GAMBLER’S RUIN WITH RANDOM STOPPING

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ABSTRACT. Let \( \{X_j, j \geq 0\} \) denote a Markov process on \([-N - 1, N + 1]\} \cup \{c\}. Suppose \( P(X_{j+1} = m + 1| X_j = m) = ph, P(X_{j+1} = m - 1| X_j = m) = (1 - p)h \), all \( j \geq 1 \) and \( |m| \leq N \), where \( p = \frac{1}{2} + \frac{h}{N} \) and \( h = 1 - c_N \) for \( c_N = \frac{1}{2}a^2/N^2 \). Define \( P(X_{j+1} = c| X_j = m) = c_N, j \geq 0, |m| \leq N \}. \{X_j\} \) terminates at the first \( j \) such that \( X_j \in \{-N - 1, N + 1, c\}. \) Let \( L = \max\{j \geq 0 : X_j = 0\}. \) On \( \Omega^0 = \{X_j \text{ terminates at } c\} \), denote by \( R^0, Y^0, \) and \( L^0 \) respectively, as the numbers of runs, short runs, and steps from \( L \) until termination. Denote \( \overline{Y}^0 = R^0 - 2Y^0 \) and \( \overline{Z}^0 = L^0 - 3R^0 + 2Y^0 \). Then \( \lim_{N \to \infty} E\{e^{i \xi(\overline{Y}^0 + \overline{Z}^0)} | \Omega^0\} = C_{a,b} \frac{\sqrt{e^2 + (a^2 + t^2)/2(cosh \sqrt{e^2 + (a^2 + t^2)/2 - cosh(2b))}}}{(2a^2 + t^2) \sinh \sqrt{e^2 + (a^2 + t^2)/2}} \), where \( c^2 = a^2 + 4b^2 \).

1. Introduction

We introduce a model of gambler’s ruin as follows. Let \( \{X_j, j \geq 0\} \) denote a Markov process on fortunes \( Z \cap \{-N - 1, N + 1\} \) together with an abstract stopping state \( c \), started from \( X_0 = 0 \). Suppose that \( P(X_{j+1} = m + 1| X_j = m) = ph \) and \( P(X_{j+1} = m - 1| X_j = m) = (1 - p)h \), all \( j \geq 1 \) and all \( |m| \leq N \), where \( 0 < p < 1 \). We define \( P(X_{j+1} = c| X_j = m) = c_N \), independent of \( j \geq 0 \) and \( |m| \leq N \), where \( h = h_N = 1 - c_N \) for \( c_N = \frac{1}{2}a^2/N^2 \), with a stopping parameter \( a > 0 \). Thus at each epoch \( j \) the Markov chain either steps up one unit with probability \( ph \), steps down one unit with probability \( (1 - p)h \), or transitions to \( c \) with small probability \( c_N \). The process \( \{X_j\} \) terminates at the first epoch \( j \) such that \( X_j \in \{-N - 1, N + 1, c\}. \) The value \( X_j \) is a model of a bettor’s fortune at epoch \( j \) in gambling with one unit bets per play such that at any play for which the fortune is in \([-N, N]\], there is a random stoppage of play at discrete rate \( \frac{1}{2}a^2/N^2 \). In case \( a = 0 \) so that \( h = 1 \), we have a classical gambler’s ruin model with no random stopping and boundaries \( \pm(N + 1) \). We define the parameter \( \xi = p(1 - p)h^2 \) which in the classical case \( h = 1 \) is simply the variance of the step variable taking values \( \pm 1 \).

In the symmetric case \( p = \frac{1}{2} \) it is natural to identify \( c \) with fortune \( m = 0 \) in case the process comes to \( c \) before one of the boundaries \( \pm(N + 1) \), and thereby continue the process until one of these boundaries is reached. With this modification, and calling the extended process \( \{\tilde{X}_j\} \), the sequence \( \{(j, \tilde{X}_j)\} \) produces a lattice path up to a random termination epoch, though not of nearest neighbor type. Indeed we are inspired by the work \([1]\) that treats nonnegative lattice paths, with no height restriction and on a fixed time scale, for which transition to the stopping state \( c \) corresponds there to a catastrophe, namely a return of the lattice path to the \( j \)-axis in one unit of time. The catastrophe is itself motivated by several contexts in \([1]\), including a discrete time queuing model, which allows for the queue to reset to zero length on events with small constant probability. In the symmetric case, by considering the absolute value process \( \{|\tilde{X}_j|\} \) as a model for queuing, with \(|\tilde{X}_j|\) being the length of the queue at epoch \( j \), we have a nonnegative Markov chain with reflection at the \( j \)-axis. We treat this queuing model in section 4.3. Note that our model in general allows a transition to the state \( c \) at the very first step, but this occurs with a negligible probability (order \( O(1/N^2) \)), and it turns out we can safely ignore this possibility.

