Laws Relating Runs and Steps in Gambler's Ruin

Gregory J. Morrow

Department of Mathematics
University of Colorado Colorado Springs

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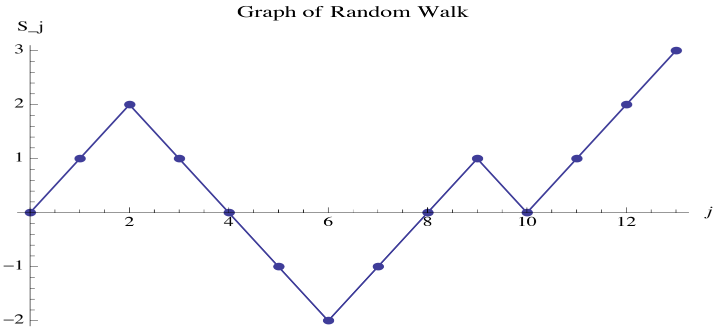
Outline

Simple Random Walk.

Simple Random Walk on the Integers: $\mathbf{S}_j$, $j = 0, 1, 2, \ldots$

- Independent fair coin tosses determine the steps.
- Example:
  $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \cdots = 0,\ 1,\ 2,\ 1,\ 0,\ -1,\ -2,\ -1,\ 0,\ 1,\ 0,\ 1,\ 2,\ 3 \ldots$



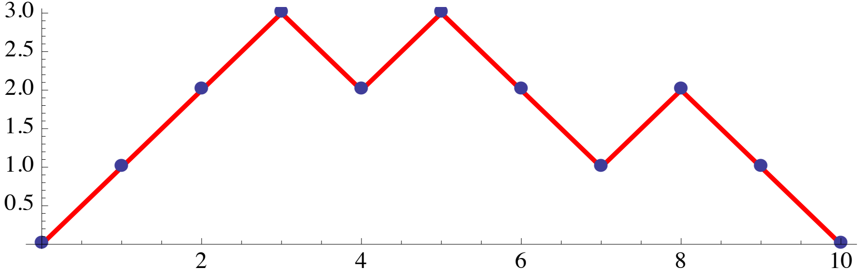
- Excursion:

A Path of First Return to Zero by Random Walk

Example Shows :

Number of Runs: $\mathbf{R} = 6$

Length: $\mathbf{L} = 10$; Height: $\mathbf{H} = 3$



Narayana Polynomials

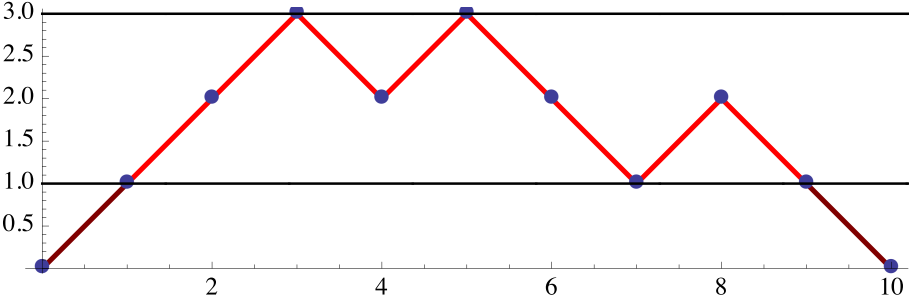
- *n*-**Dyck Path** with *k* **Peaks**: Non-negative, Length $2n$, Zero at Ends

  Counted by: $\mathcal{N}_{n,k} := \binom{n}{k-1}\binom{n-1}{k-1}/k$

- **Narayana Polynomial**:

$$\mathcal{N}_n(s) := \sum_{k=1}^{n} \mathcal{N}_{n,k} s^{k-1}; \ \ \mathcal{N}_1(s) = 1.$$

$$\mathcal{N}_2(s) = 1 + s,\ \mathcal{N}_3(s) = 1 + 3s + s^2,\ \mathcal{N}_4(s) = 1 + 6s + 6s^2 + s^3.$$



Joint G. F. of Runs and Length: $p = \frac{1}{2}$

By raising a $(n-1)$-**Dyck Path** to Level 1,

$$P_{n,k} := P(\mathbf{R} = 2k, \mathbf{L} = 2n) = 2 \times 4^{-n} \mathcal{N}_{n-1,k}, \quad P_{1,1} = \frac{1}{2}.$$

Therefore

$$\text{(G.F.)} \quad E\{y^{\mathbf{R}} z^{\mathbf{L}}\} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P_{n,k} y^{2k} z^{2n} = \frac{y^2 z^2}{2} \left(1 + \sum_{n=1}^{\infty} \mathcal{N}_n(y^2)(z^2/4)^n \right).$$

