf{H} \geq N)}{1 - u \cdot K_N P(\mathbf{H} < N)}$$

Upward and Downward Conditional G.F. $g_{m,n}$

Focus on Construction for Meander

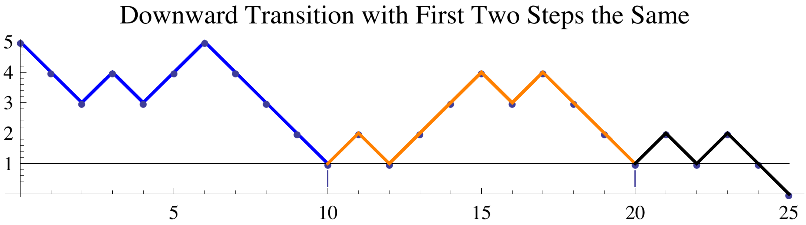
$$\Gamma'_{m,n} := \text{Conditional First Passage Path to Level } n,$$

$$\text{given } \mathbf{X}_0 = m, \ \& \ \text{Path "One-Sided"} : \mathbf{X}_j \in [m, n]$$

- $\mathbf{L}'_{m,n} := \#(\textbf{Steps} \text{ along } \Gamma'_{m,n})$. Similarly: $\mathbf{R}'_{m,n}$, $\mathbf{V}'_{m,n}$ (**Runs, Short Runs**)
- **Define**:

$$g_{m,n}(r, y, z) := E\{r^{\mathbf{R}'_{m,n}} y^{\mathbf{V}'_{m,n}} z^{\mathbf{L}'_{m,n}} \, | \varepsilon_1 = \varepsilon_2\}.$$

- **Picture**: A Path for $g_{5,0}$



Recurrence Relation for $g_{m,n}$

Example : Decompose Downward Transition for $g_{n,0}$ by:

Return to Level 1 After Each Future Maximum.

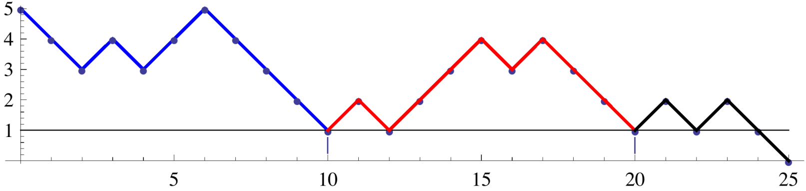
- $\rho_{m,n} :=$ Probability of One-sided Transition
- $k_m^+ :=$ G.F. over $(UD)^\ell$ of Continuation Seq. $(UD)^\ell UU$ at Level m

  $k_n^- :=$ Corresponding G.F. at Level n with Roles of $U, D,$ Reversed
- $z \cdot h_m^+ :=$ G.F. over Termination Seq. $(UD)^\ell D$, at Level m

  $z \cdot h_n^- :=$ Corresponding G.F. at Level n with Roles of $U, D,$ Reversed
- $\lambda_{m,n} := \sum_{\ell=0}^{\infty} \left(c_{m,n} \cdot \rho_{m,n} k_m^+ g_{m,n} \cdot \rho_{n,m} k_n^- g_{n,m}\right)^\ell$, $m < n$; $\lambda_{n,m} = \lambda_{m,n}$
- **Recurrence**

$$\textbf{(g)} \qquad g_{n,0} = c \cdot g_{n,1} \cdot \lambda_{1,n} \cdot \lambda_{1,n-1} \cdots \lambda_{1,3} \cdot z \cdot h_1^+$$

Downward Transition with First Two Steps the Same



Denominators w_n^* of $g_{0,n}$: Homogeneous case $b = a$

Consider First : $b = a$. **Define Denominator Polys** $w_n^* = w_n^*(a)$

- $v_n = w_n^*$ satisfy (**F**) $v_{n+1} = \beta v_n - x v_{n-1}$,

  such that : $w_1^* = 1$, & w_n^* Serves as Denominator of $g_n := g_{0,n}$.

