Laws Relating Runs, Long Runs, and Steps in Gambler's Ruin, with Persistence in Two Strata

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Abstract Define a certain gambler's ruin process $\mathbf{X}_j,\ \ j \geq 0$, such that the increments $\varepsilon_j := \mathbf{X}_j - \mathbf{X}_{j-1}$ take values ± 1 and satisfy $P(\varepsilon_{j+1} = 1 | \varepsilon_j = 1, |\mathbf{X}_j| = k) = P(\varepsilon_{j+1} = -1 | \varepsilon_j = -1, |\mathbf{X}_j| = k) = a_k$, all $j \geq 1$, where $a_k = a$ if $0 \leq k \leq f-1$, and $a_k = b$ if $f \leq k < N$. Here $0 < a, b < 1$ denote persistence parameters and $f, N \in \mathbb{N}$ with $f < N$. The process starts at $\mathbf{X}_0 = m \in (-N, N)$ and terminates when $|\mathbf{X}_j| = N$. Denote by $\mathscr{R}'_N$, $\mathscr{U}'_N$, and $\mathscr{L}'_N$, respectively, the numbers of runs, long runs, and steps in the meander portion of the gambler's ruin process. Define $X_N := \left(\mathscr{L}'_N - \frac{1-a-b}{(1-a)(1-b)}\mathscr{R}'_N - \frac{1}{(1-a)(1-b)}\mathscr{U}'_N\right)/N$ and let $f \sim \eta N$ for some $0 < \eta < 1$. We show $\lim_{N\to\infty} E\{e^{itX_N}\} = \hat{\varphi}(t)$ exists in an explicit form. We obtain a companion theorem for the last visit portion of the gambler's ruin.

Key words: runs, generating function, excursion, gambler's ruin, last visit, meander, persistent random walk, generalized Fibonacci polynomial.
2010 Mathematics Subject Classification: Primary: 60F05. Secondary: 05A15.

1 Introduction

Define a gambler's ruin process $\{\mathbf{X}_j,\ \ j \geq 0\}$, with values in $\mathbb{Z} \cap [-N, N]$, such that the increments $\varepsilon_j := \mathbf{X}_j - \mathbf{X}_{j-1}$ take values ± 1 and satisfy $P(\varepsilon_{j+1} = 1 | \varepsilon_j = 1, |\mathbf{X}_j| = k) = P(\varepsilon_{j+1} = -1 | \varepsilon_j = -1, |\mathbf{X}_j| = k) = a_k$, all $j \geq 1$, where $a_k = a$ if $0 \leq k \leq f-1$, and $a_k = b$ if $f \leq k < N$. Here $0 < a, b < 1$ denote persistence parameters and $f, N \in \mathbb{N}$ with $f < N$. The process starts at some fixed level $m \in (-N, N)$ and terminates at an epoch j when $|\mathbf{X}_j| = N$. For initial probabilities, take $\pi_+ = P(\varepsilon_j = 1) = \pi_- = P(\varepsilon_j = -1) = \frac{1}{2}$. We call the two ranges of values $|k| \leq f-1$ and $f \leq |k| < N$ as strata for the two persistence parameter values a and b, respectively. In gambling, $\mathbf{X}_j$

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denotes a fortune after j games on which the gambler makes unit bets. If $a,b > \frac{1}{2}$, then any run of fortune tends to keep going in the same direction. Thus for example a win [loss] resulting in fortune k for some $|k| \leq f-1$ is followed by another win [loss] with probability a, whereas a change in fortune occurs with probability $1-a$. Henceforth we shall simply refer to $\{\mathbf{X}_j = \mathbf{X}_j^N\}$ as the gambler's ruin process, with or without mention of the parameters a, b, f, and N. Note that $\{\mathbf{X}_j\}$ is the classical fair gambler's ruin process in case $a = b = \frac{1}{2}$, with symmetric boundaries N and $-N$. For the *homogeneous* case $a = b$, the increments $\{\varepsilon_j, j \geq 0\}$, form a strictly stationary process with zero means, where the correlation between ε_j and ε_{j+1} is $2a-1$. If $a = b$ and also $N = \infty$ then $\{\mathbf{X}_j^\infty\}$ becomes a symmetric persistent (or correlated) random walk on $\mathbb{Z}$ that is recurrent by [19, Thm. 8.1].

Physical models of persistence often consider the velocity of a particle either staying the same or being changed according to a random collision process [1, 17, 21]; in our model the velocity only takes values ± 1. Our introduction of strata corresponds to a change in medium over which the persistence parameter, or likelihood of the velocity staying the same, would deterministically change. In [21], the authors obtain a Wiener limit for the normalized sum of velocities under a random environment, that includes our deterministic model. Our aim is different since we want results for discrete statistics that have no analogue in the Wiener process. In this context our stratified model seems to be new.

We define a nearest neighbor path of length n in $\mathbb{Z}$ to be a sequence $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$, where $\Gamma_j \in \mathbb{Z}$ and $\delta_j := \Gamma_j - \Gamma_{j-1}$ satisfies $|\delta_j| = 1$ for all $j = 1, \ldots, n$. We also call n the number of steps of Γ. We connect successive lattice points $(j-1, \Gamma_{j-1})$ and (j, Γ_j) in the plane by straight line segments, and term this connected union of straight line segments the *lattice path*. See Figures 1–2. We define the number of runs along Γ as the number of inclines, either straight line ascents or descents, of maximal extent along the lattice path; the length of a run is the number of steps in such a maximal ascent or descent. A *long run* is itself a run that consists of at least two steps; in gambling terminology a long run means that the run of fortune does not immediately change direction. A *short run* is on the other hand a run of length exactly one, so every run is either a long run or a short run. In Figure 2, the lattice path shown has 15 runs, with 7 short runs and 8 long runs. An *excursion* is a nearest neighbor path that starts and ends at $m = 0$, $\Gamma_0 = \Gamma_n = 0$, but for which $\Gamma_j \neq 0$ for $1 \leq j \leq n-1$. A positive excursion is an excursion whose graph lies above the x–axis save for its endpoints. For a positive excursion path, the number of runs is just twice the number of peaks, where a peak at lattice point (j, Γ_j) corresponds to $\delta_j = 1$ and $\delta_{j+1} = -1$.

The *last visit* is defined as

$$\mathscr{L}_N := \max\{j \geq 0 : j = 0, \text{ or } \mathbf{X}_j = 0 \text{ for some } j \geq 1\}. \tag{16.1}$$

The *meander* is the portion of the process that extends from the epoch of the last visit $\mathscr{L}_N$ until the gambler's ruin process terminates. So the meander process never returns to the level $m = 0$. See Figure 1. It is shown by [14] that, for $a = b = \frac{1}{2}$, if $\mathscr{R}_N$ denotes the total number of runs over all excursions of the absolute value process

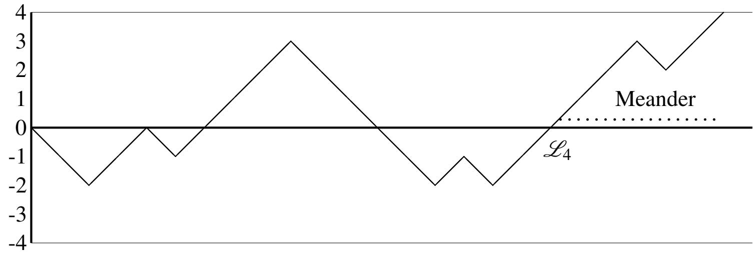

Fig. 1: Last Visit and Meander; $N = 4$.

$\{|\mathbf{X}_j|\}$ until the last visit, then, with the order N scaling, it holds that $(\mathscr{L}_N - 2\mathscr{R}_N)/N$ converges in law. Also, if $\mathscr{R}'_N$ and $\mathscr{L}'_N$ denote respectively the number of runs and steps over the meander portion of the process, then $(\mathscr{L}'_N - 2\mathscr{R}'_N)/N$ converges in law to a density $\varphi(x) = (\pi/4)\text{sech}^2(\pi x/2)$, $-\infty < x < \infty$, with characteristic function $\int_{-\infty}^{\infty} \varphi(x)e^{ixt}dx = t/\sinh(t)$. We first generalize this result for the meander case. Let $\mathscr{R}'_N$, $\mathscr{V}'_N$, $\mathscr{L}'_N$, denote respectively the numbers of runs, short runs, and steps, in the meander portion of the gambler's ruin; for the lattice path of Figure 1 we have $\mathscr{R}'_4 = 3$, $\mathscr{V}'_4 = 1$, $\mathscr{L}'_4 = 6$. Define the following scaled random variable over the meander:

$$X_N := \frac{1}{N}\left(\mathscr{L}'_N - \frac{2-a-b}{(1-a)(1-b)}\mathscr{R}'_N + \frac{1}{(1-a)(1-b)}\mathscr{V}'_N\right). \tag{16.2}$$

Theorem 16.1. *Let $f = \eta N$ for some fixed $0 < \eta < 1$. Denote $\kappa_1 := \frac{\eta\sigma_1}{1-b}$ and $\kappa_2 := \frac{(1-\eta)\sigma_2}{1-a}$, with $\sigma_1 = \sqrt{a+b^2-2ab}$ and $\sigma_2 = \sqrt{b+a^2-2ab}$. Let X_N be defined by* (16.2). *Then, $\lim_{N\to\infty} E\{e^{itX_N}\} = \hat{\varphi}(t)$, where*

$$\begin{aligned}(b\kappa_1\sigma_2 + a\kappa_2\sigma_1)t/\hat{\varphi}(t) &:= a\sigma_1\cosh(\kappa_1 t)\sinh(\kappa_2 t)\\ &+ b\sigma_2\sinh(\kappa_1 t)\cosh(\kappa_2 t) + i(b-a)^2\sinh(\kappa_1 t)\sinh(\kappa_2 t).\end{aligned} \tag{16.3}$$

In Theorem 16.1, we obtain that $\varphi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\hat{\varphi}(t)dt$ is real since the complex conjugate of $\varphi(x)$ is equal to itself; observe this by making a change of variables $t \to -t$ after conjugation of the integral.

We have a bivariate result for the homogeneous case as follows. Define

$$Y_{1,N} := \frac{1}{N}\left(\mathscr{R}'_N - \frac{1}{(1-a)}\mathscr{V}'_N\right); \; Y_{2,N} := \frac{1}{N}\left(\mathscr{L}'_N - \frac{1}{(1-a)}\mathscr{R}'_N\right) - Y_{1,N}. \tag{16.4}$$

Corollary 16.1. *Suppose $a = b$. Then the limiting joint characteristic function of the random variables $Y_{1,N}$ and $Y_{2,N}$ is:*

$$\lim_{N\to\infty} E\{e^{isY_{1,N}+itY_{2,N}}\} = \frac{\sqrt{(1-a)s^2+at^2}}{\sinh(\sqrt{(1-a)s^2+at^2})}. \tag{16.5}$$

Remark 16.1. Let $a=b$, and define $X_{\zeta,N} := \frac{1}{N}\left(\mathscr{L}'_N - \frac{1+\zeta}{(1-a)}\mathscr{R}'_N + \frac{\zeta}{(1-a)^2}\mathscr{V}'_N\right)$. Then by setting $s=(1-a-\zeta)t/(1-a)$ in (16.5) we obtain, for all $\zeta\in\mathbb{R}$,

$$\lim_{N\to\infty} E\{e^{itX_{\zeta,N}}\} = \frac{A_\zeta t}{\sinh(A_\zeta t)};\ A_\zeta := \sqrt{[(2\zeta-1)a+(1-\zeta)^2]/(1-a)}.$$

Example 16.1. As a special case of Theorem 16.1, consider $b=1-a$ and $\eta=a$. Then $\kappa_i=\sigma_j=\sigma:=\sqrt{1-3a+3a^2}$, for all $i,j=1,2$. In this case we have

$$\hat{\varphi}(t) = \sigma^2 t/\left\{\sinh(\sigma t)[\sigma\cosh(\sigma t)+i(1-2a)^2\sinh(\sigma t)]\right\}. \tag{16.6}$$

The complex factor of the denominator of $\hat{\varphi}(t)$ in (16.6) is equal to zero if and only if $e^{2\sigma t} = \frac{-\sigma+(1-2a)^2 i}{\sigma+(1-2a)^2 i}$. The smallest root is $t=\frac{i}{2\sigma}(\pi-\arctan\frac{2\sigma(1-2a)^2}{\sigma^2-(1-2a)^4})$, with $\sigma^2-(1-2a)^4 = a(1-a)(5-16a+16a^2)>0$. Thus we can analytically continue $\hat{\varphi}(t)$ to a suitably chosen ball of positive radius ε_0 about the origin such that $\sup_{|\xi|\le\varepsilon_0}\|\hat{\varphi}(\cdot+i\xi)\|_2<\infty$. It follows by [18, Thm. IX.13] that the inverse Fourier transform $\varphi(x)$ of $\hat{\varphi}(t)$ has exponential decay, meaning $e^{\varepsilon|x|}\varphi(x)$ is square integrable for any $\varepsilon<\varepsilon_0$. However the probability density, $\varphi(x)$, is not symmetric in x under (16.6) with $a\neq\frac{1}{2}$; see Figure 4 at the end of the paper; see also [15] for computational details.

We now introduce the definitions of the excursion statistics to further describe our results. For the definitions in this paragraph we assume $\mathbf{X}_0=0$ and $N=\infty$. Define the index j, or step, of first return of $\{\mathbf{X}_j\}$ to the origin by $\mathbf{L}:=\inf\{j\ge1:\mathbf{X}_j=0\}$. Define the excursion sequence from the origin by $\boldsymbol{\Gamma}:=\{\mathbf{X}_j, j=0,\dots,\mathbf{L}\}$; again $\mathbf{L}$ is the number of steps of $\boldsymbol{\Gamma}$. Define the *height* $\mathbf{H}$ of the excursion $\boldsymbol{\Gamma}$ as the maximum absolute value of the path over this excursion:

$$\mathbf{H} := \max\{|\mathbf{X}_j| : j=1,\dots,\mathbf{L}\}. \tag{16.7}$$

Also define $\mathbf{R}$ as the number of runs along $\boldsymbol{\Gamma}$, and further define $\mathbf{V}$ as the number of short runs along $\boldsymbol{\Gamma}$. Thus officially $\mathbf{U}:=\mathbf{R}-\mathbf{V}$ is the number of long runs along $\boldsymbol{\Gamma}$. In Figure 1 there are 4 excursions until the last visit to the origin, with respective heights: 2, 1, 3, 2. The numbers of runs in the excursions of the absolute value process $\{|\mathbf{X}_j|\}$ until the last visit of Figure 1, wherein negative excursions are reflected into positive excursions, are: 2, 2, 2, 4. The corresponding numbers of short runs in this last visit portion of the absolute value process are: 0, 2, 0, 2.

The first motivation of the present paper is to show how the method of [14] extends to the three statistics, runs, short runs, and steps, in the homogeneous setting ($a=b$). As a particular result we find the following Corollary 16.2, which connects the present work with a certain combinatorial domain in the study of Dyck paths. Note that the generating function method which drives the present study depends heavily on a *return to the level* 1 type recurrence approach that has been applied extensively in the field of lattice path combinatorics; see [2, 3, 4, 7, 9, 10, 12]. Let P_a denote the probability for the homogeneous model with persistence parameter a. We obtain the following symmetry for the joint distribution of the excursion statistics.

Corollary 16.2. *Let $a = b$ and assume $\mathbf{X}_0 = 0$ and $N = \infty$. Then for all $n \geq 2$ there holds:*

$$(1-a)P_a(\mathbf{L} = 2n, \mathbf{R} = 2k, \mathbf{U} = \ell) = aP_{1-a}(\mathbf{L} = 2n, \mathbf{L} - \mathbf{R} = 2k, \mathbf{U} = \ell). \quad (16.8)$$

In particular if $a = \frac{1}{2}$, then $E\{e^{ir\mathbf{R}}e^{is\mathbf{U}}e^{it\mathbf{L}}\} - E\{e^{ir(\mathbf{L}-\mathbf{R})}e^{is\mathbf{U}}e^{it\mathbf{L}}\} = \frac{1}{2}e^{2it}(e^{2ir} - 1)$.

Corollary 16.2 extends the known result for the simple symmetric random walk that $P(\mathbf{L} = 2n, \mathbf{R} = 2k) = P(\mathbf{L} = 2n, \mathbf{L} - \mathbf{R} = 2k)$, $n \geq 2$. Our proof depends on algebraic manipulation of the generating function; see Section 3.5.1.

The second motivation is to extend the persistence model to the case of two distinct strata $a \neq b$. This *full model*, together with its solution, has interesting features, which include:

1. its intrinsic value as physical model; cf. [1, 21],
2. completely explicit formulae throughout for key polynomials, identities, and generating functions;
3. new limiting distributions for a scaling of order N in both the meander and the last visit portions of the gambler's ruin.

We finally state a companion result to Theorem 16.1, again for the full model, that gives a scaling limit of order N over the last visit portion of the gambler's ruin. Let $\mathscr{R}_N$ and $\mathscr{V}_N$ denote the total number of runs and short runs of the absolute value process $\{|\mathbf{X}_j|\}$ until the epoch of the last visit, $\mathscr{L}_N$, defined by (16.1). Define $\mathscr{M}_N$ as the number of consecutive excursions of height at most $N-1$ of the absolute value process $\{|\mathbf{X}_j|\}$ until $\mathscr{L}_N$. In Figure 1, we have $\mathscr{M}_4 = 4$, $\mathscr{R}_4 = 10$, $\mathscr{V}_4 = 4$, $\mathscr{L}_4 = 18$. Define:

$$\mathscr{X}_N := \frac{1}{N}\left(\mathscr{L}_N - \frac{2-a-b}{(1-a)(1-b)}\mathscr{R}_N + \frac{1}{(1-a)(1-b)}\mathscr{V}_N - \frac{a(b-a)}{(1-a)(1-b)}\mathscr{M}_N\right). \quad (16.9)$$

Theorem 16.2. *Let $f \sim \eta N$, as $N \to \infty$, for some fixed $0 < \eta < 1$. Let $\mathscr{X}_N$ be defined by* (16.9)*. Let also κ_j, $j = 1, 2$, and σ_j, $j = 1, 2$, be as defined in Theorem 16.1. Let $\hat{\varphi}$ be defined by* (16.3)*. Then, $\lim_{N\to\infty} E\{e^{it\mathscr{X}_N}\} = \hat{\psi}(t)/\hat{\varphi}(t)$, where*

$$(ab\sigma_1\sigma_2)/\hat{\psi}(t) := ab\sigma_1\sigma_2\cosh(\kappa_1 t)\cosh(\kappa_2 t) + a^2\sigma_1^2\sinh(\kappa_1 t)\sinh(\kappa_2 t) \\ + ia\sigma_1(b-a)^2\cosh(\kappa_1 t)\sinh(\kappa_2 t).$$

Theorem 16.1, Corollary 16.1, and Theorem 16.2, are proved in Section 4, naturally following Section 3 on building blocks for the proofs.