Break up

$$\mathcal{N} := 1 + \sum_{n=1}^{\infty} \mathcal{N}_n(s) t^n$$

Into Pure Excursions Plus Dyck Paths with At Least One "Middle" Zero.

$$\Longrightarrow \quad \mathcal{N} = 1 + t\mathcal{N} + st(\mathcal{N}^2 - \mathcal{N})$$

$\Longrightarrow$ **Fact (Joint G.F. of Runs and Steps):**

- $E\{y^{\mathbf{R}} z^{\mathbf{L}}\} = 1 + xu - \sqrt{(xu-1)^2 - 4x}, \quad u := y^2 - 1, \; x := z^2/4.$

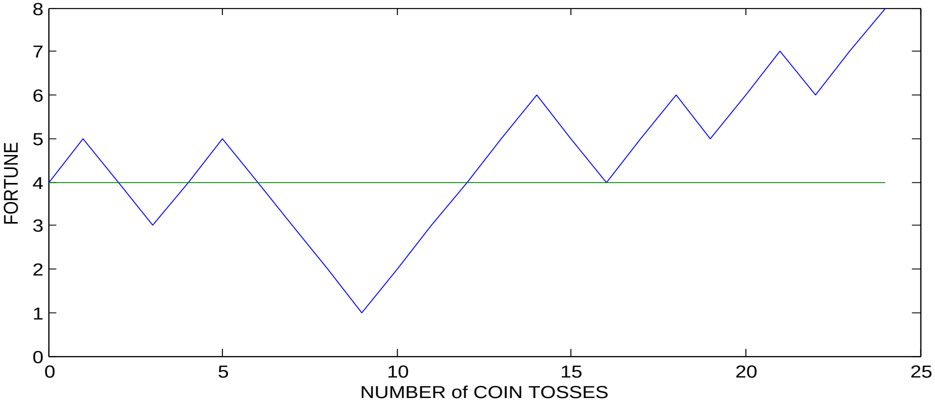
Example for Gambler's Ruin

Initial Fortune N. Play Independent Fair Games: Win 1 or Lose 1.

Terminate Game when **Fortune** First Equals $2N$ or 0.

Equivalently, Terminate at Epoch: $\inf\{j : |\mathbf{S}_j| = N\}$.

In Example, the "Last Visit" is: $\mathcal{L} = 16$, with $N = 4$.



Goal: Explicitly Find $K_N(y,z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < N\}$.

- Let $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \ldots$ be Independent Copies of $\mathbf{L}$.
- Similarly $\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \ldots$ are Independent Copies of $\mathbf{R}$
- Let $\mathcal{M} = \mathcal{M}_N := \#$ Excursions Until the Last Visit $\mathcal{L}$.
- $\mathcal{M}$ is a Geometric random variable:

$$P(\mathcal{M} = m) = P(\mathbf{H} < N)^m P(\mathbf{H} \geq N), \; m = 0, 1, 2, \ldots.$$

- Write: $K_N(y,z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < N\}$,

$$\mathcal{L} := \sum_{m=0}^{\mathcal{M}} \mathbf{L}^{(m)}, \;\; \mathcal{R} := \sum_{m=0}^{\mathcal{M}} \mathbf{R}^{(m)}.$$

- Hence, **Application:**

$$E\{y^{\mathcal{R}} z^{\mathcal{L}}\} = \sum_{m=0}^{\infty} P(\mathcal{M} = m)\,(K_N(y,z))^m = \frac{P(\mathbf{H} \geq N)}{1 - P(\mathbf{H} < N) K_N(y,z)}.$$

Non-increasing Future Maxima in an Excursion

Future Maxima for Excursion Shown:

$M_1 :=$ Max. Height Over Excursion. Here, $M_1 = 4$.

$M_2 :=$ Max. Height **After** Return from M_1 to Level 1.

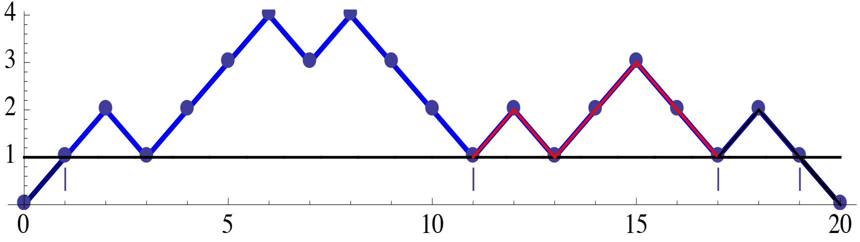
Here, $M_2 = 3$, $M_3 = 2$.