- Here, $x = x_a$ and $\beta = \beta_a$ Determined by: (**g**), (**F**), and

  (**Interlacing**) $v_n^2 - v_{n+1}v_{n-1} = \beta^{-1}x^{n-1}(v_2v_1 - v_3v_0)$. ... conseq. of (**F**)

- w_0^* by back iteration; $\Longrightarrow$ $w_n^* = (1 - w_0^*) \cdot q_n(x_a, \beta_a) + w_0^* \cdot w_n(x_a, \beta_a)$

- (**g**) $\Longrightarrow$ $g_{n+1} = c \cdot \lambda_n \cdot g_n^2 / g_{n-1}$; $\lambda_n := \lambda_{0,n}$

Lemma

If $b = a$, *Then:* $\lambda_n = \frac{(w_n^*)^2}{w_{n-1}^* w_{n+1}^*}$; $g_n = c \cdot r \cdot z^n \tau_a^{n-2} / w_n^*$; $n \geq 2$.

- $x_a := a^2 z^2 \tau_a^2$, $\tau_a := 1 + (1-a)^2 r^2 z^2 y(1-y)$, β_a is explicit too.
- Numerator Polys q_n^* Satisfy (**F**), $\ni$: $q_1^* = y^2$, and

  (**Commutation**) $[w^*(a), q^*]_n := w_n^* q_{n+1}^* - q_n^* w_{n+1}^* = a^2 z^2 x_a^{n-1}$.

Denominators $\overline{w}_{m,n}$ of $g_{m,n}$: Full model

Definition

- $\overline{w}_{m,m+\ell} := w^*_\ell(a), \quad m+\ell \le f; \qquad \overline{w}_{m,m+\ell} := w^*_\ell(b), \quad f \le m$
- $\overline{w}_{f-\ell,f+1} := \frac{1-b}{1-a} w^*_{\ell+1}(a) + \frac{b-a}{1-a} w^*_\ell(a), \quad 1 \le \ell \le f$
- $\overline{w}_{m,f+2} := \beta(a,b)\overline{w}_{m,f+1} - x(a,b)\overline{w}_{m,f}, \quad m \le f-1$
- $\overline{w}_{m,f+j+1} := \beta_b \overline{w}_{m,f+j} - x_b \overline{w}_{m,f+j-1}, \quad m \le f-1,\ j \ge 2$

Define Downward Denominator $\overline{w}_{n,m}$ by switching the roles of a and b.

Lemma

$$\overline{w}_{f-\ell,f+j} = \begin{bmatrix} q^*_j(b) & w^*_j(b) \end{bmatrix} M \begin{bmatrix} w^*_\ell(a) \\ w^*_{\ell+1}(a) \end{bmatrix},$$ *M an explicit 2×2 matrix.*

Define Interlacing Bracket $[\,\overline{w}\,]_{m.n} := \overline{w}_{m,n}\overline{w}_{m+1,n+1} - \overline{w}_{m,n+1}\overline{w}_{m+1,n}, \quad m \le n-2$

Lemma

(1) $[\,\overline{w}\,]_{f-\ell,f+j} = a^2 r^2 z^4 (1-a)(1-b) x_a^{\ell-2} \cdot x(a,b) \cdot x_b^{j-1}, \quad \ell \ge 2,\ j \ge 1.$

(2) $\lambda_{m.n} = \overline{w}_{m,n}\overline{w}_{m+1,n+1} / \{\overline{w}_{m,n+1}\overline{w}_{m+1,n}\}$ *and* $g_{m,n}$ *has Closed Formula .*

Joint Distribution of Excursion Stats: Case $b = a$

Theorem

Let $b = a$. Then $K_{N+1} = E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}} \mid \mathbf{H} \leq N\} = c \cdot r^2 z^2 \left(\frac{q_N^*(a)}{w_N^*(a)}\right)$

Corollary

Let $b = a$. Define $\alpha_a := \sqrt{\beta_a^2 - 4x_a}$.
Then, (1) $K(r, y, z; a) := E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}}\} = (1 - \frac{1}{2}\beta_a - \frac{1}{2}\alpha_a)/(1 - a)$

- Define the Excursion Statistic $\mathbf{U} := \#\text{Long Runs} = \mathbf{R} - \mathbf{V}$

Corollary

Let P_a denote the Probability for Homogeneous case. Then, for $n \geq 2$,
$\frac{1-a}{a} P_a(\mathbf{L} = 2n, \mathbf{R} = 2k, \mathbf{U} = \ell) = P_{1-a}(\mathbf{L} = 2n, \mathbf{L} - \mathbf{R} = 2k, \mathbf{U} = \ell)$

Proof: (1) $\Longrightarrow$ $K(ru, 1/u, z; \frac{1}{2}) - K(u/r, 1/u, rz; \frac{1}{2}) = \frac{1}{2} z^2 (r^2 - 1)$. $\square$