To further state our motivations we need some definitions. A nearest neighbor lattice path is a sequence of vertices \( \{(j, x_j), j = 0, \ldots, k\} \in \mathbb{Z}^2 \) such that \(|x_{j+1} - x_j| = 1, j = 0, \ldots, k - 1, \) for some \( k \geq 1 \). We call \( x_j \) the level of the path at epoch \( j \). An excursion path satisfies the additional
requirements that \( x_0 = x_k = 0 \) (so \( k \) is even), and \( x_j \neq 0 \), for all \( 0 < j < k \). A nonnegative excursion is an excursion that lies above the \( j \)-axis save for its endpoints. A run of a nearest neighbor lattice path is either an ascending incline or a descending incline of maximal length along the linearly interpolated path. Thus the number of runs of a nearest neighbor lattice path is one more than the number of turns of the path, where a turn simply corresponds to a change in direction: ascent to descent or vice versa. The number of steps of a lattice path is simply the length of the discrete time interval, or \( k \), in the above nearest neighbor description. A long run is itself a run that consists of at least two steps; in gambling terminology a long run means that the run of fortune does not immediately change direction. A short run is on the other hand a run of length exactly one, so every run is either a long run or a short run. In Figure 1, the lattice path shown has 26 steps and 13 runs, with 5 short runs and 8 long runs. An earlier work for gambler’s ruin with boundaries, [7], calculates the probability of ruin on a infinite time scale with both catastrophes and windfalls, and with constant probabilities of each of these events besides constant win and loss probabilities on each play, by utilizing a difference equation. Our main aim is to handle random stopping as well as certain statistics of the gambler’s ruin path (runs and short runs) that so far have not been handled by the difference equation method. Even without random stopping, our results involving the runs statistics have so far relied on generalized Fibonacci recurrences to compute probability generating functions for them, [9, 10]. We shall see that the random stopping gives rise to Fibonacci recurrences with an extra (driving) term that is not observed for the nearest neighbor only models. A nice feature of our approach is that joint probability generating functions of runs, short runs, and steps take on explicit closed forms, and limit distributions follow as the parameter \( N \) tends to infinity. While we will develop explicit generating function formulae for general \( p \), in our application to Theorem 1.1 we will consider the asymptotically symmetric case \( p = \frac{1}{2} + \frac{b}{N} \), for a constant \( b \). Relative to the symmetric case, for certain problems such as Theorem 1.1(b), the presence of \( b \) produces a nonlinear effect on a limit law, while in other problems the parameter \( b \) simply increases the effective size of \( a \). The nonlinear effect comes about for statistics along paths that are interrupted by random stopping.

We introduce some definitions. Denote by \( \mathcal{L} = \mathcal{L}_N \) the epoch of last visit to the fortune \( m = 0 \), namely
\[
\mathcal{L} = \max\{j \geq 0 : X_j = 0\}. \tag{1.1}
\]
In Figure 1 the last visit is depicted relative to the absolute value process \( \{|X_j|\} \) and for the case when this process terminates at the boundary \( N + 1 = 4 \). We define the meander as the part of the absolute value path from the epoch of last visit until the process terminates, where we may call \( |c| = c \). Define also
\[
L = \inf\{j \geq 1 : X_j = 0 \text{ or } X_j \in \{-(N + 1), N + 1, c\} \}. \tag{1.2}
\]
Thus \( L \) is either the first time until a return to fortune \( m = 0 \), namely an excursion time, or the termination time, whichever is smallest. Define \( E_0 = \{X_L = 0\} \); on \( E_0 \) the process \( \{X_j\} \) makes an excursion from \( m = 0 \) back to \( m = 0 \) for a first time without terminating. Thus on \( E_0 \) we have that
\( \mathbf{L} \) is the number of steps in the excursion. Also on \( E_0 \), define \( \mathbf{R} \) and \( \mathbf{V} \), respectively, as the number of runs and short runs in the excursion of the absolute value process path \( \{ (j, |X_j|), j = 0, 1, \ldots, \mathbf{L} \} \).