Within Segment i, the Path Forms:

(a) A Non-negative First Passage from Level 1 to M_i,

(b) An Upside-Down Non-negative First Passage.

Excursion with 3 Segments for Future Maxima



Conditional G. F. of $\mathbf{R}$ and $\mathbf{L}$ Given $\mathbf{H} = N$

- $\mathbf{L}_N :=$ # Steps Until a First Passage to $\mathbf{H} = N$.
- $\mathbf{R}_N :=$ # Runs Until a First Passage to $\mathbf{H} = N$.
- Conditional Joint G. F. of $\mathbf{R}_N$ and $\mathbf{L}_N$,
  Given **Non-negative Passage** to $\mathbf{H} = N$:

$$g_N(y,z) := E\{y^{\mathbf{R}_N} z^{\mathbf{L}_N} \mid \mathbf{S}_j \geq 0, j = 1, \ldots, \mathbf{L}_N\}.$$

- $zg_N(y,z)$ Is the Joint Distribution of Runs and Steps over **Gambler's Ruin Meander.**
- By **Non-negative Passage** to level N from Level 1, Followed by an Upside Down **Non-negative Passage** to Level N+1, the Conditional G. F. of $\mathbf{R}$ and $\mathbf{L}$ Given $\mathbf{H} = N + 1$ is:

$$(1) \quad G_{N+1}(y,z) := E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} = N + 1\} = zg_N g_{N+1}.$$

Recurrence Formula for $G_N(y, z)$.

- Define
$$\rho_N := P(\mathbf{S}_j \geq 0, j = 1, \ldots, \mathbf{L}_N).$$
- By Classical Gambler's Ruin Solution, if $p = \frac{1}{2}$, $\quad \rho_N = \frac{1}{N+1}$.
- Future Maxima Decomposition $\Longrightarrow$

$$G_{N+1} = Cz^2 \sum_{\ell=1}^{\infty} (\rho_N g_N)^{2\ell} \prod_{a=1}^{N-1} \sum_{\ell_a=0}^{\infty} (\rho_a g_a)^{2\ell_a},$$

$[\ell := \max\{i : M_1 = \cdots = M_i = N+1\}.$]

$$\Longrightarrow \quad (2) \qquad G_{N+1} = Cz^2 g_N^2 \prod_{a=1}^{N} \frac{1}{1 - [\rho_a g_a]^2}.$$

- (1) and (2) $\Longrightarrow$

$$(3) \qquad g_{N+1} = z^{-1} g_N^{-1} G_{N+1} = Czg_N \prod_{a=1}^{N} \frac{1}{1 - [\rho_a g_a]^2}.$$

Recursion for g_N.

- Equation (3) may be Rewritten:

$$(3)' \qquad g_{N+1} = C\frac{g_N}{g_{N-1}}\frac{1}{1-[\rho_N g_N]^2}\left(z g_{N-1}\prod_{a=1}^{N-1}\frac{1}{1-[\rho_a g_a]^2}\right).$$

Lemma

$$(*): \qquad g_{N+1} = C_N\frac{g_N^2}{g_{N-1}} * \frac{1}{1-[\rho_N g_N]^2},\ N \geq 2,$$

where, if $p=\frac{1}{2}$, $C_N = 1-\rho_N^2 = \frac{(N+2)N}{(N+1)^2}$, *and* $g_1 = yz$, $g_2 = \frac{3}{4}\frac{yz^2}{1-y^2z^2/4}$.

- For $p=\frac{1}{2}$, Write **Denominator** w_2 of g_2 by

$$w_2 = 1 - x(u+1),\ \text{for}\ \ x := \frac{1}{4}z^2,\ \ u := y^2-1.$$

Denominators w_N of g_N

- **Introduce** $u := y^2 - 1$ and $x := \xi z^2$, for $\xi := p(1-p)$.
- For $p = \frac{1}{2}$, **Lemma** $(*) \implies$ $w_2 = (1-x) - ux$,

$$w_3 = (1-2x) - (2x - x^2)u + x^2u^2,$$

$$w_4 = (1 - 3x + x^2) - (3x - 4x^2)u + (3x^2 - x^3)u^2 - x^3u^3.$$

- In General, **Denominators** w_N of g_N Satisfy the **Identity**:

$$(**) \qquad w_N^2 - x^N(u+1) = w_{N-1}w_{N+1}; \;\; w_0 = w_1 = 1.$$

- For $p = \frac{1}{2}$, and $u = 0$: $w_N = N^{\text{th}}$ Fibonacci Polynomial, F_N.

$$F_{N+1} = F_N - xF_{N-1}.$$

- Via Mathematica and OEIS, **Guess General Formula** for w_N.