Note Taylor Expansion: $\frac{1}{2} K(ru, 1/u, 2z; \frac{1}{2}) = \cdots +$

$$z^{16} \left\{ \begin{array}{l} (r^2 + r^4 + r^6 + r^8 + r^{10} + r^{12} + r^{14})u^2 + \\ (10r^4 + 16r^6 + 18r^8 + 16r^{10} + 10r^{12})u^3 + (10r^4 + 46r^6 + 63r^8 + 46r^{10} + 10r^{12})u^4 \\ + (36r^6 + 68r^8 + 36r^{10})u^5 + (6r^6 + 23r^8 + 6r^{10})u^6 + 2r^8 u^7) \end{array} \right\} + \cdot$$

Scaling Limit of Order N: Meander of Gambler's Ruin

Denote $\mathcal{V}' :=$ #Short Runs over Meander; $\mathcal{R}'$, $\mathcal{L}'$: Runs, Steps;

$$\mathcal{X}'_N := \left(\mathcal{L}' - \frac{2-a-b}{(1-a)(1-b)}\mathcal{R}' + \frac{1}{(1-a)(1-b)}\mathcal{V}'\right)/N.$$

Theorem

Let $f \sim \eta N$ for some fixed $0 < \eta < 1$. Denote $\sigma_1 := \sqrt{a + b^2 - 2ab}$, and $\sigma_2 := \sqrt{b + a^2 - 2ab}$. Write $\kappa_1 := \frac{\eta\sigma_1}{1-b}$ and $\kappa_2 := \frac{(1-\eta)\sigma_2}{1-a}$.

Then $\lim_{N\to\infty} E\{e^{it\mathcal{X}'_N}\} = \hat{\varphi}(t)$,

$$\hat{\varphi}(t) := \frac{(b\kappa_1\sigma_2 + a\kappa_2\sigma_1)t}{a\sigma_1\cosh(\kappa_1 t)\sinh(\kappa_2 t) + b\sigma_2\sinh(\kappa_1 t)\cosh(\kappa_2 t) + i(b-a)^2\sinh(\kappa_1 t)\sinh(\kappa_2 t)}$$

Put $\mathcal{Y}'_{1,N} := \left(\mathcal{R}' - \frac{1}{(1-a)}\mathcal{V}'\right)/N$; $\mathcal{Y}'_{2,N} := \left(\mathcal{L}' - \frac{1}{(1-a)}\mathcal{R}'\right)/N - \mathcal{Y}'_{1,N}$.

- As a consequence of the Proof of Theorem,

Corollary

Let $b = a$. Then

$$\lim_{N\to\infty} E\{e^{is\mathcal{Y}'_{1,N} + it\mathcal{Y}'_{2,N}}\} = \sqrt{(1-a)s^2 + at^2}/\sinh(\sqrt{(1-a)s^2 + at^2})$$

Extend Numerator Polynomials: $\overline{q}_n$

Definition

Define $\overline{q}_n = \overline{q}_n(r, y, z; a, b)$ for all $n \geq 1$ by:

(1) $\overline{q}_n := q_n^*(a),\ 1 \leq n < f;$ (2) $\overline{q}_f := \frac{1-b}{1-a} q_f^*(a) + \frac{b-a}{1-a} q_{f-1}^*(a);$

(3) $\overline{q}_{f+1} := \beta(a,b)\overline{q}_f - x(a,b)\overline{q}_{f-1};$ (4) $\overline{q}_{f+j+1} := \beta_b \overline{q}_{f+j} - x_b \overline{q}_{f+j-1},\ j \geq 1.$

Lemma

Let $M^{2\times 2}$ be as before. If $j \geq 1,\quad \overline{q}_{f+j-1} = \begin{bmatrix} q_j^*(b) & w_j^*(b) \end{bmatrix} M \begin{bmatrix} q_{f-1}^*(a) \\ q_f^*(a) \end{bmatrix}$

Lemma

$[\,\overline{w}, \overline{q}\,]_n := \overline{w}_{n,0}\overline{q}_{n+1} - \overline{q}_n \overline{w}_{n+1,0} = -|M|[w^*(a), q^*(a)]_{f-1}[w^*(b), q^*(b)]_{n-f}$

Theorem

$K_{N+1}(r, y, z; a, b) = E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}} \mid \mathbf{H} \leq N\} = c \cdot r^2 z^2 \left(\overline{q}_N / \overline{w}_{1,N+1}\right)$

Scaling Limit: Last Visit portion of Gambler's Ruin

Denote $\mathcal{V} :=$ #Short Runs until $\mathcal{L}$; $\mathcal{M} :=$ #Excursions until $\mathcal{L}$;

$$\mathcal{X}_N := \left(\mathcal{L} - \frac{2-a-b}{(1-a)(1-b)}\mathcal{R} + \frac{1}{(1-a)(1-b)}\mathcal{V} - \frac{a(b-a)}{(1-a)(1-b)}\mathcal{M}\right)/N$$

Theorem

Let $f \sim \eta N$ for some fixed $0 < \eta < 1$. Let σ_j, κ_j, and $\hat{\varphi}(t)$ as before.