2 Elements of the proof

Recall the definitions of the excursion statistics in (16.7) and following. We define the conditional joint probability generating function of the excursion statistics for runs, short runs, and steps given the height is at most N by

$$K_N(a,b) := E\{r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}} | \mathbf{X}_0 = 0,\ \mathbf{H} \le N\}. \tag{16.10}$$

To calculate (16.10), our proofs feature bivariate Fibonacci polynomials $\{q_n(x,\beta)\}$ and $\{w_n(x,\beta)\}$, defined as follows.

Definition 16.1. Define sequences $q_n(x,\beta)$ and $w_n(x,\beta)$ generated by the following recurrence relations, valid for $n \ge 1$.

$$q_{n+1} = \beta q_n - x q_{n-1},\ q_0 = 0, q_1 = 1; w_{n+1} = \beta w_n - x w_{n-1},\ w_0 = 1, w_1 = 1. \tag{16.11}$$

Here, $\beta, x \in \mathbb{C}$. The polynomials $q_n(x,\beta)$ generalize the univariate Fibonacci polynomials $F_n(x) = q_n(x,1)$, see [10, p. 327]; also $w_n(x,1) = F_{n+1}(x)$. In the case of steps alone in the classical fair gamblers ruin problem ($a = b = \frac{1}{2}$; $\beta = 1$ and $x = \frac{1}{4}z^2$ in (16.11)), the $\{q_n = F_n(x)\}$ are classically *numerator* polynomials, and the $\{w_n = F_{n+1}(x)\}$ are the *denominator* polynomials for the excursion generating function of height less than n, namely $K_{n-1}(\frac{1}{2},\frac{1}{2})$ with $r = y = 1$ in (16.10), [6], [10, Sec. V.4.3]. Here numerator and denominator refer to the convergent of a continued fraction representation of K_∞. See [4] for an interesting direction on excursions with different step sets besides the classical steps ± 1.

We write an *interlacing* property of any two term recurrence $v_{n+1} = \beta v_n - x v_{n-1}$, $n \ge 1$, with coefficients β and x independent of n:

$$v_{n+1}v_{n-1} - v_n^2 = \beta^{-1} x^{n-1} (v_3 v_0 - v_2 v_1),\ \beta \ne 0; \tag{16.12}$$

see [14, Eqs. (2.7)–(2.8)]. Note that when $v_0 = 0,\ v_1 = 1$, the polynomials $v_n = v_n(\beta, -x)$ are called the generalized Fibonacci polynomials in β and $-x$, and by standard generating function techniques, the fundamental sequences (16.11) have closed formulae given as follows: cf. [20, Eqs. (2.1) and (2.3)]; or [14, Eqs. (2.11)–(2.12)]. Define $\alpha := \sqrt{\beta^2 - 4x}$. Then, for all $n \ge 1$, and with $q_0(x,\beta) = 0$,

$$q_n(x,\beta) = \frac{2^{-n}}{\alpha}\left((\beta+\alpha)^n - (\beta-\alpha)^n\right); w_n(x,\beta) = q_n(x,\beta) - x q_{n-1}(x,\beta). \tag{16.13}$$

The formula for w_n follows from that of q_n, for $n \ge 1$, since $q_1 - x q_0 = 1 = w_1$, and $q_2 - x q_1 = \beta - x = w_2$.

We need some additional notation to describe our method as follows. For any pair of integers $m, n \in (-N, N)$ with $m \ne n$ we define the following *first passage* length for the process $\{\mathbf{X}_j\}$ that starts at $\mathbf{X}_0 = m$:

$$\mathbf{L}_{m,n} := \inf\{j \ge 1 : \mathbf{X}_j = n \text{ or } |\mathbf{X}_j| = N\}. \tag{16.14}$$

For any starting level $\mathbf{X}_0 = m$, let $\boldsymbol{\Gamma}_{m,n} := \{\mathbf{X}_j, j = 0, \ldots, \mathbf{L}_{m,n}\}$ denote the ordinary first passage path from level m to either level n or to the boundary of the gambler's ruin process. For our key definition (16.15), additional conditions are placed on the first passage path to make it *one–sided*.

Denote by $\mathbf{R}_{m,n}$ the number of runs and by $\mathbf{V}_{m,n}$ the number of short runs, respectively, along $\boldsymbol{\Gamma}_{m,n}$, where $\mathbf{L}_{m,n}$ denotes the number of steps along this path. For $n > m$, define $g_{m,n} = g_{m,n}(a,b)$ as the following *upward* conditional joint probability generating function for these counting statistics given two conditions on the path: (1) the path is a one–sided first passage path that starts at m and stays at or above level m until it reaches level n, and (2) the first two steps of this path are both in the positive direction. If still $n > m$ then we also define the analogous *downward* conditional joint generating function $g_{n,m}$:

$$g_{m,n} := E(r^{\mathbf{R}_{m,n}} y^{\mathbf{V}_{m,n}} z^{\mathbf{L}_{m,n}} | \varepsilon_1 = \varepsilon_2, \mathbf{X}_0 = m, \mathbf{X}_j \geq m,\ j = 0, \ldots, \mathbf{L}_{m,n}).$$
$$g_{n,m} := E(r^{\mathbf{R}_{n,m}} y^{\mathbf{V}_{n,m}} z^{\mathbf{L}_{n,m}} | \varepsilon_1 = \varepsilon_2, \mathbf{X}_0 = n, \mathbf{X}_j \leq n,\ j = 0, \ldots, \mathbf{L}_{n,m}). \qquad (16.15)$$

The condition that the first two steps be in the same direction in the definition (16.15) arises due to the inclusion of the statistic $\mathbf{V}_{m,n}$ in the analysis. The path in Figure 2 is a downward, first passage path from level 5 to level 0.

Let $n > m$. In the formulation of the recurrence for $g_{m,n}$, we must take account of the unconditional probability that a first passage from level m to level n remains at or above the starting level; we must also define the corresponding probability $\rho_{n,m}$, as follows.

$$\rho_{m,n} := P(\mathbf{X}_j \geq m, j = 0, \ldots, \mathbf{L}_{m,n} | \mathbf{X}_0 = m);$$
$$\rho_{n,m} := P(\mathbf{X}_j \leq n, j = 0, \ldots, \mathbf{L}_{n,m} | \mathbf{X}_0 = n). \qquad (16.16)$$

For $a = b = \frac{1}{2}$, the probability $\rho_{n,0} = \rho_{0,n}$ is determined by the classical solution of the probability of ruin started from fortune n on the interval $[0, n+1]$. For $a = b$, $\rho_{m,n}$ depends only on $k = n - m$ and is determined by $\rho_{m,m+\ell} = \frac{1}{2}(\ell - (\ell - 1)a)^{-1}$, see [13, Eq. (2.4)].

There are many calculations used to establish various formulae by the help of certain key definitions. We reserve the phrase *direct calculation* to mean that computer algebra (*Mathematica* [22]) is used to help verify the results. The companion document [15] to the present paper provides details of the verifications. In our approach, the complication of a second stratum is solved by finding the right formulae and then rendering a proof; we often utilize induction based on the proposed formulae. Our proofs may be termed elementary, since we use path decompositions to establish explicit formulae for the conditional generating functions $g_{m,n}$.

Our method for the full model is to show that the appropriate denominators $\{\overline{w}_{m,n}\}$ of the conditional generating functions $g_{m,n}$, together with certain singly–indexed numerators $\{\overline{q}_n\}$, give rise also to a nice representation of (16.10); see Theorem 16.3, in which our approach involves conditioning on the height $\mathbf{H} = n$ of an excursion. For the homogeneous case of Proposition 16.5, the formula for (16.10) follows in a standard way of dealing with a finite continued fraction. In the homogeneous case an alternative approach based on the format of [10], Proposition V.3, could probably be devised. Yet we need a closed formula for the one–sided first passage generating function $g_{0,N}$ to handle the meander in the full model, and this leads us to take an approach via recurrences proper, not only for $g_{m,n}$ but for $\overline{w}_{m,n}$.

Accordingly, by Propositions 16.3 and 16.4, we obtain our main results with the help of trigonometric substitutions and direct calculations.

3 Proofs of the building blocks

3.1 Recurrence for $g_{m,n}$

We first establish the general recurrence relations governing the upward and downward generating functions of (16.15). The condition *initial two steps the same* on the trajectory of the lattice path yields immediately that

$$g_{m,m+2}(a,b) := rz^2,\ g_{m+2,m}(a,b) := rz^2,\ m \geq 0. \tag{16.17}$$

The path decomposition of [14] handles runs and steps; here we extend that approach for short runs as well. It is convenient to focus on $g_{n,0}$ with some $n \geq 3$; see the definition (16.15). The Figure 2 is an illustration of one lattice path counted by $g_{5,0}$. Let U or D stand for one step up or down, respectively, in a lattice path, and let $(UD)^\ell$ be shorthand for $UDUD\cdots$ with ℓ repetitions of the pattern UD for some $\ell \geq 0$. Since any downward lattice path from n to 0 must first reach the level $m = 1$, we have an initial factor $g_{n,1}$ in a product formula for $g_{n,0}$.

Any section of a lattice path for $g_{n,1}$, which must end in DD, is followed by a sequence of steps of the form $(UD)^\ell UU$, or by a *terminal* sequence $(UD)^\ell D$. To handle transitions that do not start UU or DD we introduce:

$$\begin{aligned} \omega(a,b) &:= 1-(1-a)(1-b)r^2y^2z^2,\ k(a,b) := (a+b-ab)/\omega(a,b), \\ \tau(a,b) &:= 1+(1-a)(1-b)r^2z^2y(1-y);\ h(a,b) := \tau(a,b)/\omega(a,b). \end{aligned} \tag{16.18}$$

Denote $\mathbf{1} = (1,1,1)$ and evaluation of any function $u(a,b)$ at (r,y,z) by $u(a,b)[r,y,z]$. For brevity we may write u_a in place of $u(a,a)$. By (16.18), $k(a,b)[\mathbf{1}] = \tau(a,b)[\mathbf{1}] = 1$. Thus $k(a,b)$ is a probability generating function; the term $h(a,b)/h(a,b)[\mathbf{1}]$ is as well. In our discussion of $g_{n,0}$, if $f \geq 3$, then $k_a = k(a,a)$ accounts for a generating factor for an *upward preamble* $(UD)^\ell$ from level $m = 1$, succeeding DD and preceding UU; in this case $k_a = c\sum_{\ell=0}^{\infty}((1-a)^2r^2y^2z^2)^\ell = c/\omega_a$, where $c = a(2-a)$. If instead $f = 2$ is the change of stratum parameter, then we obtain $k(a,b)$ in place of k_a due to the fact that now a change in direction at level $m = 2$ occurs with probability $(1-b)$ while a change in direction at level $m = 1$ occurs with probability $(1-a)$. To handle the dependence on f, we define

$$[a,b]_m^+ := \begin{cases} (a,a), \text{ if } m \leq f-2 \\ (a,b), \text{ if } m = f-1, \\ (b,b), \text{ if } m \geq f; \end{cases} \qquad [a,b]_n^- := \begin{cases} (a,a), \text{ if } n \leq f-1, \\ (a,b), \text{ if } n = f, \\ (b,b), \text{ if } n \geq f+1. \end{cases} \tag{16.19}$$

Let us suppose that the continuation of the path after the first downward passage to level $m=1$ is not yet passing into a terminal sequence, so takes the form $(UD)^k UU\ldots$. Starting thus from UU the path makes an upward first passage to level n again (or not), and the pattern "up to level n and down to level 1" repeats for an indefinite number of times, $\ell \geq 0$. To handle the probability associated with the turning of the path downward from a level it will no longer exceed in the future of the path, or in turning from the bottom level $m=1$ to upwards (in the return to level 1), we define the *turning probability at altitude m* by

$$\gamma_m := \begin{cases} 1-a, \text{ if } m \leq f-1, \\ 1-b, \text{ if } m \geq f. \end{cases} \tag{16.20}$$

By definition (16.16), it now follows that $g_{n,0} = cg_{n,1}\lambda_{1,n}\lambda_{1,n-1}\cdots\lambda_{1,3}zh[a,b]_1^+$, with $\lambda_{1,n} := \sum_{\ell=0}^{\infty}(4\gamma_1\gamma_n\rho_{1,n}\rho_{n,1}k[a,b]_1^+k[a,b]_n^- g_{1,n}g_{n,1})^\ell$, or

$$\lambda_{1,n} = \frac{1}{1-4\gamma_1\gamma_n\rho_{1,n}\rho_{n,1}k[a,b]_1^+k[a,b]_n^- g_{1,n}g_{n,1}}.$$

Here the factor of 4 arises due to the fact that the stationary probabilities for first step up and down, namely $\pi_+ = \frac{1}{2}$ and $\pi_- = \frac{1}{2}$, get replaced by γ_1 and γ_n respectively in $\rho_{1,n}$ and $\rho_{n,1}$. The factor $k[a,b]_n^-$ takes account of a *downward preamble* succeeding UU and preceding DD from the maximum possible level $M_2 \geq 3$ in the remainder of the downward lattice path. Here the successive maximum levels $n = M_1 \geq M_2 \geq \cdots \geq M_r$ over the whole future of the path, determined in turn from the points of each of its returns to level $m=1$ from the previous such maximum, are the *future maxima* (cf. [14]) of a downward path from level $n \geq 3$ to level $m=0$. See Figure 2, in which we have $M_1 = 5$, and $M_2 = 4$, $M_3 = 3$; there is no second future maximum of level 4, for example, because there is no return to level 1 between the two peaks at level 4, but instead we see a downward preamble $(DU)^1$ at M_2. By definition we have $M_r \geq 3$, and the downward path goes into a *terminal* sequence after a return to level 1 from M_r; the terminal sequence is of form $(UD)^k D$; see Figure 2. Eventually, in the beginning, the path will never rise to level n again; but to lower future maxima at levels $3 \leq m \leq n-1$; thus the product $\lambda_{1,n}\lambda_{1,n-1}\ldots\lambda_{1,3}$. The factor $zh[a,b]_1^+$ corresponds to the terminal sequence. Now replace $m=1$ by $m \geq 1$ for a final destination level $m-1$, to obtain the following downward recurrence relation for any $m < n-1$:

$$g_{n,m-1} = czh[a,b]_m^+ g_{n,m} \prod_{j=m+2}^{n} \lambda_{m,j}. \tag{16.21}$$

for a normalization constant c such that $g_{n,m-1}[\mathbf{1}] = 1$. Here we officially define $\lambda_{m,j} = \lambda_{m,j}(a,b)$:

$$\lambda_{m,j} := \frac{1}{1-4\gamma_m\gamma_j\rho_{m,j}\rho_{j,m}k[a,b]_j^- k[a,b]_m^+ g_{m,j}g_{j,m}}, \; m+2 \leq j. \tag{16.22}$$

By symmetric arguments we also obtain the upward recurrence relation for any $m < n-1$:

$$g_{m,n+1} = czh[a,b]_n^- g_{m,n} \prod_{j=m}^{n-2} \lambda_{j,n}, \tag{16.23}$$

where c denotes a generic normalization constant. Each factor $\lambda_{m,n}/\lambda_{m,n}[\mathbf{1}]$ defined by (16.22) is a probability generating function for a class of paths starting and ending at the same level n (say), with probability of first step given by the turning probability (at n). Each path besides the empty path makes a positive number of consecutive down–up transitions of type "a first passage downward transition to m followed immediately by a first passage upward transition to n". See Figure 3, where a path starts at level $n = 3$ at the first marker γ_n, and makes exactly 2 down–up transitions between $n = 3$ and $m = 0$, and ends at the third marker γ_n. We obtain the same generating function if the paths instead start and end at m, with up–down transitions.

By (16.21)–(16.23) we retrieve a closed recurrence for $g_{m,n}$. Indeed, by (16.23), for $m < n-2$, we simply have $g_{m,n+1}/g_{m+1,n+1} = c_1 g_{m,n}\lambda_{m,n}/g_{m+1,n}$, for a normalization constant c_1. Hence,

$$g_{m,n+1} = c_1 g_{m,n} g_{m+1,n+1} (g_{m+1,n})^{-1} \lambda_{m,n}, \quad n-m \geq 3. \tag{16.24}$$

Similarly, by applying (16.21), $g_{n,m-1}/g_{n-1,m-1} = c_2 g_{n,m}\lambda_{m,n}/g_{n-1,m}$, for a normalization constant c_2 and $m < n-2$. Hence,

$$g_{n,m-1} = c_2 g_{n,m} g_{n-1,m-1} (g_{n-1,m})^{-1} \lambda_{m,n}, \quad n-m \geq 3. \tag{16.25}$$

Observe that the factor $\lambda_{m,n}$ of (16.22), with $m+2 < n$, appears exactly the same in both (16.21) and (16.23), and again in (16.24)–(16.25).

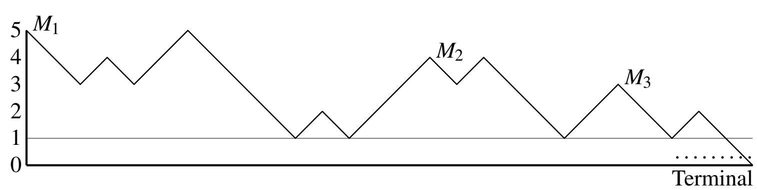

Fig. 2: Downward Transition with Future Maxima $M_1 = 5$, $M_2 = 4$, $M_3 = 3$.