Define the event \( E_c = \{ \mathbf{X}_L = c \} \); on \( E_c \) the stopping state \( c \) is reached on the first attempt at excursion. On \( E_c \) we say we have a stopped excursion, that reaches \( c \) before ever returning to \( m = 0 \) or to a boundary. Finally, denote \( E_N \) as the event that on the first attempt at excursion the chain terminates at one of the boundaries \( \pm (N + 1) \). Thus \( E_0, E_c, \) and \( E_N \) are mutually disjoint and exhaustive.

On \( E_0 \), define \( \mathbf{H} = \max\{ |X_j|, j = 0, 1, \ldots, \mathbf{L} \} \), while on \( E_c \), define \( \mathbf{H} = \max\{ |X_j|, j = 0, 1, \ldots, \mathbf{L} - 1 \} \). So \( \mathbf{H} \) denotes the height of an excursion on \( E_0 \), while on \( E_c \) it represents the maximum level of that portion of the absolute value process up until one step before reaching \( c \). On \( E_N \), define \( \mathbf{H} = N + 1 \).

Note that \( E_N \) and \( E_c \) are both unusual events, each with probability of order \( O(1/N) \), as we shall see. That is because it is difficult to exit the large region \( [-N, N] \cap \mathbb{Z} \) before coming back to \( m = 0 \), or else to make a long stopped excursion (of order \( O(N^2) \) steps) to the stopping state \( c \). Define now \( \Omega^0 \) as the event that the process \( \{ \mathbf{X}_j \} \) terminates at state \( c \). Define \( \Omega' \) as the complement of \( \Omega^0 \), that is the event that the process \( \{ \mathbf{X}_j \} \) terminates at one the boundaries.

Denote by \( \mathcal{M} = \mathcal{M}_N \) the number of consecutive bona fide excursions of the absolute value process \( \{ |X_j| \} \), each of height at most \( N \), until the last visit \( \mathcal{L} \). Let \( \mathbf{R}_\nu \) and \( \mathbf{V}_\nu \) be the number of runs and short runs, respectively, in the \( \nu \)th excursion of the absolute value process path \( \{ (j, |X_j|) \} \).

Define \( R_N = \sum_{\nu=1}^M \mathbf{R}_\nu \) and \( V_N = \sum_{\nu=1}^M \mathbf{V}_\nu \). Here, if \( \mathcal{M}_N = 0 \), then the empty sums are taken as zero. Equivalently \( R_N \) and \( V_N \) are the total number of runs and short runs, respectively, of the last visit portion of the absolute value path.

On the event \( \Omega^0 \) that the process terminates at state \( c \), denote by \( R^0_N, V^0_N, \) and \( \mathcal{L}^0_N \) as the numbers of runs, short runs, and steps, respectively, of the absolute value process \( \{ (j, |X_j|) \} \) starting from epoch \( \mathcal{L} \) to the stopping state. For convenience, we shall count a new run on a final step to \( c \) if the step just before this final transition is away from the \( j \)-axis, but this will not count as a short run. Thus we may think of the final transition to \( c \) as in the direction of \( m = 0 \), but with a single long step, which we count as a single step. So, if the final nearest neighbor lattice path step, just before the actual final step to state \( c \), is toward the \( j \)-axis, then the actual final step may be regarded as a long run continuation towards the \( j \)-axis, and therefore we count exactly one run and no short run on this continued incline toward the \( j \)-axis. On the complementary event \( \Omega' \), denote by \( R'_N, V'_N, \) and \( \mathcal{L}'_N \) as the numbers of runs, short runs, and steps, respectively, of the absolute value process \( \{ (j, |X_j|) \} \) from epoch \( \mathcal{L} \) to the terminal epoch.