Then $\lim_{N\to\infty} E\{e^{it\mathcal{X}_N}\} = \hat{\psi}(t)/\hat{\varphi}(t)$,

$$\hat{\psi}(t) := \frac{ab\sigma_1\sigma_2}{ab\sigma_1\sigma_2\cosh(\kappa_1 t)\cosh(\kappa_2 t) + a^2\sigma_1^2\sinh(\kappa_1 t)\sinh(\kappa_2 t) + ia\sigma_1(b-a)^2\cosh(\kappa_1 t)\sinh(\kappa_2 t)}$$

- Proof of "Last Visit" Thm Involves Second Order Cancellation in Denominator of Characteristic Function Expansion.
- *Mathematica* applied throughout to make "Direct Calculations".
- Key Closed Formula for $g_{m,n}$ Established by Induction

Combine Results for Last Visit and Meander

Corollary

$\lim_{N\to\infty} E\{e^{it(\mathcal{X}_N+\mathcal{X}'_N)}\} = \hat{\psi}(t)$

Conclusion

- The Future Maxima Decomposition is amenable to the study of the conditional Joint Generating Function, given the Height of a Random Walk Excursion, even with Persistence in Strata
- Applications:
  - Distributional Symmetry for Runs, Long Runs, and Steps in Homogeneous case.
  - New Order N Scaling Limits for Both Last Visit and Meander, and thereby for entire Gambler's ruin process
- Outlook
  - One may attempt analogous results for the "Non-sequential" setting, where the ArcSine Law is already known for a fixed interval $[0, N]$ of Steps along the x-axis.

Bibliography

- N.G. de Bruijn, D.E. Knuth, and S.O. Rice, The average height of planted plane trees, *Graph Theory and Computing* , Ronald C. Read, ed., Academic Press, New York (1972), p. 15-22.
- E. Deutsch, Dyck path enumeration, *Discrete Mathematics* **204** (1999) 167-202.
- W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. I, 3rd ed.* Wiley, New York (1968).
- P. Flagolet and R. Sedgewick, *Analystic Combinatorics*. Cambridge University Press (2009).
- C. Mohan, The gambler's ruin problem with correlation, *Biometrika* **42** (1955) 486-493.
- G.J. Morrow, Laws relating runs and steps in gambler's ruin, *Stoch. Proc. Appl.* **125** (2015) 2010-2025.
- D. Szasz, B. Toth, Persistent random walks in a one-dimensional random environment, *J. Stat. Phys* **37** (1984) 27-38.

Addendum: Formula for $\overline{K}(r,u,z) := K(ru, 1/u, z; \frac{1}{2})$

Recall the Joint G.F. of Excursions Stats **R**, **V**, **L** in Homogeneous Case

Corollary

Let $b = a$. *Define* $\alpha_a := \sqrt{\beta_a^2 - 4x_a}$.
Then (1) $K(r,y,z;a) := E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}}\} = (1 - \frac{1}{2}\beta_a - \frac{1}{2}\alpha_a)/(1-a)$

- Assume now in Addition that $a = \frac{1}{2}$.
- Then by Direct Calculation the Joint G.F. of **R**, **U**, **L** is given by

$$\overline{K}(r,u,z) := \frac{1}{16}\left(16 - 4z^2 + 4r^2z^2 + r^2z^4 - 2r^2uz^4 + r^2u^2z^4 - S\right),$$

- With Main Term S given by:

$$S := \sqrt{(4 + 2z + 2rz + rz^2 - ruz^2)(4 + 2z - 2rz - rz^2 + ruz^2)}$$

$$\times\sqrt{(4 - 2z + 2rz - rz^2 + ruz^2)(4 - 2z - 2rz + rz^2 - ruz^2)}.$$

- It Holds that : $S(r,u,z) = S(1/r, u, rz)$.