We introduce some notation for the basic method to calculate (16.10), which consists of conditioning on $\{\mathbf{H} = n\}$. In the remainder of this section we assume $f \geq 3$. Let G_n denote the conditional joint probability generating function of the number of runs, short runs, and steps in an excursion given that the height is $\mathbf{H} = n$ for some $1 \leq n < N$:

$$G_n := E(r^{\mathbf{R}} y^{\mathbf{V}} z^{\mathbf{L}} | \mathbf{H} = n, \mathbf{X}_0 = 0), \quad n \geq 1. \tag{16.26}$$

In definition (16.26), the condition is that after the first step from $m=0$, the path does not return to the x-axis until it terminates, but that also, for a positive excursion, the path reaches the specified height, n, as a maximum. Now we work with positive excursions. We consider an initial sequence $U(UD)^{\ell}UU$ that brings a lattice path for the first time to level $m=3$ while never returning to level $m=0$. The joint generating function for the numbers of runs, short runs, and steps, for only the part $U(UD)^{\ell}$ of this initial sequence is simply $J_a := a(2-a)zh_a$, with h_a defined by (16.18). Now, to make a positive excursion that starts at level $m=0$ and reaches a level $n \geq 3$ for a first time, we also consider any upward path $\Gamma^{+}_{1,n}$ for $g_{1,n}$ that starts at level $m=1$ with UU. We link the initial sequence $U(UD)^{\ell}UU$ and $\Gamma^{+}_{1,n}$ together by making them overlap on the end UU of the initial sequence and beginning of $\Gamma^{+}_{1,n}$. Thus the factor of G_n corresponding to a lattice path first reaching level $n \geq 3$ is given by $J_a g_{1,n}$. The remaining factor corresponds to a downward preamble from level n followed by a downward path from level n to level 0. Hence,

$$G_n = a(2-a)zh_a g_{1,n} k[a,b]^{-}_{n} g_{n,0},\ n \geq 3,\ f \geq 3. \tag{16.27}$$

Moreover, by symmetry we have $g_{0,-n} = g_{0,n}$ for all $n \geq 2$. Hence, the joint generating function of the meander statistics is:

$$E\{r^{\mathscr{R}'_N} y^{\mathscr{V}'_N} z^{\mathscr{L}'_N}\} = a(2-a)zh_a g_{1,N},\ \ f \geq 3. \tag{16.28}$$

3.2 Formula for $\rho_{m,n}$

In this section we establish a formula for $\rho_{m,n}$ as defined by (16.16). Note that $1 - \rho_{1,N}$ is the probability of ruin for the gambler's ruin persistence model with two strata on $[0,N]$ in case $\mathbf{X}_0 = 1$. The novelty of our approach, based on induction, is unnecessary if $a=b$, since by [13] a difference equation will solve the probability of ruin in this case.

The method we use to establish a formula is based first on the future maxima construction of Section 3.1, only in (16.16) there is no condition on upward or downward paths starting UU or DD. In place of $\lambda_{m,j}$ of (16.22), here define:

$$u_{m,j} := \frac{1}{1-4\gamma_m\gamma_j\rho_{m,j}\rho_{j,m}},\ m+1 \leq j. \tag{16.29}$$

Let $m<n$. By the way we developed the formulae (16.21)–(16.23), we have

$$\text{(i)}\quad \rho_{m,n+1} = (1-\gamma_n)\rho_{m,n}\prod_{j=m}^{n-1} u_{j,n};$$

$$\text{(ii)}\quad \rho_{n,m-1} = (1-\gamma_m)\rho_{n,m}\prod_{j=m+1}^{n} u_{m,j}. \tag{16.30}$$

The factor $(1-\gamma_n)$ in (16.30)(i) gives the probability (a or b) of the last step in any one–sided first passage path from level m to level n; a similar comment applies to (16.30)(ii). See Figure 3. By the same method as shown to obtain (16.24)–(16.25), we have by (16.29)–(16.30) that

$$\begin{aligned} &\text{(i)} \quad \rho_{m,n+1} = \frac{\rho_{m,n}\rho_{m+1,n+1}}{\rho_{m+1,n}} u_{m,n}; \\ &\text{(ii)} \quad \rho_{n,m-1} = \frac{\rho_{n,m}\rho_{n-1,m-1}}{\rho_{n-1,m}} u_{m,n}. \end{aligned} \tag{16.31}$$

With the help of (16.31), we will now develop a closed recurrence for $\rho_{m,n}$. We first

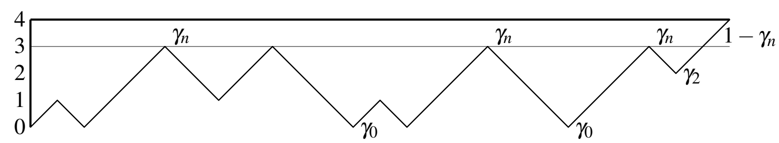

Fig. 3: Illustration of $\rho_{0,n+1} = (1-\gamma_n)\rho_{0,n}\prod_{j=0}^{n-1} u_{j,n}$ for $n=3$. For the path shown, $u_{0,n}$ has 2 down–up transitions and $u_{1,n}$ has none, while $u_{2,n}$ has 1 down–up transition.

make a definition to establish a convenient form of $\rho_{m,n}$.

Definition 16.2. Let $\rho_{m.n}$ be defined by (16.16). We define a denominator term $\Pi_{m,n}$ for $\rho_{m,n}$ as follows, with $m<n$ in all cases:

(I) (1) $\rho_{m,n} = \frac{1}{2}\frac{b/a}{\Pi_{m,n}}$, $m \le f-1$; (2) $\rho_{m,n} = \frac{1}{2}\frac{1}{\Pi_{m,n}}$, $f \le m$.

(II) (1) $\rho_{n,m} = \frac{1}{2}\frac{b/a}{\Pi_{n,m}}$, $n \le f-1$; (2) $\rho_{n,m} = \frac{1}{2}\frac{1}{\Pi_{n,m}}$, $f \le n$.

Proposition 16.1. *The terms $\Pi_{m,n}$ determined by Definition 16.2 satisfy:*

(I) *Between strata formulae:*

(1) $\Pi_{f-\ell,f+j} = j + \ell(\frac{b}{a}) - (\ell+j-1)b$, $\ell \ge 1$, $j \ge 0$;
(2) $\Pi_{f+j,f-\ell} = (j+1) + (\ell-1)(\frac{b}{a}) - (\ell+j-1)b$, $\ell \ge 1$, $j \ge 0$.

(II) *Within stratum formulae:*

(1) $\Pi_{m,m+\ell} = \Pi_{m+\ell,m} = \frac{b}{a}\{\ell-(\ell-1)a\}$, $m < m+\ell \le f-1$;
(2) $\Pi_{m,m+j} = \Pi_{m+j,m} = j-(j-1)b$, $f \le m < m+j$.

Remark 16.2. If $a=b$, we have $\Pi_{m,m+\ell} = \Pi_{m+\ell,m} = \ell-(\ell-1)a$ in all cases of Proposition 16.1, consistent with Definition 16.2 and [13, Eq. (2.4)].

Proof (of Proposition 16.1). By Remark 16.2, and Definition 16.2, the within stratum formulae (II)(1),(2) hold in general. The proof of the between strata cases proceeds by induction on $n-m$, where we assume $n>m$ throughout. Recall that the first step ε_1 of the gambler's ruin process is determined by $\pi_+ = P(\varepsilon_j = 1) = \frac{1}{2}$. We have $\rho_{m,m+1} = \rho_{m,m-1} = \frac{1}{2}$, so the case $n-m=1$ is easily checked. We next verify the cases $n-m=2$ for $\Pi_{m,n}$ and $\Pi_{n,m}$ in (I)(1),(2). We apply (16.29)–(16.30) with $u_{m,m+1} = (1-\gamma_m\gamma_{m+1})^{-1}$. Thus for all m, $\rho_{m,m+2} = \frac{1}{2}(1-\gamma_{m+1})/(1-\gamma_m\gamma_{m+1})$. In particular, by (16.20), $\rho_{f-1,f+1} = \frac{1}{2}b/(1-(1-a)(1-b)) = \frac{1}{2}(b/a)/\left(1+\frac{b}{a}-b\right)$. This gives the correct form for the denominator in (I)(1) by Definition 16.2(I)(1). We apply direct calculation to check the other between strata cases. Thus all the cases $n-m=2$ have been verified.

Assume by induction that all statements of the proposition hold for $2 \le n-m \le k$ for some $k \ge 2$. We wish to show the following induction step:

$$\begin{aligned}&\text{Both } (i): \ \Pi_{m,n+1}, \text{ and } (ii): \ \Pi_{n,m-1}, \text{ conform to statements (I)(1) and (I)(2),}\\ &\quad \text{respectively, for all } m \le f-1 \text{ and } n \ge f, \text{ with } n-m=k+1. \end{aligned} \tag{16.32}$$

There are two boundary cases, $\Pi_{f-k-1,f}$ and $\Pi_{f+k,f-1}$, that aren't covered formally by this scheme. However both of these cases actually fall under the within stratum regime. For example, in the calculation of $\rho_{f-k-1,f}$, the one-sided first passage path from $f-k-1$ to f never oscillates between levels $f-1$ and f, so the probability $\rho_{f-k-1,f}$ is governed by a single stratum design. Hence, $\rho_{f-k-1,f} = \frac{1}{2}/(k+1-ka) = \frac{1}{2}(b/a)/\left\{(k+1)\left(\frac{b}{a}\right)-kb\right\}$, consistent with (I)(1). For the other boundary case, by similar reasoning, $\rho_{f+k,f-1} = \frac{1}{2}/(k+1-kb)$, consistent with (I)(2). So the boundary cases have been resolved for all k.

We proceed with our argument for establishing (16.32). By our ranges for m and n we have $\gamma_m = 1-a$ and $\gamma_n = 1-b$ throughout. By Definition 16.2(I), we compute $u_{m,n}$ by (16.29) under (16.32) as follows.

$$u_{m,n} = \left\{1-\gamma_m\gamma_n\frac{b/a}{\Pi_{m,n}\Pi_{n,m}}\right\}^{-1} = \frac{\Pi_{m,n}\Pi_{n,m}}{\Pi_{m,n}\Pi_{n,m}-\gamma_m\gamma_n(b/a)}. \tag{16.33}$$

Now write $m = f-\ell$ and $n = f+j$ for some $\ell \ge 1$ and $j \ge 0$ with $\ell + j = k \ge 2$. By the induction hypothesis we can write $\Pi_{m,n} = j+\ell(\frac{b}{a})-(\ell+j-1)b$, and also $\Pi_{n,m} = (j+1)+(\ell-1)(\frac{b}{a})-(\ell+j-1)b$. Now, by direct calculation, we have a simple identity for the denominator of the right hand side of (16.33):

$$\Pi_{m,n}\Pi_{n,m}-\gamma_m\gamma_n(b/a) = \left\{j+1+\ell(\frac{b}{a})-(\ell+j)b\right\}\Pi_{m+1,n}, \tag{16.34}$$

where we applied the induction hypothesis for $\Pi_{m,n}$, $\Pi_{n,m}$, and $\Pi_{m+1,n}$. Now rewrite (16.31)(i) by applying Definition 16.2 and (16.33)–(16.34), as follows. We have that $\rho_{m,n+1}$ is given by:

$$\left[\frac{\frac{1}{2}b/a}{\Pi_{m,n}}\right]\left[\frac{\frac{1}{2}b/a}{\Pi_{m+1,n+1}}\right]\left[\frac{\frac{1}{2}b/a}{\Pi_{m+1,n}}\right]^{-1}\frac{\Pi_{m,n}\Pi_{n,m}}{\left\{j+1+\ell(\frac{b}{a})-(\ell+j)b\right\}\Pi_{m+1,n}}, \tag{16.35}$$

with the caveat that if $m+1=f$, then the factor b/a in the second and third factors on the left is replaced by 1. Finally we use that, by the induction hypothesis and all statements of the proposition themselves, we have $\Pi_{m+1,n+1}=\Pi_{n,m}$ for all $n>m$ under (16.32). Therefore, simply by cancellation of 3 Π–factors, (16.35) yields $\rho_{m,n+1}=\frac{\frac{1}{2}b/a}{\{j+1+\ell(\frac{b}{a})-(\ell+j)b\}}$. Thus by Definition 16.2(I)(1), the induction step (16.32) has been verified for case (*i*). The argument for the downward case (*ii*) is wholly similar to the upward case (*i*). In fact by direct calculation the relevant identity in place of (16.34) is the same except with $\Pi_{n-1,m}$, in place of $\Pi_{m+1,n}$. And in (16.35) the roles of b/a and 1 are reversed. Thus $\rho_{f+j,f-\ell-1}=\frac{1}{2}/\left\{j+1+\ell(\frac{b}{a})-(\ell+j)b\right\}$, as required. Therefore the induction step (16.32) has been verified.

3.3 The denominators $\overline{w}_{m,n}$ of $g_{m,n}$

We first consider the homogeneous case $a=b$, and establish formulae for the denominators $w_n^*(a)$ of $g_{0,n}(a,a)$ defined by (16.15), where of course $g_{m,n}(a,a)$ depends only on $|n-m|$. We will abbreviate $g_n:=g_{0,n}$ without confusion for this homogeneous case. Denote $\rho_n:=\rho_{0,n}=\frac{1}{2}/(n-(n-1)a)$, by Definition 16.2 and Proposition 16.1, and $\lambda_n:=\lambda_{m,m+n}=\{1-4(1-a)^2k_a^2\rho_n^2g_n^2\}^{-1}$, defined by (16.22). By (16.24), we have

$$g_{n+1}=c_1g_n^2g_{n-1}^{-1}\lambda_n,\ \ n\geq 3;\ \ g_3=czh_ag_2\lambda_2. \tag{16.36}$$

We now establish that, for a certain sequence of polynomials

$$\{w_n^*(a)=w_n^*(a;r,y,z),n\geq 1\},$$

with constant coefficient 1, we have

$$g_n=C_{n,a}\omega_arz^n\tau_a^{n-2}/w_n^*(a),n\geq 2;\ C_{n,a}:=a^{n-2}(n-(n-1)a)/(2-a). \tag{16.37}$$

The proposed formula (16.37) holds for $n=2$ with $w_2^*(a):=\omega_a$, since $C_{2,a}=1$. We also define $w_1^*(a):=1$. Motivated by the idea that $w_n^*(a)$ satisfies a Fibonacci recurrence, we introduce

$$x_a:=a^2z^2\tau_a^2;\ \ \beta_a:=1+z^2(a^2-(1-a)^2r^2(y^2+a^2(1-y)^2z^2)), \tag{16.38}$$

and we define

$$w_{n+1}^*(a)=\beta_aw_n^*(a)-x_aw_{n-1}^*(a),\ n\geq 2. \tag{16.39}$$

The form of (16.38) used to make the definition (16.39) may be guessed by taking account of (16.12) together with the proposed form (16.37). That is, we already have defined $w_1^*(a)$ and $w_2^*(a)$, consistent with (16.37), and we can derive g_3 via (16.36). So we will have thereby guessed $w_3^*(a)$. We can likewise predict $w_4^*(a)$. But (16.12) gives that the appropriate x_a for (16.39) is

$$x_a = \left(w_3^*(a)^2 - w_4^*(a)w_2^*(a)\right) / \left(w_2^*(a)^2 - w_3^*(a)w_1^*(a)\right).$$

Once we have x_a, we find β_a via (16.39), and we also extend the definition (16.39) to $n = 0$ by solving (16.39) backwards: $w_0^*(a) := (\beta_a - \omega_a)/x_a$. We define also the associated numerators $\{q_n^*(a)\}$ defined by the Fibonacci recurrence $q_{n+1}^*(a) = \beta_a q_n^*(a) - x_a q_{n-1}^*(a)$, $n \geq 1$, with initial conditions

$$q_0^*(a) := -(1-y)(1+y+(1-a)^2 r^2 y^2 z^2 (1-y))/\tau_a^2, \;\; q_1^*(a) := y^2. \tag{16.40}$$

By the choice of $q_1^*(a)$ we obtain the form $K_1 = r^2 z^2 q_1^*(a)/w_1^*(a)$ for (16.10). By the choice of $q_0^*(a)$ we obtain by direct computation an interlacing form $w_{n+1}^* q_n^* - w_n^* q_{n+1}^* = a^2 z^2 x_a^{n-1}$ at $n = 0$. By direct computation to check the initial conditions for Fibonacci recurrences, we have:

$$\begin{aligned} q_n^*(a) &= c_1 q_n(x_a, \beta_a) + c_2 w_n(x_a, \beta_a), \;\; c_2 := q_0^*(a), c_1 = y^2 - c_2; \\ w_n^*(a) &= c_1' q_n(x_a, \beta_a) + c_2' w_n(x_a, \beta_a), \;\; c_2' := w_0^*(a), c_1' = 1 - c_2'. \end{aligned} \tag{16.41}$$

We first verify (16.37) for $n = 3$. By (16.18) and (16.23), we have $g_3 = czh_a g_2 \lambda_2 = cz(\tau_a/\omega_a) rz^2 (1 - a^2(1-a)^2 r^2 z^4/\omega_a^2)^{-1}$, since $\rho_2 = \frac{1}{2}/(2-a)$ and $k_a = a(2-a)/\omega_a$. This yields $g_3 = C_{3,a}\omega_a rz^3 \tau_a / w_3^*(a)$, by direct computation. Now assume by induction that (16.37) holds with m in place of n for all $2 \leq m \leq n$, for some $n \geq 3$. Then by (16.36) we have $g_{n+1} = c_{n+1}[\omega_a rz^n \tau_a^{n-2}/w_n^*]^2 [\omega_a rz^{n-1} \tau_a^{n-3}/w_{n-1}^*]^{-1} \lambda_n$, where c_{n+1} incorporates both the constant c_1 of (16.36) and the factor $C_{n,a}^2/C_{n-1,a}$. By direct substitution of the induction hypothesis, and taking care to write the term g_n^2 that appears in λ_n in terms of x_a via $\tau_a^2 = a^{-2} z^{-2} x_a$, so that

$$\lambda_n = \left(1 - a^2(1-a)^2 r^2 z^4 x_a^{n-2}/w_n^*(a)^2\right)^{-1},$$

we obtain

$$g_{n+1} = c_{n+1}\omega_a rz^{n+1} \tau_a^{n-1} w_{n-1}^*(a)/\{w_n^*(a)^2 - a^2(1-a)^2 r^2 z^4 x_a^{n-2}\}. \tag{16.42}$$

To compute the denominator in this last expression, we note the following.

Lemma 16.1. *Let $w_n^*(a)$ be defined as the solution to* (16.39). *Then for all $n \geq 1$ we have:* $w_n^*(a)^2 - w_{n+1}^*(a) w_{n-1}^*(a) = a^2(1-a)^2 r^2 z^4 x_a^{n-2}$.

Proof. By the definition (16.39) and by (16.12) we have:

$$w_n^*(a)^2 - w_{n+1}^*(a) w_{n-1}^*(a) = -\beta_a^{-1} x_a^{n-1} (w_3^*(a) w_0^*(a) - w_2^*(a) w_1^*(a)). \tag{16.43}$$

By direct calculation, $w_3^*(a) w_0^*(a) - w_2^*(a) w_1^*(a) = -a^2 r^2 z^4 (1-a)^2 \beta_a / x_a$. Hence the lemma follows by substitution of this last formula into (16.43).