We define a pair of half–integer valued statistics for the last visit portion of gambler’s ruin by \( Y_N = R_N - 2V_N + \frac{1}{2} \mathcal{M}_N \) and \( Z_N = \mathcal{L}_N - 3R_N + 2V_N + \frac{1}{2} \mathcal{M}_N \). We also define the pair of statistics \( Y^0_N = R^0_N - 2V^0_N \) and \( Z^0_N = \mathcal{L}^0_N - 3R^0_N + 2V^0_N \), and finally the pair \( Y'_N = R'_N - 2V'_N \) and \( Z'_N = \mathcal{L}'_N - 3R'_N + 2V'_N \). Define the overall process statistics \( Y_N = Y_N + Y^0_N \cdot \Omega^0 + Y'_N \cdot \Omega' \) and \( Z_N = Z_N + Z^0_N \cdot \Omega^0 + Z'_N \cdot \Omega' \). In the asymptotically symmetric case we obtain joint limit laws after normalizing these pairs of statistics by \( N \) as follows.

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Theorem 1.1. Let $p = \frac{1}{2} + \frac{b}{N}$ for some constant $b$. Denote $c^2 = a^2 + 4b^2$. Let $s, t \in \mathbb{R}$, and denote $\varphi(s, t) = \sqrt{c^2 + (s^2 + t^2)/2}$. Then we have the following limiting joint characteristic functions:

\begin{align*}
(a) \lim_{N \to \infty} \mathbb{E}\{e^{i\frac{1}{N}(sY_N + tZ_N)}\} &= \frac{c}{\tanh(c)} \frac{\tanh \varphi(s, t)}{\varphi(s, t)}, \\
(b) \lim_{N \to \infty} \mathbb{E}\{e^{i\frac{1}{N}(sY_N + tZ_N)} \mid \Omega^0\} &= C_{a,b} \frac{\varphi(s, t) (\cosh \varphi(s, t) - \cosh(2b))}{(2a^2 + s^2 + t^2) \sinh \varphi(s, t)} \\
(c) \lim_{N \to \infty} \mathbb{E}\{e^{i\frac{1}{N}(sY_N + tZ_N)} \mid \Omega'\} &= \frac{\sinh(c)}{c} \frac{\varphi(s, t)}{\sinh \varphi(s, t)} \\
(d) \lim_{N \to \infty} \mathbb{E}\{e^{i\frac{1}{N}(sY_N + tZ_N)}\} &= \frac{2a^2 \cosh \varphi(s, t) + (s^2 + t^2) \cosh(2b)}{(2a^2 + s^2 + t^2) \cosh \varphi(s, t)}
\end{align*}

where $C_{a,b} = \frac{2a^2 \sinh(c)}{c (\cosh(c) - \cosh(2b))}$.

Remark 1.2. In the symmetric case $b = 0$, the limiting joint characteristic function in Theorem 1.1(b) reduces to $\frac{1}{2}C_{a,0} \frac{\tanh^4 \sqrt{a^2 + (s^2 + t^2)^2}}{2a^2 + s^2 + t^2}$ due to the half angle identity $\frac{\cosh \varphi - 1}{\sinh \varphi} = \tanh \frac{\varphi}{2}$. By (4.36) the events $\Omega^0$ and $\Omega'$ are macroscopic, with $\lim_{N \to \infty} \mathbb{P}(\Omega') = \frac{\mathrm{cosh}(2b)}{\mathrm{cosh}(c)}$.

Remark 1.3. Consider a specific linear combination $\frac{1}{\sqrt{N}}(k_1 Y_N + k_2 Z_N)$, where for simplicity $k_1^2 + k_2^2 = 2$, for the normalized statistical pair in Theorem 1.1(b). The probability density of the measure determined by the limiting conditional characteristic function given $\Omega^0$ for this linear combination may be found via the Mittag–Leffler partial fraction expansions of $\frac{\tanh(u)}{u}$ and $\frac{u}{\sinh(u)}$. This is shown in Section 4.1, culminating in Example 4.2. The inversion of the limiting characteristic function in the same context for part (a) is included in this example. The densities of Example 4.2 for parts (a)–(b) are not bounded, as illustrated in Figure 4. This method for inverting a limiting characteristic function via partial fraction expansion and term-wise inversion isn’t tractable for the corresponding cases of Theorem 1.1(c)–(d). Instead, we may apply the residue theorem directly to implement Fourier inversion and obtain a bounded density for the limiting distribution of $\frac{1}{\sqrt{N}}(k_1 Y_N + k_2 Z_N)$ in part (d), since the limiting characteristic function is integrable. We illustrate this approach in section 4.1.1.