Denominators w_N

Define

$$w_N(x,u) := \sum_{j\geq 0}\sum_{k\geq 0}\binom{N-k-j}{k}\binom{N-k-1}{j}(-x)^k(-ux)^j,\ \ N\geq 1.$$

Lemma

$$\sum_{N=1}^{\infty} w_N(x,u)t^N = \frac{t(1-xt)}{1+(xu-1)t+xt^2}.$$

- **Lemma** $\Longrightarrow$ Closed Form for w_N:

$$w_N = \frac{2^{-N}}{\alpha}\{(\beta+\alpha)^N-(\beta-\alpha)^N-2x[(\beta+\alpha)^{N-1}-(\beta-\alpha)^{N-1}]\},$$

$$\beta := 1-xu,\ \ \alpha := \sqrt{(xu-1)^2-4x}.$$

- **Identity** $(**)$ for Denominators Established Algebraically By Applying **Closed Form**.

Numerators $q_N(u)$

- Define **Numerators**:

$$q_N := \frac{2^{-N}}{\alpha}\{(\beta+\alpha)^N - (\beta-\alpha)^N\}$$

- **Commutation Identity** Follows Algebraically from Closed Forms:

$$(***) \qquad q_N w_{N+1} - w_N q_{N+1} = -x^N.$$

Theorem

$$\frac{C}{y^2 z^2} E\{y^{\mathbf{R}} z^{\mathbf{L}} \mid \mathbf{H} \le N\} = \sum_{a=1}^{N} G_a P(\mathbf{H}=a) = \sum_{a=1}^{N} \frac{x^{a-1}}{w_{a-1} w_a} = \frac{q_N}{w_N}.$$

- Proof of Theorem: $(***) \Longrightarrow \frac{q_N}{w_N} + \frac{x^N}{w_N w_{N+1}} = \frac{q_{N+1}}{w_{N+1}}$.

Corollary

$$E\{y^{\mathbf{R}} z^{\mathbf{L}} | \mathbf{H} < \infty\} = \frac{1+\sqrt{1-4\xi}}{4\xi}(1 + xu - \sqrt{(xu-1)^2 - 4x}).$$

Distribution of Runs.

Corollary

If $p = \frac{1}{2}$, then

$$P(\mathbf{R} = 2n) = \frac{1}{\pi} \int_0^{\pi} \frac{\sin^2(\theta) d\theta}{(5 + 4\cos\theta)^{n+1}}, \; n \geq 2; \; P(\mathbf{R} = 2) = 2/3.$$

- Proof: Expand $q_N(\frac{1}{4}, u)/w_N(\frac{1}{4}, u)$ by Partial Fractions.

Application to Gambler's Ruin: $p = \frac{1}{2}$

- $\mathcal{R} = \mathcal{R}_N :=$ Sum of Runs over Excursions in Gambler's Ruin.
- Formula $E\{y^{\mathcal{R}} z^{\mathcal{L}}\} = \frac{P(\mathbf{H} \geq N)}{1 - P(\mathbf{H} < N) K_N(y,z)}$, $z = 1$, and $y = e^{-\lambda/N^2} \Longrightarrow$

Corollary

$$\lim_{N \to \infty} E\{e^{-\lambda \mathcal{R}_N / N^2}\} = \frac{\tanh \sqrt{\lambda}}{\sqrt{\lambda}} = \int_0^{\infty} f(x) e^{-\lambda x} dx,$$

$f(x) = (\pi x)^{-1/2} \sum_{\nu=-\infty}^{\infty} (-1)^{\nu} e^{-\nu^2/x}, x > 0.$

Scaling Limit for $2\mathcal{R}-\mathcal{L}$ in Gambler's Ruin: $p=\frac{1}{2}$.