Up to the form of the constant $C_{n,a}$, relation (16.37) now follows by induction from (16.42) and Lemma 16.1. To verify the constant, we need only verify the claim:

$w_n^*(a)[\mathbf{1}] = a^{n-1}(n-(n-1)a)$. This is easily verified by induction, (16.39), and direct computation. Hence we have verified (16.37).

Now turn to the full model. We recursively define an array of functions $\{\overline{w}_{m,n} = \overline{w}_{m,n}(a,b)\}$ such that $\overline{w}_{m,n}$ will turn out to be the denominator polynomial with constant term 1 for the rational expression of $g_{m,n}$. We first define initial cases:

$$\begin{aligned}&\overline{w}_{m,m+2} := \omega[a,b]_m^+,\ m \geq 0;\ \ \overline{w}_{n,n-2} := \omega[a,b]_n^-,\ n-2 \geq 0;\\ &\overline{w}_{m,m+1} = \overline{w}_{m+1,m} = 1,\ m \geq 0.\end{aligned} \tag{16.44}$$

For example, if $m \leq f-2$, then $[a,b]_m^+ = (a,a)$, so $\overline{w}_{m,m+2} := \omega_a$. We require a generalization of x_a, and β_a of (16.38) to make our definition of $\{\overline{w}_{m,n}\}$ for two strata, as follows. Define

$$x(a,b) := b^2z^2\tau^2(a,b);\ \ \beta(a,b) = \beta_b - (b-a)b^2(1-b)r^2(1-y)^2z^4. \tag{16.45}$$

for β_b defined by (16.38). Here we note that $\tau(a,b)$ is symmetric in a and b, so $x(b,a) = a^2z^2\tau^2(a,b)$ for $\tau(a,b)$ defined by (16.18).

Definition 16.3. Denote $\overline{w}_{m,n} = \overline{w}_{m,n}(a,b)$.

(I) Define the upward denominator $\overline{w}_{m,n}$ for all $n-m \geq 2$ by:

(1) $\overline{w}_{m,m+\ell} := w_\ell^*(a),\ m < m+\ell \leq f;\ \overline{w}_{m,m+\ell} := w_\ell^*(b),\ f \leq m < m+\ell;$
(2) $\overline{w}_{f-\ell,f+1} := \frac{1-b}{1-a}w_{\ell+1}^*(a) + \frac{b-a}{1-a}w_\ell^*(a),\ 1 \leq \ell \leq f;$
(3) $\overline{w}_{m,f+2} := \beta(a,b)\overline{w}_{m,f+1} - x(a,b)\overline{w}_{m,f},\ m \leq f-1;$
(4) $\overline{w}_{m,f+j+1} := \beta_b\overline{w}_{m,f+j} - x_b\overline{w}_{m,f+j-1},\ m \leq f-1,\ j \geq 2.$

(II) Define the downward denominator $\overline{w}_{n,m}$ for all $n-m \geq 2$ by:

(1) $\overline{w}_{m+\ell,m} := w_\ell^*(a),\ m < m+\ell \leq f-1;\ \overline{w}_{m+\ell,m} := w_\ell^*(b),\ f-1 \leq m;$
(2) $\overline{w}_{f+j,f-2} := \frac{1-a}{1-b}w_{j+2}^*(b) + \frac{a-b}{1-b}w_{j+1}^*(b),\ 0 \leq j;$
(3) $\overline{w}_{n,f-3} := \beta(b,a)\overline{w}_{n,f-2} - x(b,a)\overline{w}_{n,f-1},\ f \leq n;$
(4) $\overline{w}_{n,f-\ell-2} := \beta_a\overline{w}_{n,f-\ell-1} - x_a\overline{w}_{n,f-\ell},\ f \leq n,\ \ell \geq 2.$

Notice that in Definition 16.3(II), we are effectively reversing the roles of a and b from (I). In case $a = b$, we simply have $\overline{w}_{m,n} = w_{|n-m|}^*(a),\ |n-m| \geq 2$. We write the first step of *crossing over the threshold of the stratum* in either upward or downward directions as a linear combination of two successive homogeneous case solutions. For the next step over the threshold we use the *mixed* parameters for x and β, and for further steps we use the appropriate homogeneous parameters for x and β. With no crossing over a stratum, the homogeneous solution is shown. Finally, Definition 16.3 and (16.44) are consistent. For example, in part (I)(2) of the definition, we find:

$$\overline{w}_{f-1,f+1} = \frac{1-b}{1-a}w_2^*(a) + \frac{b-a}{1-a}w_1^*(a) = \frac{1-b}{1-a}\omega(a,a) + \frac{b-a}{1-a} = \omega(a,b).$$

3.3.1 Interlacing identity and closed formula for $\overline{w}_{m,n}$.

To establish a formula for $g_{m,n}$, we will employ an interlacing identity for the denominators $\overline{w}_{m,n}$. Define the interlacing bracket:

$$[\overline{w}]_{m,n} := \overline{w}_{m,n}\overline{w}_{m+1,n+1} - \overline{w}_{m,n+1}\overline{w}_{m+1,n},\ m \leq n-2. \tag{16.46}$$

It actually suffices to consider only the upward direction for $[\overline{w}]_{m,n}$, since by Lemma 16.2, the natural corresponding downward definition,

$$[\overline{w}]_{n,m} := \overline{w}_{n,m}\overline{w}_{n-1,m-1} - \overline{w}_{n,m-1}\overline{w}_{n-1,m}, \quad m \leq n-2,$$

satisfies $[\overline{w}]_{n,m} = [\overline{w}]_{m,n}$.

Proposition 16.2. *The following identities hold for* $[\overline{w}]_{m,n}$*:*

(1) $[\overline{w}]_{f-\ell,f+j} = a^2r^2z^4(1-a)(1-b)x_a^{\ell-2}x(a,b)x_b^{j-1},\ \ \ell \geq 2,\ j \geq 1$;
(2) $[\overline{w}]_{f-\ell,f} = a^2r^2z^4(1-a)(1-b)x_a^{\ell-2},\ \ \ell \geq 2$;
(3) $[\overline{w}]_{f-1,f+j} = b^2r^2z^4(1-a)(1-b)x_b^{j-1},\ \ j \geq 1$;
(4) $[\overline{w}]_{m,m+\ell} = a^2r^2z^4(1-a)^2x_a^{\ell-2},\ m+\ell \leq f-1$;
(5) $[\overline{w}]_{m,m+j} = b^2r^2z^4(1-b)^2x_b^{j-2};\ f \leq m,\ \ j \geq 2$.

Proof. By Definition 16.3(I)(1) and by Lemma 16.1 we have that statements (4)–(5) of the proposition hold. Next fix $\ell \geq 2$ and notice that the case $j = 0$ in (1) is similar to the case of statement (2), the difference being $x(a,b) \neq x_b$. We will first verify (2). Thus we write, using the Definition 16.3 and (16.46), that $[\overline{w}]_{f-\ell,f}$ is given by:

$$w_\ell^*(a)\{\frac{1-b}{1-a}w_\ell^*(a) + \frac{b-a}{1-a}w_{\ell-1}^*(a)\} - \{\frac{1-b}{1-a}w_{\ell+1}^*(a) + \frac{b-a}{1-a}w_\ell^*(a)\}w_{\ell-1}^*(a). \tag{16.47}$$

The $w_\ell^*(a)w_{\ell-1}^*(a)$ terms cancel in (16.47). Thus by (16.47) and Lemma 16.1, $[\overline{w}]_{f-\ell,f} = \frac{1-b}{1-a}\left(w_\ell^*(a)^2 - w_{\ell+1}^*(a)w_{\ell-1}^*(a)\right) = a^2(1-a)(1-b)r^2z^4x_a^{\ell-2}$. Hence statement (2) is proved.

We now turn to statement (1). Fix $\ell \geq 2$ and let $j \geq 0$. Denote $[a,b]_0 = (a,b)$ and $[a,b]_j = (b,b)$ for $j \geq 1$. Thus, by Definition 16.3(I)(3)–(4) and (16.46),

$$\begin{aligned}[\overline{w}]_{f-\ell,f+j+1} &= \overline{w}_{f-\ell,f+j+1}\left\{\beta[a,b]_j\overline{w}_{f-\ell+1,f+j+1} - x[a,b]_j\overline{w}_{f-\ell+1,f+j}\right\} \\ &\quad - \left\{\beta[a,b]_j\overline{w}_{f-\ell,f+j+1} - x[a,b]_j\overline{w}_{f-\ell,f+j}\right\}\overline{w}_{f-\ell+1,f+j+1}.\end{aligned} \tag{16.48}$$

Now the terms of (16.48) involving $\beta[a,b]_j$ cancel and we obtain from (16.48) and (16.46) that

$$[\overline{w}]_{f-\ell,f+j+1} = x[a,b]_j[\overline{w}]_{f-\ell,f+j}. \tag{16.49}$$

Now put $j = 0$ in (16.49) and conclude by (2) and (16.49) that statement (1) holds for the initial case $j = 1$ for the given fixed $\ell \geq 2$. Now for the same fixed index ℓ, take statement (1) as an induction hypothesis for induction on $j \geq 1$. We have just established this induction hypothesis for $j=1$. Thus verify by (16.49) again that the induction step holds since $x[a,b]_j = x(b,b) = x_b$ for all $j \geq 1$. Thus statement (1) is proved.

Finally we turn to statement (3). We note that (16.48)–(16.49) continues to hold by Definition 16.3(I)(3) with $\ell = 1$ as long as $j \geq 1$. Now we compute by (16.44)–(16.45), Definition 16.3(I)(3), and the interlacing bracket definition (16.46) that, since by (16.44), $\overline{w}_{f-1,f+1} = \omega(a,b)$, while by Definition 16.3, $\overline{w}_{f,f+2} = w_2^*(b) = \omega(b,b)$,

$$\begin{aligned}[\overline{w}]_{f-1,f+1} &= \omega(a,b)\omega(b,b) - (\beta(a,b)\omega(a,b) - x(a,b)\cdot 1)\cdot 1 \\ &= \omega(a,b)\left(\omega(b,b) - \beta(a,b)\right) + x(a,b) = b^2(1-a)(1-b)r^2z^4,\end{aligned} \tag{16.50}$$

where at the last step we make a direct calculation based on the definitions in (16.18) and (16.45). Now take statement (3) as an induction hypothesis for induction on $j \geq 1$. By (16.50) have established this induction hypothesis for $j=1$. Thus verify by (16.49) with $\ell = 1$ and $j \geq 1$ that the induction step holds since $x[a,b]_j = x(b,b) = x_b$ for all $j \geq 1$. So, statement (3) is proved.

We turn to the task of obtaining a closed formula for $\overline{w}_{m,n}$. By Definition 16.3(I)(4), given $m = f - \ell < f$, $\overline{w}_{m,f+1}$ and $\overline{w}_{m,f+2}$ form the initial conditions for a recurrence $\overline{w}_{m,f+j+1} := \beta_b \overline{w}_{m,f+j} - x_b \overline{w}_{m,f+j-1}$, $j \geq 2$. Put $m = f - \ell$ for some $\ell \geq 1$. We denote the vector of these upward initial conditions across the stratum threshold by the 2×1 vector $\mathbf{W}(\ell)$. Then we define a 2×2 matrix $Q(b)$, and for each $\ell < f$, a 2×1 vector $\mathbf{d} = \mathbf{d}(\ell)$ by

$$Q(b) := \begin{bmatrix} q_1^*(b) & w_1^*(b) \\ q_2^*(b) & w_2^*(b) \end{bmatrix}, \; \mathbf{W}(\ell) := \begin{bmatrix} \overline{w}_{f-\ell,f+1} \\ \overline{w}_{f-\ell,f+2} \end{bmatrix} = Q(b)\mathbf{d};\; \mathbf{d} := \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix}. \tag{16.51}$$

By Definitions 16.3(I)(1–3), we can write each term of the right side of the recurrence of (I)(3) using (I)(1–2) in terms of $w_\ell^*(a)$ and $w_{\ell+1}^*(a)$ as follows: $\overline{w}_{f-\ell,f+1} = \frac{1-b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{1-a} w_\ell^*(a)$ and $\overline{w}_{f-\ell,f} = w_\ell^*(a)$, so

$$\overline{w}_{f-\ell,f+2} = \beta(a,b)\left(\frac{1-b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{1-a} w_\ell^*(a)\right) - x(a,b) w_\ell^*(a).$$

We combine terms with the notation $\kappa(a,b) := \left(\frac{b-a}{1-a}\right)\beta(a,b) - x(a,b)$. Thus

$$\mathbf{W}(\ell) = B \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}; \quad B = \begin{bmatrix} \frac{b-a}{1-a} & \frac{1-b}{1-a} \\ \kappa(a,b) & \frac{1-b}{1-a}\beta(a,b) \end{bmatrix}. \tag{16.52}$$

By equating the two expressions for the vector $\mathbf{W}(\ell)$ in (16.51) and (16.52), we recover

$$\mathbf{d}(\ell) = \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix} = M \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}; \; M := Q(b)^{-1} B. \tag{16.53}$$

Here it is clear that the entries of the matrix $M = (\mu_{i,j})$, with $\mu_{i,j} = \mu_{i,j}(a,b)$ $1 \leq i,j \leq 2$, do not depend on ℓ. We note by direct calculation from (16.51) that

$\det(Q(b)) = -b^2z^2$, so we have a straightforward formula for M via (16.51) and (16.53).

Proposition 16.3. *Let* $d_1(\ell)$ *and* $d_2(\ell)$ *be defined by* (16.51)–(16.53). *Then*

$$\overline{w}_{f-\ell,f+j} = d_1(\ell)q_j^*(b) + d_2(\ell)w_j^*(b),\ \ell \geq 1,\ j \geq 1. \tag{16.54}$$

Proof. Fix $\ell \geq 1$. By Definition 16.3(I)(4), for all $j \geq 2$ it holds that $\overline{w}_{f-\ell,f+j+1} = \beta_b\overline{w}_{f-\ell,f+j} - x_b\overline{w}_{f-\ell,f+j-1}$. But if we denote the right side of (16.54) by v_j, then also $v_{j+1} = \beta_b v_j - x_b v_{j-1},\ j \geq 2$, because by construction each of $\{q_j^*(b)\}$ and $\{w_j^*(b)\}$ satisfy the same two term recurrence, and the coefficients $d_1(\ell)$ and $d_2(\ell)$ in (16.54) are independent of j. Also by definition (16.51), for any given $\ell \geq 1$, (16.54) holds for $j = 1$ and $j = 2$, that is, $v_j = \overline{w}_{f-\ell,f+j},\ j = 1,2$. Hence we have $v_j = \overline{w}_{f-\ell,f+j}$ for all $j \geq 1$. Since ℓ was arbitrary the proof is complete.

Lemma 16.2. *For all* $1 \leq m < n$, *there holds:* $\overline{w}_{m,n} = \overline{w}_{n-1,m-1}$.

Proof. Notice that the lemma holds in the initial cases $n - m = 1,2$ by (16.44). Also, if $f \leq m < n$ or $1 \leq m < n \leq f$ then the statement holds by (I)(1) and (II)(1) in Definition 16.3. So consider now $\overline{w}_{f-\ell,f+j}$ for $1 \leq \ell < f$ and $j \geq 1$. Our method is to prove that the statement: $(\mathrm{H})_{\ell,j}$ $\overline{w}_{f-\ell,f+j} = \overline{w}_{f+j-1,f-\ell-1}$, holds for both the initial cases $\ell = 1$ and $\ell = 2$, and all $j \geq 1$.

We first establish $(\mathrm{H})_{\ell,j}$ for $\ell = 1$ and all $j \geq 1$. On the one hand, write $\overline{w}_{f-1,f+j}$ by (16.54) with $\ell = 1$, and on the other hand, write $\overline{w}_{f+j-1,f-2}$ by Definition 16.3(II)(2), as follows.

$$\overline{w}_{f-1,f+j} = d_1(1)q_j^*(b) + d_2(1)w_j^*(b); \overline{w}_{f+j-1,f-2} = \frac{1-a}{1-b}w_{j+1}^*(b) + \frac{a-b}{1-b}w_j^*(b). \tag{16.55}$$

By (16.51), (16.53) and direct calculation, we have that $d_1(1) = \mu_{1,1}w_1^*(a) + \mu_{1,2}w_2^*(a) = -(1-a)(1-b)r^2z^2$, and $d_2(1) = \mu_{2,1}w_1^*(a) + \mu_{2,2}w_2^*(a) = 1$. Therefore, by substitution into (16.55), we find that the two expressions in (16.55) are equal if and only if

$$-(1-b)^2r^2z^2q_j^*(b) = w_{j+1}^*(b) - w_j^*(b). \tag{16.56}$$

By direct computation we check that (16.56) is true at both $j = 1$ and $j = 2$. Thus since $\{q_j^*(b)\}$ and $\{w_j^*(b)\}$ each satisfy the same Fibonacci recurrence, (16.56) holds for all $j \geq 1$.

Next we establish that $(\mathrm{H})_{\ell,j}$ holds with $\ell = 2$ and all $j \geq 1$. Write $\overline{w}_{f-2,f+j}$ by (16.54) with $\ell = 2$, and write $\overline{w}_{f+j-1,f-3}$ by Definition 16.3(II)(3), as follows.

$$\begin{aligned}
&\text{(i)} \quad \overline{w}_{f-2,f+j} = d_1(2)q_j^*(b) + d_2(2)w_j^*(b);\\
&\text{(ii)} \quad \overline{w}_{f+j-1,f-3} = \beta(b,a)\overline{w}_{f+j-1,f-2} - x(b,a)\overline{w}_{f+j-1,f-1},
\end{aligned} \tag{16.57}$$

with $\overline{w}_{f+j-1,f-2} = \frac{1-a}{1-b}w_{j+1}^*(b) + \frac{a-b}{1-b}w_j^*(b)$; $\overline{w}_{f+j-1,f-1} = w_j^*(b)$. By (16.51) and (16.53) we directly verify that $d_1(2) = -(1-a)(1-b)r^2z^2\beta(b,a)$; $d_2(2) =$

$\beta(b,a) - x(b,a)$. To verify that the expressions (i) and (ii) in (16.57) are equal, we substitute $d_1(2)$ and $d_2(2)$, and obtain, after a little algebra in which $x(b,a)x_j^*(b)$ cancels on the two sides, the condition

$$-(1-b)^2 r^2 z^2 \beta(b,a) q_j^*(b) = \beta(b,a)\left(w_{j+1}^*(b) - w_j^*(b)\right), \quad \text{for all } j \geq 1.$$

This is obviously equivalent to the condition (16.56). Hence the two expressions in (16.57) are equal for all $j \geq 1$, so $(\mathrm{H})_{\ell,j}$ holds also at $\ell = 2$ for all $j \geq 1$.