By the proof of Theorem 1.1 we obtain explicit limiting univariate Laplace transforms for the nonnegative statistics involving runs alone, short runs alone, or steps alone after scaling by $N^2$ as follows. Corollary 1.4 extends [9, Cor. 1], which handles the classical fair gambler’s ruin model (c = 0) for the runs statistic and the steps statistic.

Corollary 1.4. Let $p = \frac{1}{2} + \frac{b}{N}$ and denote $c^2 = a^2 + 4b^2$. For all $\lambda \geq 0$ there hold:

\begin{align*}
(a) \lim_{N \to \infty} \mathbb{E}\{e^{-\lambda R_N/N^2}\} &= \frac{c}{\tanh(c)} \frac{\tanh(\sqrt{c^2 + \lambda})}{\sqrt{c^2 + \lambda}}, \\
(b) \lim_{N \to \infty} \mathbb{E}\{e^{-\lambda R_N/N^2} \mid \Omega^0\} &= C_{a,b} \frac{\sqrt{c^2 + \lambda} \left( \cosh \sqrt{c^2 + \lambda} - \cosh(2b) \right)}{(a^2 + \lambda) \sinh \sqrt{c^2 + \lambda}} \\
(c) \lim_{N \to \infty} \mathbb{E}\{e^{-\lambda R_N/N^2} \mid \Omega'\} &= \frac{\sinh(c)}{c} \frac{\sqrt{c^2 + \lambda}}{\sinh(\sqrt{c^2 + \lambda})}, \\
(d) \lim_{N \to \infty} \mathbb{E}\{e^{-\lambda (R_N + R_N^0 + (\lambda - \lambda')/N^2)}\} &= \frac{a^2 \cosh \sqrt{c^2 + \lambda} + \lambda \cosh(2b)}{(a^2 + \lambda) \cosh \sqrt{c^2 + \lambda}}.
\end{align*}
Moreover the same transforms hold in parts (a)–(d) with $2V_N$, $2V_0$, and $2V'_N$, respectively, in place of $R_N$, $R_0$, and $R'_N$. The same is true also with $\frac{1}{2}L_N$, $\frac{1}{2}L_0$, and $\frac{1}{2}L'_N$, respectively, in place of $R_N$, $R_0$, and $R'_N$.

Our outline for the paper is as follows. In section 2 we review the generating function method via Fibonacci recurrences to establish closed form generating function identities. In section 3 we show that the fundamental approach of section 2 can be generalized for the present model while still maintaining explicit generating function formula. Specifically, in section 3 we develop extended Fibonacci recurrences with a driving term to solve the problem of computing the joint probability generating function of runs, short runs, and steps over stopped excursions. There we make crucial use of the Markov property to establish Lemma 3.1. In section 4 we prove Theorem 1.1, focusing especially on Theorem 1.1(b), because this case arises due to the advent of random stopping. We illustrate the theme for stopping in Example 4.2, and prove Corollary 1.4 in section 4.2. In Theorem 4.4 we establish a joint limit distribution for the pair $(\frac{1}{N}X_{L-1}, \frac{1}{N}H)$ given $E_c$.

2. Method of Proof

Our basic strategy for the proofs is to decompose generating functions according to the events \{H = n\}, 1 ≤ n ≤ N. Our method is based on calculation of the following two joint probability generating functions in turn. Define

\[
\begin{align*}
G_n &= G_n(r, y, z) = \mathbb{E}\left(r^R y^V z^L \mid E_0 \cap \{H = n\}\right), \ 1 \leq n \leq N; \\
K_N &= K_N(r, y, z) = \mathbb{E}\left(r^R y^V z^L \mid E_0 \cap \{1 \leq H \leq N\}\right).
\end{align*}
\]

In this section we focus on the basic step to calculate $G_n$ by means of a certain joint generating function defined by (2.3). The method we follow was first established for the case of runs and steps alone in [9]. A key feature of the method even without the presence of random stopping is to find an explicit formula for the generating function (2.3). The formula we seek is given by Proposition 2.2. The condition that the process exits via the boundaries ±(N + 1) during an attempted excursion is straightforward to handle via the nonnegative first passage calculation that we start to develop in the following.