- $\mathcal{L}_N :=$ $\#$ Steps over Excursions in Gambler's Ruin ="Last Visit".

$$\lim_{N\to\infty} E\{e^{-\lambda \mathcal{L}_N/N^2}\} = \frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}}$$

- Expect $2\mathcal{R}_N - \mathcal{L}_N$ to have order N.
- Denote

$$\mathcal{R}'_N := \#\text{Runs over Meander},\quad \mathcal{L}'_N := \#\text{Steps over Meander},$$

$$X_N := 2\mathcal{R}_N - \mathcal{L}_N - \mathcal{M}_N,\quad X'_N := 2\mathcal{R}'_N - \mathcal{L}'_N.$$

Corollary

$\lim_{N\to\infty} E\{e^{it(X_N+X'_N)/N}\} = \frac{1}{\cosh(t)} = \int_{-\infty}^{\infty} e^{itx} s_1(x)dx,$

$\lim_{N\to\infty} E\{e^{itX'_N/N}\} = \frac{\sinh(t)}{t} = \int_{-\infty}^{\infty} e^{itx} s_2(x)dx,$

$$s_1(x) = \frac{1}{2} sech(\pi x/2),\ s_2(x) = \frac{\pi}{4} sech^2(\pi x/2),\ -\infty < x < \infty.$$

Integer Sequences

From OEIS A059231,

- $a(n) := \#$ Lattice Paths $(0,0)$ to $(2n,0)$ using steps from

  $$S = \{(k,k) \text{ or } (k,-k) : k \geq 1\}, \text{ and never crossing the } x\text{-axis.}$$

- Paths Counted by $a(n)$ are so-called "Catalan Rook Paths".
- $a(n) = 1, 5, 29, 185, 1257, \ldots.$
- "Twin" Generating Function to $K(y) := E\{y^{\mathbf{R}}\}$:

$$A(t) := \sum_{n=0}^{\infty} a(n)t^n = \frac{1}{8t}\left(1 + 3t - \sqrt{1 - 10t + 9t^2}\right) \text{ [Kung + de Mier]}$$

$$\text{Relationship:} \quad K(y) = 2A(1/y^2).$$

Corollary

$$a(n) = \frac{1}{2} 3^{2n+1} P(\mathbf{R} = 2(n+1)), \; n \geq 0.$$

Conclusion

- The Future Maxima Decomposition is amenable to the study of the conditional Joint Generating Function of Runs and Steps given the Height of a Random Walk Excursion.
- Applications:
  - A $(\mathrm{Sech})^2$-type Scaling Distribution Holds for Twice the Number of Runs Minus the Number of Steps in the Gambler's Ruin Meander.
  - A Binomial Identity Arises from the Denominator Polynomials $w_N(x, u)$ in the Special Case $x = \frac{1}{4}$.
- Outlook
  - Understand the Relationship Between "Catalan Rook Paths" Enumerated by $a(n) = 1, 5, 29, \ldots,$ and the Distribution of Runs.

Bibliography

- H. Bateman and A. Erdelyi (1953). *Higher Transcendental Functions,* **1**. McGraw-Hill, New York, 1953-55.
- N.G. de Bruijn, D.E. Knuth, and S.O. Rice (1972). The average height of planted plane trees, *Graph Theory and Computing* , Ronald C. Read, ed., Academic Press, New York (1972), p. 15-22.
- W. Feller (1968). *An Introduction to Probability Theory and Its Applications, Vol. I, 3rd ed.* Wiley, New York.
- P. Flagolet and R. Sedgewick (2009). *Analystic Combinatorics*. Cambridge University Press.
- J. P. S. Kung and A. de Mier (2013). Catalan lattice paths with rook, bishop and spider steps, *Journal of Combinatorial Theory, Series A* **120**, 379-389.
- *On-Line Encyclopedia of Integer Sequences*. http://oeis.org/
- J. Riordan (1968). *Combinatorial Identities.* Wiley, New York.

Alternative Formulae for w_N and Q_N with $x = \frac{1}{4}$.

- Special Case: $x = \frac{1}{4}$.

$$w_N(\frac{1}{4}, u) = 2^{-N} \sum_{j=0}^{N-1} \left[\binom{N+j}{2j+1} + \binom{N+j-1}{2j} \right] \left(\frac{-u}{2} \right)^j .$$

- Note Form of $q_N(\frac{1}{4}, u)$:

$$q_N(\frac{1}{4}, u) := 2^{-N+1} \sum_{j=0}^{N-1} \binom{N+j}{2j+1} \left(\frac{-u}{2} \right)^j .$$

- Binomial Identity Arising from Two Representations of $w_N(\frac{1}{4}, u)$:

$$\sum_{k \geq 0} \binom{N-k-j}{k} \binom{N-k-1}{j} \left(-\frac{1}{4} \right)^k$$

$$= 2^{j-N} \left(\binom{N+j}{2j+1} + \binom{N+j-1}{2j} \right), \quad \text{all } j \geq 0, N \geq 1.$$