Finally, fix any $j \geq 1$. We appeal to (16.53) and (16.54) and to Definition 16.3(II)(4), to obtain, for any $\ell \geq 3$,

$$\begin{aligned} \overline{w}_{f-\ell,f+j} &= \left(\mu_{1,1} w_\ell^*(a) + \mu_{1,2} w_{\ell+1}^*(a)\right) q_j^*(b) \\ &\quad + \left(\mu_{2,1} w_\ell^*(a) + \mu_{2,2} w_{\ell+1}^*(a)\right) w_j^*(b); \\ \overline{w}_{f+j-1,f-\ell-1} &= \beta_a \overline{w}_{f+j-1,f-\ell} - x_a \overline{w}_{f+j-1,f-\ell+1}. \end{aligned} \tag{16.58}$$

For any $\ell \geq 1$, write $u_\ell := \overline{w}_{f-\ell,f+j}$ and $v_\ell := \overline{w}_{f+j-1,f-\ell-1}$ for the two lines of (16.58). Since u_ℓ is a linear combination of two successive terms of the sequence $\{w_\ell^*(a)\}$, it follows that,$\{u_\ell\}$ itself satisfies the recursion $u_{\ell+1} = \beta_a u_\ell - x_a u_{\ell-1}, \ell \geq 2$. But also $\{v_\ell\}$ satisfies the same recurrence. Moreover, we proved that $(\mathrm{H})_{\ell,j}$ holds for $\ell = 1$ and $\ell = 2$, so we have $u_1 = v_1$, and $u_2 = v_2$. Therefore we have $u_\ell = v_\ell$ for all $\ell \geq 1$. Thus by (16.58), $(\mathrm{H})_{\ell,j}$ is proved for all $\ell \geq 1$. Since $j \geq 1$ was arbitrary, $(\mathrm{H})_{\ell,j}$ is true for all $\ell, j \geq 1$.

Lemma 16.3. *The following identities hold.*

(1) $\overline{w}_{f-\ell,f+j}[\mathbf{1}] = a^\ell b^{j-1} \Pi_{f-\ell,f+j}$, *for all* $\ell \geq 1,\ j \geq 1$.
(2) $\overline{w}_{f+j,f-\ell}[\mathbf{1}] = a^{\ell-1} b^j \Pi_{f+j,f-\ell}$, *for all* $\ell \geq 2,\ j \geq 0$.
(3) $q_\ell^*(a)[\mathbf{1}] = \ell a^{\ell-1}, \quad w_\ell^*(a)[\mathbf{1}] = a^{\ell-1}(\ell - (\ell-1)a)$*; for all* $\ell \geq 1$.

Proof. At $(r,y,z) = \mathbf{1}$ we have $\beta_a = 2a$ and $x_a = a^2$. Thus $\alpha = 0$ in (16.13). Therefore by (16.13), $q_\ell^*(a)[\mathbf{1}] = \lim_{\alpha\to 0} \frac{2^{-\ell}}{\alpha}\{(2a+\alpha)^\ell - (2a-\alpha)^\ell\} = \ell a^{\ell-1}$. Thus, by the second formula of (16.13), we obtain $w_\ell^*(a)[\mathbf{1}]$ by $x_a[\mathbf{1}] = a^2$, so (3) is proved. Now apply (16.54), also at $(r,y,z) = \mathbf{1}$. By (16.53) and direct calculation, $d_1(\ell)[\mathbf{1}] = -(1-a)(1-b)\ell a^{\ell-1}$, and $d_2(\ell)[\mathbf{1}] = a^{\ell-1}[\ell - (\ell-1)a]$. Now plug in $q_j^*(b)[\mathbf{1}]$ and $w_j^*(b)[\mathbf{1}]$ from (3), into (16.54) to obtain formula (1) from Proposition 16.3 after direct simplification. The proof of (2) follows from (1) and Lemma 16.2, in view of Definition 16.2.

3.4 Closed formula for $g_{m,n}$

Proposition 16.4. *We have the following formulae for* $g_{m,n}$.

(I) *The formulae for upward between–strata cases,* $j \geq 1$ *and* $\ell \geq 2$*:*

(1) $g_{f-\ell,f+j} = \frac{\omega(a,a)}{2-a} rz^{j+\ell}\tau(a,b)[a\tau(a,a)]^{\ell-2}[b\tau(b,b)]^{j-1} \times \left(a\Pi_{f-\ell,f+j}/\overline{w}_{f-\ell,f+j}\right),$

(2) $g_{f-1,f+j} = \frac{\omega(a,b)}{a+b-ab} rz^{j+1}[b\tau(b,b)]^{j-1}\left(a\Pi_{f-1,f+j}/\overline{w}_{f-1,f+j}\right);$

(II) *The formulae for downward between–strata cases,* $j \geq 1$ *and* $\ell \geq 2$*:*

(1) $g_{f+j,f-\ell} = \frac{\omega(b,b)}{2-b} rz^{j+\ell}\tau(a,b)[a\tau(a,a)]^{\ell-2}[b\tau(b,b)]^{j-1} \times \left(a\Pi_{f+j,f-\ell}/\overline{w}_{f+j,f-\ell}\right),$

(2) $g_{f,f-\ell} = \frac{\omega(a,b)}{a+b-ab} rz^{\ell}[a\tau(a,a)]^{\ell-2}\left(a\Pi_{f,f-\ell}/\overline{w}_{f,f-\ell}\right);$

(III) *The formulae for within stratum cases:*

(1) $g_{m,m+\ell} = g_{m+\ell,m} = \frac{\omega(a,a)}{2-a} rz^{\ell}[a\tau(a,a)]^{\ell-2}\left(\frac{a}{b}\Pi_{m,m+\ell}/w^*_{\ell}(a)\right),$ $m < m+\ell \leq f-1$*;*

(a) $g_{f-\ell,f} = \frac{\omega(a,a)}{2-a} rz^{\ell}[a\tau(a,a)]^{\ell-2}\left(\frac{a}{b}\Pi_{f-\ell,f}/w^*_{\ell}(a)\right),\ \ell \geq 1$*;*

(2) $g_{m,m+j} = g_{m+j,m} = \frac{\omega(b,b)}{2-b} rz^{j}[b\tau(b,b)]^{j-2}\left(\Pi_{m,m+j}/w^*_{j}(b)\right),$ $f \leq m < m+j$*;*

(a) $g_{f+j,f-1} = \frac{\omega(b,b)}{2-b} rz^{j+1}[b\tau(b,b)]^{j-1}\left(\Pi_{f+j,f-1}/w^*_{j+1}(b)\right),\ j \geq 1.$

Furthermore, the following identity holds for all $n \geq m+2$*, where* $\lambda_{m,n}$ *is defined by* (16.22).

$$\lambda_{m,n} = \frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}{\overline{w}_{m,n+1}\overline{w}_{m+1,n}}. \tag{16.59}$$

Remark 16.3. Since $\tau(a,b)[\mathbf{1}] = 1$ for all a and b, one easily checks by Definition 16.2 and Lemma 16.3 that the formulae of Proposition 16.4 all yield the evaluation $g_{m,n}[\mathbf{1}] = 1$. The factor $\Pi_{m,n}$ appears in Lemma 16.3 the same as it does in the statements (I)–(II), so these factors cancel at **1**.

Remark 16.4. All formulae in (III) hold by (16.37) for the homogeneous case. For example, in the statement (III)(1), we have $\frac{a}{b}\Pi_{m,m+\ell} = \ell - (\ell-1)a$, so there is no dependence on b.

Proof (of Proposition 16.4). Recall by (16.17) that $g_{m,m+2} = g_{m+2,m} = rz^2$. One can easily check by Definitions 16.2 and (16.44) that in each of (I)(1) with $j = 1$, and (II)(2) with $\ell = 2$, the formulae reduce to rz^2. By (16.24)–(16.25), we must calculate a term $\lambda_{m,n}$ defined by (16.22). The term $\lambda_{m,n}$ is the same in both (16.24) and (16.25), so we only consider $m < n$ in (16.22). The structure of the proof is to first establish (16.59) for $n = m+2$ and to establish the initial cases $n - m = 3$ of the statements (I)–(II) of the proposition. Following this, an induction step will be established for all cases at once, wherein an inductive step for (16.59) shall be the main stepping stone of the proof.

Thus consider first $n := m+2$ in (16.22). We consider 4 cases: (i) $n \leq f-1$; (ii) $m = f-2,\ n = f$; (iii) $m = f-1,\ n = f+1$; (iv) $m \geq f$. We verify by (16.18)–(16.20), Definition 16.2, Proposition 16.1, Proposition 16.2, and direct calculation, that in all cases (i)–(iv), $\lambda_{m,n} = \frac{\omega[a,b]^+_m \omega[a,b]^+_{m+1}}{\omega[a,b]^+_m \omega[a,b]^+_{m+1} - [\overline{w}]_{m,n}}$. Verification of this initial identity by direct calculation suffices for (16.59), since for the numerator we have by (16.44) that $\overline{w}_{m,n} = \omega[a,b]^+_m$ and $\overline{w}_{m+1,n+1} = \omega[a,b]^+_{m+1}$, and since for the denominator we have by Definition (16.46) that $\overline{w}_{m,n}\overline{w}_{m+1,n+1} - [\overline{w}]_{m,n} = \overline{w}_{m+1,n}\overline{w}_{m,n+1}$.

We turn to the initial conditions for (I)–(II). There are again four cases to consider. We conform with the notation of (16.21) and (16.23). For the upward cases we write the lower index m and the upper index $m+3$. For the downward cases we write the upper index $m+2$ and the lower index $m-1$. The cases are (I.1) $m = f-2$, $m+3 = f+1$; (I.2) $m = f-1$, $m+3 = f+2$; (II.1) $m+2 = f+1$, $m-1 = f-2$; (II.2) $m+2 = f$, $m-1 = f-3$. We use direct calculation of $g_{m,m+3}$ or $g_{m+2,m-1}$ for the upward and downward cases, respectively. Besides the formulae (16.21) and (16.23), we use $\lambda_{m,m+2}$ given by (16.59), where $\overline{w}_{m+1,m+2} = 1$ by definition (16.44). Since the denominator of $\lambda_{m,m+2}$ in each case is $\overline{w}_{m,m+3} = \overline{w}_{m+2,m-1}$ by Lemma 16.2, we compute $p_{m,m+3} := (1/c)g_{m,m+3}\overline{w}_{m,m+3}$ and $p_{m+2,m-1} := (1/c)g_{m+2,m-1}\overline{w}_{m,m+3}$ in the upwards and downwards cases respectively. Schematically, since $g[\mathbf{1}] = 1$, we have $p/p[\mathbf{1}] = g\overline{w}/\overline{w}[\mathbf{1}] = \mathit{numerator}/\overline{w}[\mathbf{1}]$, where *numerator* stands for the stated formula without the denominator $\overline{w}$. By cancellation of the Π–factors as in Remark 16.3, we match $p/p[\mathbf{1}]$ with $(\mathit{numerator}/\Pi)/(\overline{w}[\mathbf{1}]/\Pi)$ for verification by direct calculation.

We now proceed by induction on all cases of the proposition at once, where we assume that all statements hold for $g_{m,n}$ and $g_{n,m}$ with $2 \le n-m \le k$, for some $k \ge 3$. By the above we have established all the initial cases, $k = 3$, for this hypothesis; as noted earlier, the case $n-m = 2$ is trivial. We now apply the formulae (16.24) and (16.25) to establish an induction step in each of the upward (I)(1),(2) and downward (II)(1),(2) cases respectively. We are allowed to use any of the statements of (III) by Remark 16.4. Notice that for the range of indices we must now consider, in all cases $n \ge f$ and $m \le f-1$, so by (16.20), $\gamma_m\gamma_n = (1-a)(1-b)$.

Consider first (I)(1). Let first (i) $n+1 = m+k+1$, for some $m \le f-2$ and $n \ge f+1$; there is another subcase (ii) $n = f$, that we handle as a special case by direct calculation below. We rewrite (16.24) for easy reference:

$$g_{m,n+1} = c_1 g_{m,n} g_{m+1,n+1} (g_{m+1,n})^{-1} \lambda_{m,n}.$$

In the definition (16.22) we have by (16.18)–(16.20) that $k[a,b]^+_m = k(a,a) = \frac{a(2-a)}{\omega(a,a)}$, $k[a,b]^-_n = k(b,b) = \frac{b(2-b)}{\omega(b,b)}$. Also, by Definition 16.2 and Proposition 16.1, $4\rho_{m,n}\rho_{n,m} = \frac{(b/a)}{\Pi_{m,n}\Pi_{n,m}} = \frac{ab}{(a\Pi_{m,n})(a\Pi_{n,m})}$. Therefore by (16.22) and the induction hypothesis (I)(1), for $g_{m,n}$, and (II)(1), for $g_{n,m}$, the expression (†) $1 - 1/\lambda_{m.n}$, is written as

$$\frac{\gamma_m\gamma_n ab}{(a\Pi_{m,n})(a\Pi_{n,m})} \frac{ab(2-a)(2-b)}{\omega(a,a)\omega(b,b)} g_{m,n}g_{n,m} = \frac{\gamma_m\gamma_n a^2 r^2 z^{2j+2\ell} b^2 \tau^2(a,b)[a^2\tau_a^2]^{\ell-2}[b^2\tau_b^2]^{j-1}}{\overline{w}_{m,n}\overline{w}_{n,m}}. \quad (16.60)$$

Now apply (16.38) and (16.45) to write x_a, $x(a,b) = b^2z^2\tau^2(a,b)$, and x_b, using all but 4 powers of z. So the numerator of the right member of (16.60) is simply the interlacing bracket $[\overline{w}]_{m,n}$ of Proposition 16.2.(1). Thus, after writing $\overline{w}_{n,m} = \overline{w}_{m+1,n+1}$ by Lemma 16.2, and applying the bracket definition (16.46), we have established that $(\dagger)$ is given by $\frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1} - \overline{w}_{m,n+1}\overline{w}_{m+1,n}}{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}$, so (16.59) holds. Finally, apply (16.24) and the induction hypothesis (I)(1) and (16.59). Since the lower index $m+1$ is the same in both the numerator and denominator of the ratio $g_{m+1,n+1}/g_{m+1,n}$, we obtain, by (I)(1) for $m+1 \leq f-2$, or by (I)(2) for $m+1 = f-1$, that $g_{m+1,n+1}/g_{m+1,n} = cz[b\tau(b,b)]\overline{w}_{m+1,n}/\overline{w}_{m+1,n+1}$. Thus,

$$g_{m,n+1} = c_1 \frac{g_{m,n}g_{m+1,n+1}}{g_{m+1,n}} \frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}{\overline{w}_{m,n+1}\overline{w}_{m+1,n}} = cz[b\tau(b,b)]\left(g_{m,n}\overline{w}_{m,n}\right)/\overline{w}_{m,n+1}.$$

Hence by plugging in the numerator $p_{m,n} := g_{m,n}\overline{w}_{m,n}$ (ignoring the constants) from the induction hypothesis for (I)(1), the induction step for (I)(1)(i), including (16.59), is complete by Remark 16.3.

To recapitulate, in general, there are two steps, where for the upward and downward cases we conform to the recurrences (16.24) and (16.25), respectively.

1. Establish (16.59) by showing that the numerator in the analogue of the right hand member of (16.60) gives a bracket $[\overline{w}_{m,n}]$ ($= [\overline{w}_{n,m}]$) from Proposition 16.2, for the parameters m,n of $\lambda_{m,n}$.
2. Establish that when the induction hypothesis is applied, the condition

$$\text{(u)} \quad \frac{p_{m,n}p_{m+1,n+1}}{p_{m,n}[\mathbf{1}]p_{m+1,n+1}[\mathbf{1}]} - \frac{p_{m+1,n}p_{m,n+1}}{p_{m+1,n}[\mathbf{1}]p_{m,n+1}[\mathbf{1}]} = 0,$$

is verified for (I), and condition

$$\text{(d)} \quad \frac{p_{n,m}p_{n-1,m-1}}{p_{n,m}[\mathbf{1}]p_{n-1,m-1}[\mathbf{1}]} - \frac{p_{n-1,m}p_{n,m-1}}{p_{n-1,m}[\mathbf{1}]p_{n,m-1}[\mathbf{1}]} = 0,$$

is verified for (II).

For all the remaining cases of the induction steps in (I)–(II), including the subcase (I)(1)(ii), we proceed by direct calculation to check the details of the these 2 Steps. In Step 1 it is implicit that the factors of $\omega(a,b)$ that occur variously by substitution from factors $k(a,b)$ in the formula for $\lambda_{m,n}$, and also from the numerators of $g_{m,n}$ and $g_{n,m}$, cancel one another in every case. This is borne out in the direct calculations, where the pattern of substitutions from the induction hypothesis is shown. By Remark 16.3, conditions (u)–(d) are equivalent to showing, for the ratio $\frac{p_{m+1,n+1}}{p_{m+1,n}} = cz\tau$, in the upward case, or $\frac{p_{n-1,m-1}}{p_{n-1,m}} = cz\tau$, in the downward case, that the factor of $z\tau$ completes the form of the numerator $p_{m,n+1}$ [respectively $p_{n,m-1}$] as one extra factor of the numerator form $p_{m,n}$ [respectively $p_{n,m}$]. Here the factor τ depends on subcases; it is $\tau(a,b)$ in subcases (I)(1)(ii), and in (II)(1)(ii): $n \geq f+2,\ m = f-1$. We show the pattern of substitutions for (u)–(d) in the direct calculations [15].

3.5 Generating function of the excursion statistics

We derive a closed formula for $K_N(a,b)$ of (16.10). Recall by (16.41) that $\{q_n^*(a)\}$ and $\{w_n^*(a)\}$ share a common Fibonacci recurrence: $v_{n+1} = \beta_a v_n - x_a v_{n-1}$, $n \geq 1$. We extend the $\{q_n^*(a)\}$ from the homogeneous model to the full model as $\{\overline{q}_n\}$, analogous to $\{\overline{w}_{0,n}\}$ of Definition 16.3, except with $\overline{q}_n$ we start the stratum crossing at index $n = f$ rather than $n = f+1$.