For 1 ≤ n ≤ N, on \{H ≥ n\}, define the first passage number of steps $L_n$ along the absolute value process \{|X_j|, j ≥ 0\} to reach level n by:

\[
L_n = \inf\{j \geq 1 : |X_j| = n\}.
\]

We define this first passage number $L_n = +\infty$ on the event \{H < n\}; thus \{L_n < \infty\} = \{H ≥ n\}. On $E_0$, denote by $R_n$ the number of runs and by $V_n$ the number of short runs, respectively, along the absolute value path \{(j, |X_j|), j = 0, 1, \ldots, L_n\}. For $n ≥ 2$, define $g_n$ as the following upward conditional joint probability generating function for these counting statistics given two conditions on the path: (1) the path is a first passage path that starts at 0 and stays at or above level $m = 0$ until it first reaches level $n$, and (2) the first two steps of this path are both in the positive direction. Thus for all $n ≥ 2$ we define

\[
g_n = \mathbb{E}\{r^R y^V z^L \mid X_0 = 0, X_1 = 1, X_2 = 2; X_j ≥ 0, j = 0, \ldots, L_n\}.
\]

Since any nonnegative nearest neighbor lattice path that starts at level $m = 0$ and ends when it first reaches level $n$ may be reflected to give a nonpositive path starting from $m = 0$ and reaching level $-n$ for the first time, and since there is a constant conversion factor $(1-p)/p^n$ to find the probability of the reflected path from the probability of the original nonnegative path, if we condition instead on a nonpositive path in (2.3) then we obtain the same generating function $g_n$, that we may refer to as a downward first passage generating function. There is a simple relationship between the generating function $g_n$ of (2.13) written without the imposition of condition (2) and the definition
We have the symmetric case \( g_n \) of (2.3) as long as \( n \geq 2 \); see (2.14). However care must be taken to realize that while \( \hat{g}_1 \) makes sense per se (and \( g_1 \) does not), it is of little use in a recursive method due to the inclusion of the short runs statistic in the analysis.

We introduce a terminology. Let \( m \leq n - 2 \). Call a path that starts at level \( m \) that stays at levels in \([m, n]\) until level \( n \) is reached for a first time as an upward first passage path from level \( m \) to level \( n \). For brevity, we also refer to such paths as making an upward transition. Similarly, call a path that starts at level \( n \) that stays at levels in \([m, n]\) until level \( m \) is reached for a first time as a downward first passage path from level \( n \) to level \( m \), or downward transition. See Figure 2 for one such downward first passage path from \( n = 5 \) to \( m = 0 \). Now even though in Figure 2 the given example of a downward path does satisfy the condition that the first two steps are down, this need not be the case for a downward first passage path in general, though it does need to be the case for paths contributing to a calculation of \( g_n \) of (2.3) by downward first passage paths. To handle this discrepancy, in (2.12) we will introduce the probability generating function \( \kappa \) corresponding to the initial part of a first passage path until there are two steps the same. For all \( \alpha \), we have closed formulae given by (2.9); see [11, (2.1) and (2.3)], or [9, (2.11)–(2.12)]. Define also \( \rho_0 = 1 \).

Define also \( \rho_0 = 1 \). Let \( x \neq 0 \), so that \( h = 1 \) and transition to \( c \) cannot occur, and if further we have the symmetric case \( p = \frac{1}{2} \), then \( \rho_n \) is determined by the classical solution for a fair gambler’s ruin started from \( m = 0 \) to come to the boundary of the interval \([-1, n]\) at state \( n \), namely \( \rho_n = 1/(n+1) \). For \( h < 1 \), even in the symmetric case we no longer have the reciprocal of a linear term in \( n \) as we shall find in Lemma 2.4. That lemma gives for example \( \rho_2 = (ph)^2/(1 - \xi) \), \( \rho_3 = (ph)^3/(1 - 2\xi) \), and \( \rho_4 = (ph)^4/(1 - 3\xi + \xi^2) \). Here we derive

\[
\rho_2 = (ph)^2 \sum_{k=0}^{\infty} (p(1 - p)h^2)^k = (ph)^2/(1 - \xi).
\]

Since the probability of a single nonnegative path for the event defining \( \rho_n \) is simply multiplied by the conversion factor \((1 - p)/p)^n\) to obtain the probability of the reflected nonpositive path for the event under \( \rho_{-n} \), we have that

\[
\rho_{-n} = ((1 - p)/p)^n \rho_n.
\]

The calculation of \( g_n \) is fundamental to our method. Our proofs of this step feature bivariate generalizad Fibonacci polynomials \( \{q_n(x, \beta)\} \) and \( \{w_n(x, \beta)\} \), defined as follows.

**Definition 2.1.** Let \( \beta, x \in \mathbb{C} \). Define sequences \( q_n(x, \beta) \) and \( w_n(x, \beta) \) generated by the following recurrence relations, valid for all \( n \geq 1 \).

\[
q_{n+1} = \beta q_n - x q_{n-1}, \quad q_0 = 0, q_1 = 1; \quad w_{n+1} = \beta w_n - x w_{n-1}, \quad w_0 = 1, w_1 = 1.
\]

The polynomials \( q_n(x, \beta) \) generalize the univariate Fibonacci polynomials \( q_n(x, 1) \), [6, p. 327]; also for the special case \( \beta = 1 \) we have \( w_n(x, 1) = q_{n+1}(x, 1) \). We have \( \{w_n(x, 1)\} = \{1, 1 - x, 1 - 2x, 1 - 3x + x^2, \ldots\} \); the numerical Fibonacci sequence arises with \( x = -1 \). We write an interlacing property of any two term recurrence \( v_{n+1} = \beta v_n - x v_{n-1}, \ n \geq 1 \), with coefficients \( \beta \) and \( x \) independent of \( n \):

\[
v_{n+1}v_{n-1} - v_n^2 = \beta^{-1} x^{-1}(v_3v_0 - v_2v_1), \ \beta \neq 0;
\]

see [9, (2.7)–(2.8)]. By standard generating function techniques the fundamental sequences (2.7) have closed formulae given by (2.9); see [11, (2.1) and (2.3)], or [9, (2.11)–(2.12)]. Define \( \alpha \) as

\[
\alpha = \sqrt{\beta^2 - 4x}. \text{ Then for all } n \geq 1 \text{ we have}
\]

\[
q_n(x, \beta) = \frac{\beta^{-n}}{\alpha} \left((\beta + \alpha)^n - (\beta - \alpha)^n\right); \quad w_n(x, \beta) = q_n(x, \beta) - x q_{n-1}(x, \beta).
\]
The second identity holds by the fact that \(q_1 - xq_0 = 1 = w_1, q_2 - xq_1 = \beta - x = w_2\), and \(q_n\) and \(w_n\) satisfy the same two term recurrence, (2.7). Moreover the following identities hold for all \(n \geq 1\).

\[
\begin{align*}
(\text{i}) & \quad w_{n+1}w_{n-1} - w_n^2 = x^{n-1}(\beta - x - 1); \\
(\text{ii}) & \quad q_{n+1}q_{n-1} - q_n^2 = -x^{-n}; \\
(\text{iii}) & \quad q_nw_{n+1} - w_nq_{n+1} = -x^n.
\end{align*}
\] (2.10)

Indeed, (2.10)(i)-(ii) follow directly from (2.7) and (2.8). By \(w_n = q_n - xq_{n-1}, n \geq 1\), we obtain \(q_nw_{n+1} - w_nq_{n+1} = x(q_{n+1}q_{n-1} - q_n^2)\), so (2.10)(iii) follows from (ii); see [9, Lemma 4].

2.1. Recurrence for \(g_n\). We first establish the general recurrence relations governing the upward generating function \(g_n\) of (2.3). Let \(\gamma\) be a nearest neighbor lattice path. Denote by \(\mu(\gamma)\) the product of the probabilities \(ph\) for a step up and \((1-p)h\) for a step down all along the lattice path. Let \(R(\gamma), V(\gamma),\) and \(L(\gamma)\) denote respectively the number of runs, short runs, and steps along \(\gamma\). For any unnormalized generating function of path statistics \(f = \sum \mu(\gamma)r^{R(\gamma)}y^{V(\gamma)}z^{L(\gamma)}\) over some collection of paths \(\gamma\), we may refer to a (conditional) probability generating function by normalizing the sum \(f\) by \(\sum \mu(\gamma)\). For this purpose we denote \(1 = (1,1,1)\) and evaluation of any expression \(f(r,y,z)\) at \((r,y,z) = 1\) by \(f[1]\). Then we form a probability generating function by normalization to the form \(f/f[1]\).