Definition 16.4. Define $\overline{q}_n = \overline{q}_n$ for all $n \geq 1$ by:

(1) $\overline{q}_n := q_n^*(a)$, $1 \leq n < f$;
(2) $\overline{q}_f := \frac{1-b}{1-a} q_f^*(a) + \frac{b-a}{1-a} q_{f-1}^*(a)$;
(3) $\overline{q}_{f+1} := \beta(a,b)\overline{q}_f - x(a,b)\overline{q}_{f-1}$;
(4) $\overline{q}_{f+j+1} := \beta_b \overline{q}_{f+j} - x_b \overline{q}_{f+j-1}$, $j \geq 1$.

Denote the single indexed bracket (cf. Casorati determinant, see [11])

$$[\overline{w}, \overline{q}]_n := \overline{w}_{n,0}\overline{q}_{n+1} - \overline{q}_n \overline{w}_{n+1,0}, \; n \geq 1; \tag{16.61}$$

We note that the homogeneous case of (16.61) is written as

$$[w^*(a), q^*(a)]_n := w_n^*(a) q_{n+1}^*(a) - q_n^*(a) w_{n+1}^*(a), \; n \geq 1.$$

Lemma 16.4. *The following identities hold:*

(1) $[w^*(a), q^*(a)]_n = a^2 z^2 x_a^{n-1}$, $n \geq 1$*;*
(2) $[\overline{w}, \overline{q}]_{f-1} = \frac{1-b}{1-a}[w^*(a), q^*(a)]_{f-1} = \frac{1-b}{1-a} a^2 z^2 x_a^{f-2}$*;*
(3) $[\overline{w}, \overline{q}]_{f+j-1} = \frac{1-b}{1-a} a^2 z^2 x_a^{f-2} x(a,b) x_b^{j-1}$*, for all* $j \geq 1$.

Before we can prove Lemma 16.4 we write a formula for $\overline{q}_n$ as follows.

Lemma 16.5. *Let* $M := Q(b)^{-1}B$*, for* $Q(b)$ *defined by* (16.51) *and* B *defined by* (16.52)*. Then,*

$$\overline{q}_{f+j-1} = \left[\, q_j^*(b)\; w_j^*(b) \,\right] M \begin{bmatrix} q_{f-1}^*(a) \\ q_f^*(a) \end{bmatrix}, \quad \textit{for all } j \geq 1. \tag{16.62}$$

Proof. The proof is almost the same as the proof of Proposition 16.3. Define $Q(b)$ as before in (16.51), but write now a revision $\mathbf{W}_q(f)$ of $\mathbf{W}(\ell)$, and write $\mathbf{W}_q(f)$ in two ways as follows.

$$\mathbf{W}_q(f) := \begin{bmatrix} \overline{q}_f \\ \overline{q}_{f+1} \end{bmatrix} = Q(b)\mathbf{d}_q(f); \quad \mathbf{W}_q(f) = B \begin{bmatrix} q_{f-1}^*(a) \\ q_f^*(a) \end{bmatrix}. \tag{16.63}$$

We have B given by (16.52), because by Definitions 16.3 and 16.4 the equations that define B in (16.63) are the same as those defining B in (16.52). By equating the two expressions for the vector $\mathbf{W}_q(f)$ in (16.63), we recover $\mathbf{d}_q(f) = M \begin{bmatrix} q_{f-1}^*(a) \\ q_f^*(a) \end{bmatrix}$.

Now the formula (16.62) follows because by (16.63) and the definition of $Q(b)$ we have established the formula for $j = 1,2$. Therefore by the fact that either side of (16.62) satisfies the same recurrence $v_{j+1} = \beta_b v_j - x_b v_{j-1}$, $j \geq 2$, we have that both sides are equal as stated.

Proof (of Lemma 16.4). We first prove statement (1). By the simple fact that $\{q_n^*(a)\}$ and $\{w_n^*(a)\}$ satisfy the same Fibonacci recurrence, we have:

$$[w^*(a), q^*(a)]_n = w_n^*(\beta_a q_n^* - x_a q_{n-1}^*) - q_n^*(\beta_a w_n^* - x_a w_{n-1}^*) = x_a [w^*, q^*]_{n-1}$$

holds for all $n \geq 1$, where we suppressed the a in q_n^* and w_n^*. By direct calculation from (16.38)–(16.40) and (16.61), $[w^*(a), q^*(a)]_0 = w_0^*(a) q_1^*(a) - q_0^*(a) w_1^*(a) = a^2 z^2 / x_a$. Therefore, since we may iterate the one–step recursion for $[w^*(a), q^*(a)]_n$, we obtain statement (1) of the lemma.

Next, by Definitions 16.3 and 16.4,

$$\begin{aligned}
[\overline{w}, \overline{q}]_{f-1} &= \overline{w}_{1,f} \overline{q}_f - \overline{q}_{f-1} \overline{w}_{1,f+1} \\
&= w_{f-1}^* \left(\frac{1-b}{1-a} q_f^* + \frac{b-a}{1-a} q_{f-1}^* \right) - q_{f-1}^* \left(\frac{1-b}{1-a} w_f^* + \frac{b-a}{1-a} w_{f-1}^* \right) \\
&= \frac{1-b}{1-a} [w^*, q^*]_{f-1}.
\end{aligned}$$

Therefore statement (2) of the lemma follows by statement (1).

Finally, note that by Lemma 16.2 we have $\overline{w}_{n,0} = \overline{w}_{1,n+1}$. Further by (16.53) and Proposition 16.3 with $\ell = f - 1$, we may write

$$\overline{w}_{1,f+j} = \begin{bmatrix} q_j^*(b) \ w_j^*(b) \end{bmatrix} M \begin{bmatrix} w_{f-1}^*(a) \\ w_f^*(a) \end{bmatrix}. \tag{16.64}$$

Now write $n = f + j - 1$ for some $j \geq 1$. Also for simplicity abbreviate $q_0 = q_{f-1}^*(a)$, $q_1 = q_f^*(a)$, $w_0 = w_{f-1}^*(a)$, $w_1 = w_f^*(a)$, $u = q_j^*(b)$, $v = w_j^*(b)$, $U = q_{j+1}^*(b)$, $V = w_{j+1}^*(b)$. By (16.61), Lemma 16.5, and (16.64),

$$[\overline{w}, \overline{q}]_n = \begin{bmatrix} u \ v \end{bmatrix} M \left\{ \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \begin{bmatrix} U \ V \end{bmatrix} M \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} - \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \begin{bmatrix} U \ V \end{bmatrix} M \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\}.$$

Denote $M = (\mu_{i,j})$, and calculate the expression under the curly brackets as follows: $(w_0 q_1 - q_0 w_1) \begin{bmatrix} \mu_{1,2} & \mu_{2,2} \\ -\mu_{1,1} & -\mu_{2,1} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$. Therefore, after multiplying through by $\begin{bmatrix} u \ v \end{bmatrix} M$, we obtain:
$[\overline{w}, \overline{q}]_n = (w_0 q_1 - q_0 w_1) \begin{bmatrix} u \ v \end{bmatrix} \begin{bmatrix} 0 & \det(M) \\ -\det(M) & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$. Putting back our variables we thus have

$$[\overline{w}, \overline{q}]_n = [w^*(a), q^*(a)]_{f-1} (-\det(M)) [w^*(b), q^*(b)]_j, \text{ valid for all } j \geq 1.$$

Also, by direct calculation, $\det(M) = -\frac{1-b}{1-a}\tau(a,b)^2$. Thus by (16.45) and statement (1), the statement (3) of the lemma is proved.

Theorem 16.3. *The conditional generating function* (16.10) *has the following formula.*

$$K_N(a,b) = C_{N,a,b}\frac{r^2z^2\overline{q}_N}{\overline{w}_{1,N+1}},\ N \geq 1,$$

where $\overline{q}_N$ *and* $\overline{w}_{1,N+1}$ *are given by Lemma 16.5 and* (16.64) *respectively, each with* $j = N-f+1$*, and where* $C_{N,a,b} = (1-a)\frac{(N-f)a+(f-1)b-(N-1)ab}{(N-f)a+(f-1)b-Nab}$.

In the homogeneous case $a = b$, Theorem 16.3 takes the following form.

Proposition 16.5. *Suppose* $a = b$*. Then* $K_N(a)$ *of* (16.10) *has the following formula.* $K_N(a) = \frac{1}{N}(N-(N-1)a)\frac{r^2z^2q_N^*(a)}{w_N^*(a)}$, $N \geq 1$, *where* $q_N^*(a)$ *and* $w_N^*(a)$ *are defined by* (16.38)–(16.41).

Proof (of Proposition 16.5 and Theorem 16.3). First let $a = b$. The proof parallels the construction of convergents to a continued fraction; see [5, Ch. III]. By $G_n(a)$ of (16.27) and by formula (16.37), for all $n \geq 3$,

$$G_n(a) = a(2-a)zh_ak_ag_{n-1}g_n = c_{n,a}r^2z^{2n}\tau_a^{2n-4}/[w_n^*(a)w_{n-1}^*(a)], \tag{16.65}$$

for $c_{n,a} := a^2(2-a)^2C_{n,a}C_{n-1,a}$, with $C_{n,a}$ given by (16.37), where we have written $h_ak_a\omega_a^2 = a(2-a)\tau_a$ by the definitions (16.18). Further, for the full model we have:

$$\begin{aligned} P(\mathbf{H} = n) &= 4a\rho_{1,n}\gamma_n\rho_{n,0},\ n \geq 2;\\ P(\mathbf{H} \geq n+1) &= 2a\rho_{1,n+1},\ n \geq 2;\\ P(H = 1) &= 2\frac{1}{2}(1-a) = 1-a. \end{aligned} \tag{16.66}$$

By first principles we find $G_1(a) = r^2y^2z^2$ and $G_2(a) = r^2z^4k_a$. Therefore by (16.10) and (16.26), $P(\mathbf{H} \leq N)K_N(a) = \sum_{n=1}^N G_n(a)P(\mathbf{H} = n)$ is written by:

$$(1-a)r^2y^2z^2 + \frac{a(1-a)}{2-a}r^2z^4k_a + \sum_{n=3}^N c_{n,a}P(\mathbf{H} = n)\frac{r^2z^{2n}\tau_a^{2n-4}}{w_n^*(a)w_{n-1}^*(a)}. \tag{16.67}$$

By (16.65)–(16.66) and direct calculation, $c_{n,a}P(\mathbf{H} = n) = (1-a)a^{2n-2}$. Also, by (16.18) and (16.38), $a^{2n-2}r^2z^{2n}\tau_a^{2n-4} = a^2r^2z^4x_a^{n-2}$ while $k_a = \frac{a(2-a)}{w_2^*(a)}$, since $\omega_a = w_2^*(a)$. Therefore by (16.67), $P(\mathbf{H} \leq N)K_N(a)$ is written:

$$(1-a)r^2z^2\left(y^2 + \frac{a^2z^2}{w_2^*(a)} + \sum_{n=3}^N \frac{a^2z^2x_a^{n-2}}{w_n^*(a)w_{n-1}^*(a)}\right). \tag{16.68}$$

By (16.38)–(16.41) and direct calculation, we have that $y^2 = q_1^*(a)/w_1^*(a)$ and $y^2 + a^2z^2/w_2^*(a) = q_2^*(a)/w_2^*(a)$. But, by Lemma 16.4(1), we may write

$$\frac{q_n^*}{w_n^*} - \frac{q_{n-1}^*}{w_{n-1}^*} = \frac{[w^*(a), q^*(a)]_{n-1}}{w_n^* w_{n-1}^*} = \frac{a^2 z^2 x_a^{n-2}}{w_n^* w_{n-1}^*}, \quad n \geq 1,$$

where we suppressed the dependence on a in q_n^* and w_n^*. Therefore the sum in (16.68) telescopes. Therefore, for all $N \geq 1$ the right side of (16.68) becomes: $(1-a)r^2 z^2 q_N^*(a)/w_N^*(a)$. Finally, apply (16.66) and Proposition 16.1 to compute $P(\mathbf{H} \leq N) = \frac{N(1-a)}{N-(N-1)a}$; so that, by (16.68), Proposition 16.5 is proved.

We now indicate the additional steps required to prove the Theorem 16.3. First, with $N = f-1$ in Proposition 16.5, and by Lemma 16.4, (16.68) yields:

$$\sum_{n=1}^{f-1} G_n P(\mathbf{H} = n) = (1-a)r^2 z^2 \left(y^2 + \sum_{n=2}^{f-1} \frac{[w^*(a), q^*(a)]_{n-1}}{w_n^*(a) w_{n-1}^*(a)} \right), \tag{16.69}$$

where here and in the rest of the proof we abbreviate $G_n = G_n(a,b)$. Next by (16.27) and Proposition 16.4, (III)(1)(a) and (II)(2),

$$G_f = a(2-a)zh_a k(a,b) g_{1,f} g_{f,0} = c_f \frac{r^2 z^{2f} (a\tau_a)^{2f-4}}{w_{f-1}^*(a) \overline{w}_{f,0}}. \tag{16.70}$$

Moreover, by (16.27) and Proposition 16.4, (I)(1) and (II)(1), for all $j \geq 1$, G_{f+j} becomes:

$$a(2-a)zh_a k(b,b) g_{1,f+j} g_{f+j,0} = c_{f+j} \frac{r^2 z^{2f+2j} (a\tau_a)^{2f-4} \tau(a,b)^2 (b\tau_b)^{2j-2}}{\overline{w}_{1,f+j} \overline{w}_{f+j,0}}. \tag{16.71}$$

In (16.70) and (16.71) the constants c_f and c_{f+1}, respectively can be determined from Lemma 16.3 since $G_n[\mathbf{1}] = 1$. Indeed we find in this way, and by Definition 16.2, Proposition 16.1, (16.66), and direct calculation that $c_f P(\mathbf{H} = f) = a^2(1-b)$ and $c_{f+j} P(H = f+j) = a^2 b^2 (1-b)$, $j \geq 1$. Thus by (16.38), (16.45), and (16.70)–(16.71), and since $w_{f-1}^*(a) = \overline{w}_{1,f}$, for all $j \geq 0$, $G_{f+j} P(\mathbf{H} = f+j)$ equals

$$a^2(1-b) \frac{r^2 z^4 x_a^{f-2}}{\overline{w}_{1,f} \overline{w}_{f,0}}, \text{ if } j = 0; \quad a^2(1-b) \frac{r^2 z^4 x_a^{f-2} x(a,b) x_b^{j-1}}{\overline{w}_{1,f+j} \overline{w}_{f+j,0}}, \text{ if } j \geq 1. \tag{16.72}$$

Therefore, by (16.69), (16.72) and Lemma 16.4, for all $j \geq 0$ there holds:

$$\sum_{n=1}^{f+j} G_n P(\mathbf{H} = n) = (1-a)r^2 z^2 \left(y^2 + \sum_{n=2}^{f+j} \frac{[\overline{w}, \overline{q}]_{n-1}}{\overline{w}_{1,n} \overline{w}_{n,0}} \right), \tag{16.73}$$

where the fraction $\frac{1-b}{1-a}$ enters to form $[\overline{w}, \overline{q}]_{n-1}$ when $n = f+j$ for $j \geq 0$ because we have factored out $(1-a)$ from the entire sum on the right. But by the definition (16.61) and Lemma 16.2 we have that $\frac{[\overline{w}, \overline{q}]_{n-1}}{\overline{w}_{1,n} \overline{w}_{n,0}} = \frac{\overline{q}_n}{\overline{w}_{1,n+1}} - \frac{\overline{q}_{n-1}}{\overline{w}_{1,n}}$. Hence the sum in (16.73) telescopes, and thereby we finally obtain $K_N(a,b) = P(\mathbf{H} \leq N)^{-1}(1-$

$a)\frac{r^2z^2\overline{q}_N}{\overline{w}_{1,N+1}}$, $N \geq f$, where $P(\mathbf{H} \leq N)^{-1} = \frac{(N+1-f)a+(f-1)b-(N-1)ab}{(N+1-f)a+(f-1)b-Nab}$ by (16.66) and direct calculation.

3.5.1 Proof of Corollary 16.2

The unconditional joint generating function of the excursion statistics is $K := E\{r^{\mathbf{R}}y^{\mathbf{V}}z^{\mathbf{L}}\}$. We develop a simple representation of K in the homogeneous case, as follows.

Corollary 16.3. *Let $a = b$ and define $\alpha_a := \sqrt{\beta_a^2 - 4x_a}$ for x_a and β_a given by (16.38). Then $K = \lim_{N\to\infty} K_N(a) = \left(1 - \frac{1}{2}\beta_a - \frac{1}{2}\alpha_a\right)/(1-a)$.*

Proof (of Corollary 16.3). Since we have explicitly seen in the proof of Proposition 16.5 that if $a = b$, then $P(\mathbf{H} \leq N) = \frac{N(1-a)}{N-(N-1)a}$, we have that the persistent random walk is recurrent: $\lim_{N\to\infty} P(\mathbf{H} \leq N) = 1$. So we obtain that $(*)$ $K = \lim_{N\to\infty} K_N(a) = (1-a)r^2z^2 \lim_{N\to\infty} q_N^*/w_N^*$. Here and in the rest of the proof we suppress dependence on a when convenient; in particular denote $x = x_a$ and $\beta = \beta_a$.

We introduce a substitution variable θ as follows:

$$\beta := \sqrt{4x}\cos\theta; \;\; \beta \pm \alpha = \sqrt{4x}(\cos\theta \pm i\sin\theta) = \sqrt{4x}e^{\pm i\theta}, \tag{16.74}$$

with $\Im\theta < 0$ for $|r| < 1, |y| < 1, z \neq 0$. The idea of the substitution (16.74) may be found in [8, p. 352]. By $(\beta+\alpha)^n - (\beta+\alpha)^n = (4x)^{n/2}e^{in\theta}(1+e^{-2in\theta})$, the formulae (16.13) may be rewritten, where by our convention for the sign of $\Im\theta$, $1+e^{-2in\theta} = 1+o(1)$, as $n \to \infty$. We then substitute these expressions into the formulae (16.41) and find that $q_n^*(a)/w_n^*(a)$ is given by:

$$\frac{(y^2-q_0^*)q_n(x,\beta)+q_0^*w_n(x,\beta)}{(1-w_0^*)q_n(x,\beta)+w_0^*w_n(x,\beta)} = \frac{y^2(1-e^{-2in\theta}) - \sqrt{x}q_0^*e^{-i\theta}(1-e^{-2i(n-1)\theta})}{1-e^{-2in\theta} - \sqrt{x}w_0^*e^{-i\theta}(1-e^{-2i(n-1)\theta})}.$$

Therefore, using $e^{-i\theta} = (\beta-\alpha)/\sqrt{4x}$, we obtain $\lim_{n\to\infty} \frac{q_n^*}{w_n^*} = \frac{y^2-q_0^*(\beta-\alpha)/2}{1-w_0^*(\beta-\alpha)/2}$. Finally, we simplify this expression by multiplying both numerator and denominator by $1-w_0^*(\beta+\alpha)/2$. The new denominator becomes $1+x_aw_0^*(a)^2 - \beta_aw_0^*(a) = (1-a)^2r^2z^2/\tau_a^2$, by direct calculation. Therefore by bringing the τ_a^2 of this last expression to the numerator we obtain that $\lim_{n\to\infty}(1-a)^2r^2z^2q_n^*/w_n^* = \tau_a^2(y^2 - q_0^*(\beta-\alpha)/2)(1-w_0^*(\beta+\alpha)/2)$, or

$$[y^2 - \beta_a(q_0^*+y^2w_0^*)/2 + x_aw_0^*q_0^*]\tau_a^2 + \alpha_a[q_0^* - y^2w_0^*]\tau_a^2/2 = I + \alpha_a II,$$

after cancellation of terms $\pm q_0^*w_0^*\alpha\beta/4$. By direct calculation we find that $I = 1 - \frac{1}{2}\beta_a$, and $II = -\frac{1}{2}$. Hence by $(*)$ the proof is complete.

We may view the excursion statistics in the case $a = b$ by the way they are weighted relative to one another. Indeed, a specific excursion path of $2n$ steps and $2k$

runs is weighted with the probability $\frac{1}{2}a^{2n-2k}(1-a)^{2k-1}$, for k peaks and $k-1$ valleys. In the unweighted case $a=\frac{1}{2}$, it is known that the joint distribution of $(\mathbf{L},\mathbf{R})$ is essentially the same as that of $(\mathbf{L},\mathbf{L}-\mathbf{R})$ [see [16, A001263; symmetry of the Narayana numbers]].

Proof (of Corollary 16.2). We establish the joint generating function identity in the unweighted case via a direct calculation. Let r, u and z belong to the unit circle. By applying Corollary 16.3 with $a=\frac{1}{2}$ we obtain the joint generating function of runs, long runs, and steps by

$$K(\frac{1}{2})[ru,1/u,z]=\frac{1}{16}\left(16-4z^2+4r^2z^2+r^2z^4-2r^2uz^4+r^2u^2z^4-S\right),$$

with S given by $S=S_1S_2S_3S_4$, for:

$$S_1=\sqrt{4+2z+2rz+rz^2-ruz^2},$$
$$S_2=\sqrt{4+2z-2rz-rz^2+ruz^2},$$
$$S_3=\sqrt{4-2z+2rz-rz^2+ruz^2},\ S_4=\sqrt{4-2z-2rz+rz^2-ruz^2}.$$

On the other hand, with the very same main term S, we have

$$K(\frac{1}{2})[u/r,1/u,rz]=\frac{1}{16}\left(16+4z^2-4r^2z^2+r^2z^4-2r^2uz^4+r^2u^2z^4-S\right).$$

The two generating functions differ by

$$K(\frac{1}{2})[ru,1/u,z]-K(\frac{1}{2})[u/r,1/u,rz]=\frac{1}{2}z^2(r^2-1).$$

The difference is mirrored only in the event that $\mathbf{L}=2$, when it happens that $\mathbf{R}=2$ and $\mathbf{U}=0$. Thus (16.8) holds for $a=\frac{1}{2}$ and $n\geq 2$.

Perhaps the simplest way to obtain (16.8) for $a\neq\frac{1}{2}$ is to apply (16.8) for the case $a=\frac{1}{2}$. Consider an excursion path Γ with $\mathbf{L}(\Gamma)=2n$ and $\mathbf{L}(\Gamma)-\mathbf{R}(\Gamma)=2k$. Then $P_a(\Gamma)=\frac{1}{2}a^{2k}(1-a)^{2n-2k-1}$. Here $\mathbf{R}(\Gamma)-1=2n-2k-1$ counts the number of turns in the path, so is the exponent of $(1-a)$ under P_a. Alternatively, $\mathbf{L}(\Gamma)-\mathbf{R}(\Gamma)$ is the total length of long runs minus the number of long runs in Γ, and this gives the exponent of a in $P_a(\Gamma)$. If $2n\geq 4$, then by the first part of the proof there are exactly as many paths Γ with the joint information $\mathbf{L}(\Gamma)=2n$, $\mathbf{L}(\Gamma)-\mathbf{R}(\Gamma)=2k$, and $\mathbf{U}(\Gamma)=\ell$, as there are paths Γ' with $\mathbf{L}(\Gamma')=2n$, $\mathbf{R}(\Gamma')=2k$, and $\mathbf{U}(\Gamma')=\ell$. Therefore, since for any such path Γ', the probability assigned by the probability measure P_{1-a} yields $P_{1-a}(\Gamma')=\frac{1}{2}a^{2k-1}(1-a)^{2n-2k}$, we have that $aP_{1-a}(\Gamma')=(1-a)P_a(\Gamma)$, for all Γ with $\mathbf{L}(\Gamma)\geq 4$. Hence (16.8) holds.

4 Proofs of Theorems 16.1 and 16.2

Proof (of Theorem 16.1). We fix $t \in \mathbb{R}$. All big oh terms in the proof will refer to the parameter $N \to \infty$ with implied constants depending only on a, b, and t. Since, by [13], for fixed $m > 0$ and $f \sim \eta N \to \infty$, $P(\mathbf{X}_j = 0 \text{ before } \mathbf{X}_j = f | \mathbf{X}_0 = m) \to 1$, we may assume that $\mathbf{X}_0 = 0$. Let

$$r_N := e^{-it(2-a-b)/((1-a)(1-b)N)},\ y_N := e^{it/((1-a)(1-b)N)},\ z_N := e^{it/N}. \qquad (16.75)$$

Since $\{(1+\frac{1}{N})X_{N+1}\}$ converges in distribution if and only if $\{X_N\}$ does, by (16.2) it suffices to establish that $E\{e^{it(1+\frac{1}{N})X_{N+1}}\} = \hat{\varphi}(t)$, as $N \to \infty$. It is clear that $a(2-a)z_N h_a[r_N, y_N, z_N] \to 1$ as $N \to \infty$. Therefore, by (16.28), we must show that $\lim_{N\to\infty} g_{0,N}[r_N, y_N, z_N]$ equals the limit in (16.3). By Proposition 16.4, (I)(1), we have a formula for $g_{0,N}$, and by Proposition 16.3 we have a formula for its denominator $\overline{w}_{0,N}$. The main work is in calculating an asymptotic expression for $\overline{w}_{0,N}[r_N, y_N, z_N]$.

We now make substitutions analogous to (16.74), one for each stratum:

$$\begin{aligned} &\cos(\theta_1) := \beta_a/\sqrt{4x_a}\ ;\ \cos(\theta_2) := \beta_b/\sqrt{4x_b}, \\ &\beta_a \pm \alpha_a = \sqrt{4x_a}e^{\pm i\theta_1}, \\ &\beta_b \pm \alpha_b = \sqrt{4x_b}e^{\pm i\theta_2}, \end{aligned} \qquad (16.76)$$

where all functions on the right sides of these expressions are composed with $[r_N, y_N, z_N]$ of (16.75). Here we write $\sqrt{4x_a}$ as a shorthand for the expression $2az\tau_a$; see (16.38). Note that the coefficients in (16.2) have been chosen such that the first order term of the Taylor expansions about $t = 0$ of the substitutions $\cos\theta_j[r_N, y_N, z_N]$, $j = 1, 2$, do in fact vanish in the following:

$$\cos\theta_1 = 1 + \frac{1}{2}\frac{\sigma_1^2 t^2}{(1-b)^2N^2} + O\left(\frac{1}{N^3}\right); \quad \cos\theta_2 = 1 + \frac{1}{2}\frac{\sigma_2^2 t^2}{(1-a)^2N^2} + O\left(\frac{1}{N^3}\right), \qquad (16.77)$$

where σ_1^2 and σ_2^2 are as defined in the statement of the theorem, and we obtain (16.77) by direct computation. Therefore by (16.77), and by applying the Taylor expansion of $\arccos(u)$ about $u = 1$, we find that θ_1 and θ_2 are both of order $1/N$ as follows:

$$\theta_1 = i\frac{\sigma_1 t}{(1-b)N} + O\left(\frac{1}{N^3}\right); \quad \theta_2 = i\frac{\sigma_2 t}{(1-a)N} + O\left(\frac{1}{N^3}\right). \qquad (16.78)$$

By Proposition 16.3,

$$\overline{w}_{0.N} = d_1(f)q^*_{N-f}(b) + d_2(f)w^*_{N-f}(b). \qquad (16.79)$$

We focus first on the coefficients $d_j(f)$, which are written in terms of $w^*_f(a)$ and $w^*_{f+1}(a)$ by (16.53). By (16.13) and (16.41), suppressing dependence on a,

$w_f^* = (1-w_0^*)q_f + w_0^*(q_f - xq_{f-1}) = q_f(x,\beta) - w_0^* x q_{f-1}(x,\beta)$. Thus by (16.13), and (16.76),

$$w_f^*(a) = 2i\alpha_a^{-1}(az\tau_a)^f \{\sin f\theta_1 - \sqrt{x_a} w_0^*(a)\sin(f-1)\theta_1\}$$
$$w_{f+1}^*(a) = 2i\alpha_a^{-1}(az\tau_a)^f \sqrt{x_a}\{\sin(f+1)\theta_1 - \sqrt{x_a} w_0^*(a)\sin f\theta_1\}; \qquad (16.80)$$

with verification by direct algebra for $q_f(x_a,\beta_a) = 2i\alpha_a^{-1}(az\tau_a)^f \sin(f\theta_1)$, and with $\sqrt{x_a}$ to stand for a factor of $(az\tau_a)$. Next denote

$$e_j = e_j(f) := \frac{d_j(f)}{\Lambda_1},\ j = 1,2,\ \text{for } \Lambda_1 := 2i\alpha_a^{-1}(az\tau_a)^f = (\sin\theta_1)^{-1}(az\tau_a)^{f-1}, \qquad (16.81)$$

since $\alpha_a = i\sqrt{4x_a}\sin\theta_1 = 2iaz\tau_a\sin\theta_1$. By (16.80)–(16.81) and direct algebra, through (16.53) we can write an expression for e_j as follows:

$$\left(\mu_{j,1} - \mu_{j,2}x_a w_0^*(a)\right)\sin f\theta_1 + \sqrt{x_a}\left[\mu_{j,2}\sin(f+1)\theta_1 - \mu_{j,1}w_0^*(a)\sin(f-1)\theta_1\right]. \qquad (16.82)$$

Next we apply the trigonometric identity for the sine of a sum or difference to $\sin(f+1)\theta_1$ and $\sin(f-1)\theta_1$ in (16.82). At this point we also introduce some abbreviations to keep the notation a bit compact. Thus write

$$\mathbf{s}_1 := \sin f\theta_1;\ \ \mathbf{c}_1 := \cos f\theta_1. \qquad (16.83)$$

We rewrite (16.82), with abbreviation $w_0^* = w_0^*(a)$, by collecting terms with a factor $\sqrt{x_a}$. Thus for each $j = 1,2$,

$$e_j = (\mu_{j,1} - \mu_{j,2}x_a w_0^*)\mathbf{s}_1$$
$$+ \sqrt{x_a}\left\{\mu_{j,2}(\mathbf{s}_1\cos\theta_1 + \mathbf{c}_1\sin\theta_1) - \mu_{j,1}w_0^*(\mathbf{s}_1\cos\theta_1 - \mathbf{c}_1\sin\theta_1)\right\}. \qquad (16.84)$$

We introduce a book–keeping notation for the coefficient t_j of the variable $\mathbf{x}_j$ in square brackets, within a linear expression $\sum_i t_i\mathbf{x}_i$ in parentheses: $[\mathbf{x}_j](\sum_i t_i\mathbf{x}_i) = t_j$. Our method for e_j is to asymptotically expand $[\mathbf{s}_1](e_j)$ and $[\mathbf{c}_1\sin\theta_1](e_j)$ by (16.84). We will treat $\sin\theta_1$ separately from the asymptotic expansions of the other terms due to the convenient fact that, by (16.78), we have $\sin\theta_1 = \theta_1 + O(N^{-3})$, and this will suffice for our purposes. Note that by (16.78) and (16.83), and $f \sim \eta N$, $\mathbf{s}_1$ and $\mathbf{c}_1$ are both $O(1)$. Further, by direct calculation, $\mu_{i,j}$ are polynomial, and $q_0^*(a)$ and $w_0^*(a)$ only involve negative powers of τ_a, where $\tau_a[\mathbf{1}] = 1$. Thus the Taylor expansions of $[\mathbf{s}_1](e_j)$ and $[\mathbf{c}_1\sin\theta_1](e_j)$ about $t = 0$ are well behaved.

We next find reduced expressions for the terms $q_{N-f}^*(b)$ and $w_{N-f}^*(b)$ of (16.79). The approach is as above, but now with b in place of a, $N-f$ in place of f, and using the second substitution θ_2 in (16.76). Similar to (16.81) we introduce

$$q^* := \frac{q_{N-f}^*(b)}{\Lambda_2},\ w^* := \frac{w_{N-f}^*(b)}{\Lambda_2};$$
$$\Lambda_2 := 2i\alpha_b^{-1}(bz\tau_b)^{N-f} = (\sin\theta_2)^{-1}(bz\tau_b)^{N-f-1}. \qquad (16.85)$$

Similar as for (16.80), by (16.13) and both lines of (16.41) applied in turn, and (16.85),

$$q^* := y^2 \sin(N-f)\theta_2 - \sqrt{x_b}q_0^*(b)\sin(N-f-1)\theta_2,$$
$$w^* := \sin(N-f)\theta_2 - \sqrt{x_b}w_0^*(b)\sin(N-f-1)\theta_2. \tag{16.86}$$

Introduce abbreviations also for the second stratum sines and cosines:

$$\mathbf{s}_2 := \sin(N-f)\theta_2; \;\; \mathbf{c}_2 := \cos(N-f)\theta_2. \tag{16.87}$$

We illustrate the book–keeping method by expanding $\sin(N-f-1)\theta_2 = \mathbf{s}_2\cos\theta_2 - \mathbf{c}_2\sin\theta_2$ to obtain by (16.86),

$$[\mathbf{s}_2]\,(q^*) = y^2 - \sqrt{x_b}q_0^*(b)\cos\theta_2; \;\; [\mathbf{c}_2\sin\theta_2]\,(q^*) = \sqrt{x_b}q_0^*(b);$$
$$[\mathbf{s}_2]\,(w^*) = 1 - \sqrt{x_b}w_0^*(b)\cos\theta_2; \;\; [\mathbf{c}_2\sin\theta_2]\,(w^*) = \sqrt{x_b}w_0^*(b).$$

To handle the asymptotic expansions for the four terms on the right side of (16.79), we expand the coefficients of $\mathbf{s}_1$, $\mathbf{c}_1\sin\theta_1$, $\mathbf{s}_2$, and $\mathbf{c}_2\sin\theta_2$ by direct computation and thereby find

$$\begin{aligned}\frac{\overline{w}_{0,N}}{\Lambda_1\Lambda_2} = O(N^{-2}) &+ \left[\left(-(1-a)(1-b) + 2(1-ab)\frac{it}{N}\right)\mathbf{s}_1\right]\left[\left(1+\frac{2}{1-a}\frac{it}{N}\right)\mathbf{s}_2\right] \\ &+ \left[\left(1-a-\frac{a(b-a)}{1-b}\frac{it}{N}\right)\mathbf{s}_1 + a\mathbf{c}_1\sin\theta_1\right] \\ &\times \left[\left(1-b-\frac{b(2-a-b)}{1-a}\frac{it}{N}\right)\mathbf{s}_2 + b\mathbf{c}_2\sin\theta_2\right].\end{aligned} \tag{16.88}$$

Since, by (16.78), $\sin\theta_1$ and $\sin\theta_2$ are of order $1/N$, observe that the two terms of order 1 on the right hand side of (16.88) are of form $\pm(1-a)(1-b)$ and therefore cancel. Also, since $\sin\theta_j = \theta_j + O(N^{-3})$ for θ_j are given by (16.78), we substitute these relations into (16.88) and collect the order $1/N$ terms to find by direct asymptotics that:

$$\frac{\overline{w}_{0,N}}{\Lambda_1\Lambda_2} = \left\{a\sigma_1\mathbf{c}_1\mathbf{s}_2 + b\sigma_2\mathbf{s}_1\mathbf{c}_2 + (b-a)^2\mathbf{s}_1\mathbf{s}_2\right\}\frac{it}{N} + O(N^{-2}). \tag{16.89}$$

To render a partial check on the book–keeping procedure for (16.89), write out a formula for $e_j = e_j(a,b,r_N,y_N,z_N,\mathbf{s}_1,\mathbf{c}_1\sin\theta_1)$ of (16.84) by leaving $\sin\theta_1$ as an auxiliary variable. Then $\sin\theta_j$ is replaced by the order $1/N$ term of (16.78), whereas $\cos\theta_j$ is defined exactly by (16.76). So, expand $e_1q^* + e_2w^*$ as

$$e_1\left([\mathbf{s}_2](q^*)\mathbf{s}_2 + [\mathbf{c}_2\sin\theta_2](q^*)\mathbf{c}_2\sin\theta_2\right) + e_2\left([\mathbf{s}_2](w^*)\mathbf{s}_2 + [\mathbf{c}_2\sin\theta_2](w^*)\mathbf{c}_2\sin\theta_2\right),$$

and apply a Taylor series about $t = 0$ to recover (16.89); see [15].

Now plug (16.89) into the formula for $g_{0,N}$ in Proposition 16.4(I)(1), apply Proposition 16.1(I)(1) to rewrite $\Pi_{0,N}$, and recall Λ_j in (16.81) and (16.85). So

$$g_{0,N} = \frac{\omega_a \tau(a,b) r z^2}{a(2-a)\tau_a} \left(\frac{\sin\theta_1 \sin\theta_2 [(N-f)a + fb - (N-1)ab]}{[a\sigma_1 \mathbf{c}_1 \mathbf{s}_2 + b\sigma_2 \mathbf{s}_1 \mathbf{c}_2 + (b-a)^2 \mathbf{s}_1 \mathbf{s}_2] \frac{it}{N} + O(N^{-2})} \right).$$

Finally, to find the limit as $N \to \infty$ of this last expression, we substitute (16.78) into the definitions (16.83) and (16.87), and again employ $\sin\theta_j \sim \theta_j$. We note: $\lim_{N\to\infty} \omega_a [a(2-a)]^{-1} r_N z_N^2 \tau(a,b) \tau_a^{-1} = 1$, since $\omega_a[\mathbf{1}] = a(2-a)$ and $\tau(a,b)[\mathbf{1}] = 1$. Since by assumption $f \sim \eta N$, we have $[(N-f)a + fb - (N-1)ab] \sim N[(1-\eta)a + \eta b - ab]$, and since by (16.78), $\theta_1\theta_2 \sim i^2 \frac{\sigma_1\sigma_2}{(1-a)(1-b)} t^2 N^{-2}$, we obtain, as $N \to \infty$,

$$g_{0,N} \sim \frac{i^2 t^2}{N} \frac{\sigma_1 \sigma_2}{(1-a)(1-b)} \frac{(1-\eta)a + \eta b - ab}{[a\sigma_1 \mathbf{c}_1 \mathbf{s}_2 + b\sigma_2 \mathbf{s}_1 \mathbf{c}_2 + (b-a)^2 \mathbf{s}_1 \mathbf{s}_2] \frac{it}{N}}.$$

Here we use implicitly that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$, so that by (16.78), (16.83) and (16.87), and by definition of κ_1 and κ_2, $\mathbf{s}_j \sim i\sinh(\kappa_j t)$, $j = 1,2$, and $\mathbf{c}_j \sim \cosh(\kappa_j t)$, $j = 1,2$. Thus we obtain, $\lim_{N\to\infty} g_{0,N}[r, s_N, t_N] = \hat{\varphi}(t)$, for $\hat{\varphi}(t)$ given by (16.3).

Proof (of Corollary 16.1). We now assume that $a = b$ and consider the random variable $sY_{1,N} + tY_{2,N}$ defined by (16.4) in place of tX_N in the proof of Theorem 16.1. By the definition (16.4) we write

$$sY_{1,N} + tY_{2,N} = \frac{1}{N} \left(t\mathscr{L}'_N + \frac{(1-a)s - (2-a)t}{(1-a)} \mathscr{R}'_N + \frac{t-s}{(1-a)} \mathscr{V}'_N \right).$$

Accordingly, define

$$r_{s,t,N} := e^{i((1-a)s-(2-a)t)/((1-a)N)},\ y_{s,t,N} := e^{i(t-s)/((1-a)N)},\ z_{s,t,N} := e^{it/N}. \quad (16.90)$$

It suffices to prove that, for each fixed pair of real numbers $s,t \in \mathbb{R}$,

$$\lim_{N\to\infty} g_{0,N}(a,a)[r_{s,t,N}, y_{s,t,N}, z_{s,t,N}]$$

exists and is given by the right side of (16.5). Define $\theta = \theta_{s,t,N}$ via $\cos\theta = \beta_a/\sqrt{4x_a}$, where the functions β_a and x_a are composed with the complex exponential terms in (16.90). It follows by making a direct calculation that

$$\cos\theta = 1 + \frac{1}{2}\frac{(1-a)s^2 + at^2}{N^2} + O\left(\frac{1}{N^3}\right); \quad \theta = i\frac{\sqrt{(1-a)s^2 + at^2}}{N} + O\left(\frac{1}{N^3}\right). \quad (16.91)$$

Since the model is homogeneous, we need only apply the first line of (16.80) with $f := N$ to obtain

$$w_N^*(a) = (\sqrt{x_a}\sin\theta)^{-1} [az\tau_a]^N \{\sin N\theta - \sqrt{x_a} w_0^*(a) \sin(N-1)\theta\}. \quad (16.92)$$

Expand $\sin(N-1)\theta = \mathbf{s}\cos\theta - \mathbf{c}\sin\theta$, for $\mathbf{s} := \sin N\theta$ and $\mathbf{c} := \cos N\theta$. Put $\Lambda := (\sin\theta)^{-1} (az\tau_a)^{N-1}$. After direct calculation we find

$$1-\sqrt{x_a}w_0^*(a)=1-a+O(N^{-1}).$$

Therefore, by (16.92), we have

$$\frac{w_N^*(a)}{\Lambda}=\mathbf{s}-\sqrt{x_a}w_0^*(a)(\mathbf{s}\cos\theta-\mathbf{c}\sin\theta)=(1-a)\mathbf{s}+O\left(\frac{1}{N}\right)$$

Note that there is no cancellation of the order 1 term in this expression. Now plug $\frac{w_N^*(a)}{\Lambda}$ into (16.37) to obtain

$$g_{0,N}=\frac{\omega_a}{a(2-a)}rz\tau_a^{-1}\frac{(N-(N-1)a)\sin\theta}{(1-a)\mathbf{s}+O(N^{-1})}.$$

Finally apply the asymptotic expression for θ in (16.91) and let $N\to\infty$.

Proof (of Theorem 16.2). By the same reasoning given at the outset of the proof of Theorem 16.1, we may assume that $\mathbf{X}_0=0$. By the fact that the absolute value process starts afresh at the end of each excursion, we have that $1+\mathscr{M}_N$ is a standard geometric random variable with success probability $P(\mathbf{H}\geq N)$. Thus

$$P(\mathscr{M}_N=\nu)=[P(\mathbf{H}<N)]^{\nu}P(\mathbf{H}\geq N),\quad \nu=0,1,2,\ldots. \tag{16.93}$$

Let $\mathbf{L}_N$, $\mathbf{R}_N$, and $\mathbf{V}_N$, respectively, be random variables for the number of steps, runs, and short runs, in an excursion, given that the height of the excursion is at most $N-1$. Therefore, in distribution, we may write:

$$\mathscr{R}_N=\sum_{\nu=0}^{\mathscr{M}_N}\mathbf{R}^{(\nu)},\quad \mathscr{V}_N=\sum_{\nu=0}^{\mathscr{M}_N}\mathbf{V}^{(\nu)},\quad \mathscr{L}_N=\sum_{\nu=0}^{\mathscr{M}_N}\mathbf{L}^{(\nu)},$$

where $\mathbf{R}^{(1)},\mathbf{R}^{(2)},\ldots$; $\mathbf{V}^{(1)},\mathbf{V}^{(2)},\ldots$; and $\mathbf{L}^{(1)},\mathbf{L}^{(2)},\ldots$, respectively, are sequences of independent copies of $\mathbf{R}_N$, $\mathbf{V}_N$, and $\mathbf{L}_N$. Since the random variables $\mathbf{R}_N$, $\mathbf{V}_N$, and $\mathbf{L}_N$ already have built into their definitions the condition $\{\mathbf{H}\leq N-1\}$, the probability generating function $K_{N-1}=E\{r^{\mathbf{R}_N}y^{\mathbf{V}_N}z^{\mathbf{L}_N}\}$ is calculated by Theorem 16.3. Thus by (16.93), and by calculating a geometric sum there holds:

$$E\{r^{\mathscr{R}_N}y^{\mathscr{V}_N}z^{\mathscr{L}_N}u^{\mathscr{M}_N}\}=\sum_{\nu=0}^{\infty}P(\mathscr{M}_N=\nu)\,(uK_{N-1})^{\nu}=\frac{P(\mathbf{H}\geq N)}{1-uP(\mathbf{H}<N)K_{N-1}[r,y,z]}. \tag{16.94}$$

We define (r_N,y_N,z_N) by (16.75), and also set $u_N:=e^{-ita(b-a)/[(1-a)(1-b)N]}$. By (16.9), it suffices to show that $\lim_{N\to\infty}E\{e^{it(1+1/N)\mathscr{X}_{N+1}}\}=\hat{\psi}(t)/\hat{\varphi}(t)$; see (16.98). We define θ_1 and θ_2 by (16.76), so that also (16.77)–(16.78) hold. By the statement of Theorem 16.3 we must replace the calculation of $\overline{w}_{0,N}$, starting with (16.79), with instead $\overline{w}_{1,N+1}$. However, by (16.53), (16.64), and (16.79), the difference in the two calculations is simply accounted for by replacing f by $f-1$ in the calculation of $\overline{w}_{0,N}$, because j in (16.64) for $\overline{w}_{1,N+1}$ is determined by $j=N+1-f=N-(f-1)$, so $\frac{1}{\Lambda_2}\overline{w}_{1,N+1}=d_1(f')q^*+d_2(f')w^*$ with $f':=f-1$ in place of f in both (16.53) and (16.85). This is reflected by the fact that, by Lemma 16.2, $\overline{w}_{1,N+1}=\overline{w}_{N,0}$.

We must now also calculate $\overline{q}_N = d_{q,1}(f)q^*_{N-f+1}(b) + d_{q,2}(f)w^*_{N-f+1}(b)$ given by Lemma 16.5, with $d_{q,j}(f) = \mu_{j,1}q^*_{f-1}(a) + \mu_{j,2}q^*_f(a)$, $j = 1,2$, defined by (16.63) in the proof of Lemma 16.5. In summary, $f' = f-1$ yields (†) $\overline{q}_N = d_{q,1}(f'+1)q^*_{N-f'}(b) + d_{q,2}(f'+1)w^*_{N-f'}(b)$. Thus, because we simply replace f by $f-1$ in the required substitutions, and since $f \sim \eta N$, we will not change the name of f. With this understanding, we may use the calculation of $\overline{w}_{0,N}$ in (16.79)–(16.89) verbatim in place of the calculation of $\overline{w}_{1,N+1}$, and we will do this without changing the names of e_j, q^*, w^* and Λ_j; see (16.81) and (16.85). Further with this understanding, by (†), with f now recouping the role of f', and with q^* and w^* defined by (16.85), we have $\frac{1}{\Lambda_2}\overline{q}_N = d_{q,1}q^* + d_{q,2}w^*$ for

$$d_{q,j} := \mu_{j,1}q^*_f(a) + \mu_{j,2}q^*_{f+1}(a). \tag{16.95}$$

Here, by (16.13), (16.41) and (16.76), in analogy with (16.80), we have

$$q^*_f(a) = 2i\alpha_a^{-1}(az\tau_a)^f\left\{y^2\sin f\theta_1 - \sqrt{x_a}q^*_0(a)\sin(f-1)\theta_1\right\}$$
$$q^*_{f+1}(a) = 2i\alpha_a^{-1}(az\tau_a)^f\sqrt{x_a}\left\{y^2\sin(f+1)\theta_1 - \sqrt{x_a}q^*_0(a)\sin f\theta_1\right\}.$$

Denote $e_{q,j} := d_{q,j}/\Lambda_1$. Therefore, by (16.95), the definition of Λ_1 in (16.81), and these equations for $q^*_f(a)$ and $q^*_{f+1}(a)$,

$$e_{q,j} = \left(y^2\mu_{j,1} - \mu_{j,2}xq^*_0\right)\sin f\theta_1 + \sqrt{x}\left\{y^2\mu_{j,2}\sin(f+1)\theta_1 - \mu_{j,1}q^*_0\sin(f-1)\theta_1\right\}, \tag{16.96}$$

where $x = x_a$ and $q^*_0 = q^*_0(a)$. Rewrite (16.96) by applying the notations (16.83). Thus $e_{q,j}$ is written, with dependence on a suppressed, by

$$(y^2\mu_{j,1} - \mu_{j,2}xq^*_0)\mathbf{s}_1 + \sqrt{x}\{y^2\mu_{j,2}(\mathbf{s}_1\cos\theta_1 + \mathbf{c}_1\sin\theta_1) - \mu_{j,1}q^*_0(\mathbf{s}_1\cos\theta_1 - \mathbf{c}_1\sin\theta_1)\} \tag{16.97}$$

In summary, by (16.95), we have $\overline{q}_N/(\Lambda_1\Lambda_2) = e_{q,1}q^* + e_{q,2}w^*$, for $e_{q,j}$ in (16.97), and Λ_j defined by (16.81) and (16.85).

To guide the asymptotic expansions of (16.97) we rewrite (16.94) by substituting the last line of the proof of Theorem 16.3:

$$E\{e^{it(1+1/N)\mathscr{X}_{N+1}}\} = \frac{P(\mathbf{H} \geq N+1)\overline{w}_{1,N+1}}{\overline{w}_{1,N+1} - (1-a)u_N r_N^2 z_N^2 \overline{q}_N}. \tag{16.98}$$

It turns out that there is a cancellation in the order of the denominator of (16.98). That is, the leading order of each of $\overline{w}_{1,N+1}/(\Lambda_1\Lambda_2)$ and $\overline{q}_N/(\Lambda_1\Lambda_2)$ will be some order 1 trigonometric factor times it/N; in fact there holds $(1-a)\overline{q}_N/\overline{w}_{1,N+1} \sim 1$, as $N \to \infty$. Define

$$\Delta_N := \overline{w}_{1,N+1} - (1-a)u_N r_N^2 z_N^2 \overline{q}_N. \tag{16.99}$$

By direct calculation we will establish that $\Delta_N/(\Lambda_1\Lambda_2) = O(N^{-2})$, and we find the exact coefficient of the order N^{-2} term.

For the asymptotics of (16.97) we may still treat $\sin\theta_1 = \theta_1 + O(N^{-3})$ by (16.78), but must render precisely the $O(N^{-2})$ term in $\cos\theta_1 = 1 + O(N^{-2})$ of (16.77). In an appendix to [15], we display the many terms of the book–keeping method for this problem. For the the present, we simply exhibit the asymptotics of (16.99) obtained by machine computation with $\sin\theta_j$ substituted by the corresponding order $1/N$ term of (16.78):

$$\frac{\Delta_N}{\Lambda_1\Lambda_2} = \frac{1}{(1-a)(1-b)}\frac{t^2}{N^2}\{-ab\sigma_1\sigma_2\mathbf{c}_1\mathbf{c}_2 - a\sigma_1(a-b)^2\mathbf{c}_1\mathbf{s}_2 + a^2\sigma_1^2\mathbf{s}_1\mathbf{s}_2\} + O\left(\frac{1}{N^3}\right). \quad (16.100)$$

Finally we compute the limit of the ratio (16.98) by the asymptotic relations (16.76), and by (16.89) and (16.100). Thus, because by (16.66) and Proposition 16.1 we have that $P(\mathbf{H} \geq N+1) \sim C_{a,b}N^{-1}$ for $C_{a,b} = ab/[(1-\eta)a + \eta b - ab]$, we find $E\{e^{it(1+1/N)\mathscr{X}_{N+1}}\}$ is asymptotic to

$$C_{a,b}N^{-1}\left\{[a\sigma_1\mathbf{c}_1\mathbf{s}_2 + b\sigma_2\mathbf{s}_1\mathbf{c}_2 + (b-a)^2\mathbf{s}_1\mathbf{s}_2]\frac{it}{N} + O\left(\frac{1}{N^2}\right)\right\} \Big/ \left\{\frac{1}{(1-a)(1-b)}[-ab\sigma_1\sigma_2\mathbf{c}_1\mathbf{c}_2 - a\sigma_1(a-b)^2\mathbf{c}_1\mathbf{s}_2 + a^2\sigma_1^2\mathbf{s}_1\mathbf{s}_2]\frac{t^2}{N^2} + O\left(\frac{1}{N^3}\right)\right\}.$$

As in the proof of Theorem 16.1 we have $\mathbf{c}_j \sim \cosh(\kappa_j t)$, and $\mathbf{s}_j \sim i\sinh(\kappa_j t)$, $j = 1,2$. Therefore, with $\tilde{C}_{a,b} := (1-a)(1-b)C_{a,b}$, we obtain that $E\{e^{it(1+1/N)\mathscr{X}_{N+1}}\}$ has the following limit as $N \to \infty$, where we refer to (16.3) and statement of Theorem 16.2 for the definitions of $\hat{\varphi}(t)$ and $\hat{\psi}(t)$:

$$\lim_{N\to\infty} E\{e^{it(1+1/N)\mathscr{X}_N}\} = \frac{\tilde{C}_{a,b}}{t} \times \frac{(b\kappa_1\sigma_2 + a\kappa_2\sigma_1)t}{\hat{\varphi}(t)} \times \frac{\hat{\psi}(t)}{ab\sigma_1\sigma_2}.$$

We have $\tilde{C}_{a,b} = ab\sigma_1\sigma_2/(a\sigma_1\kappa_2 + b\sigma_2\kappa_1)$, so the proof is complete.

Corollary 16.4. *Assume $a = b$. Define*

$$Z_1 = \frac{1}{N}\left(\mathscr{R}_N - \frac{1}{(1-a)}\mathscr{V}_N + a\mathscr{M}_N\right);$$
$$Z_2 = \frac{1}{N}\left(\mathscr{L}_N - \frac{1}{(1-a)}\mathscr{R}_N + \frac{a}{1-a}\mathscr{M}_N\right) - Z_1.$$

Then, $\lim_{N\to\infty} E\{e^{i(sZ_1+tZ_2)}\} = \frac{\tanh(\sqrt{(1-a)s^2+at^2})}{\sqrt{(1-a)s^2+at^2}}$.

Proof. One simplifies the lines of proof of Theorem 16.2. We leave details in an appendix to [15].

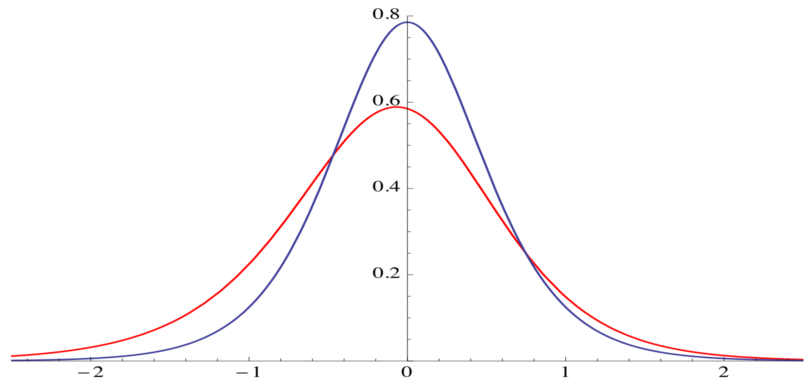

Fig. 4: The density $\varphi(x)$ whose transform $\hat{\varphi}(t) = \int_{-\infty}^{\infty} e^{itx}\varphi(x)\,dx$ is given by (16.6) for $a = \frac{1}{4}$, and the density $\frac{\pi}{4}\operatorname{sech}^2(\pi x/2)$, that is instead determined by $a = \frac{1}{2}$ and corresponds to simple random walk. Numerically, the mean of φ is $\int_{\infty}^{\infty} x\varphi(x)\,dx = -\frac{1}{4}$, and $\arg\max_x \varphi(x) = -0.131619$.

Acknowledgements The author wishes to thank the referee who gave extensive suggestions that led to many improvements in the presentation. The companion document [15] would not have come into the public domain without the referee's helpful (and exuberant!) insight.

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