The condition that the initial two steps are the same for lattice paths in the definition of \(g_n\) of (2.3) yields immediately that

\[g_2 = rz^2.\] (2.11)

It is convenient to focus on a downward path decomposition for \(g_n\) with some \(n \geq 3\). We introduce the following notation. Let \(U\) or \(D\) stand for one step up or down, respectively, in a lattice path, and let \((UD)^k\) be shorthand for \(UDUD \cdots\) with \(k\) repetitions of the pattern \(UD\) for some \(k \geq 0\). Since any downward lattice path from \(n\) to 0 must first reach the level \(m = 1\), and because \(n - 1 \geq 2\) we have an initial factor \(g_{n-1}\) in a product formula for \(g_n\).

Still assuming \(n \geq 3\), any section of a downward lattice path from level \(n\) to \(m = 1\) to for \(g_{n-1}\) must end in \(DD\). Following this, we have either a sequence of steps of the form \((UD)^kUU\), for some \(k \geq 0\), or a terminal sequence \((UD)^kD\), for some \(k \geq 0\). See Figure 2 where we have a transition \((UD)^1UU\) at the first point where level \(m = 1\) is reached. Define a turning point of a lattice path as step up to step down or vice versa. When two lattice paths are concatenated at a common point that is the terminal point of the first path and the initial point of the second path, such that the joining point is a turning point for the whole path, then the runs statistics and short runs statistics each add along the two paths. To handle generating function factors arising from certain key turning points, we introduce:

\[
\omega = 1 - \xi r^2y^2z^2; \quad \kappa = (1 - \xi)/\omega; \quad \tau = 1 + \xi r^2z^2y(1-y); \quad \eta = \tau\kappa.
\] (2.12)

By (2.12), \(\kappa[1] = \tau[1] = 1\). We may easily see that \(\kappa\) is the probability generating function for all paths of the form \((UD)^k\), for some \(k \geq 0\), and also for all paths of the form \((DU)^k\), for some \(k \geq 0\). The role of \(\kappa\) may be seen as follows. Define

\[\hat{g}_n = E(r^{R_n}y^{V_n}z^{L_n}|X_0 = 0, X_j \geq 0, j = 0, \ldots, L_n), \quad n \geq 2.\] (2.13)

The probability generating function \(\hat{g}_n\) plays a similar role as \(g_n\) except, contrary to the definition (2.3), the paths defining \(\hat{g}_n\) do not require the first two steps to be the same. Then, because any upward path for \(\hat{g}_n\) with \(n \geq 2\) in (2.13) can be decomposed into a preamble \((UD)^k\) for some \(k \geq 0\) followed by a path that starts with \(UU\), and because the change from the preamble to \(UU\) is a turning point, we have

\[\hat{g}_n = \kappa g_n, \quad \text{for all } n \geq 2,\] (2.14)
where $\kappa$ is defined by (2.12). Coming back to the definition (2.12), we observe that $z\eta$ is the probability generating function of all terminal sequences $(UD)^k D$, $k \geq 0$, for downward first passage paths as follows:

$$(1 - p)hz(1 + \xi^2 y z^2 + \xi^2 r^4 y^3 z^4 + \cdots) = (1 - p)hz \cdot \frac{\omega + \xi \tau^2 y z^2}{\omega} = \text{const. } z\tau/\omega. \quad (2.15)$$

Here and in the sequel, const. denotes a generic constant, independent of the generating function variables. By symmetry in $\xi$ after the interchange of the roles of $ph$ and $(1 - p)h$, we have that $z\eta$ is also the probability generating function of all terminal sequences $(DU)^k U$, $k \geq 0$, for upward first passage paths from $-n$ to 0. Suppose still $n \geq 3$, and that the continuation of a path in a downward representation of $g_n$ after the first downward passage to level $m = 1$ is not yet passing into a terminal sequence. Then the path makes an upward first passage form level $m = 1$ to level $n$ again (or not), and the pattern “upward transition from level $m = 1$ to level $n$ followed by downward transition to level $m = 1$” repeats for an indefinite number of times, $\ell \geq 0$. After the path would no longer reach level $n$ again from the starting level $m = 1$, and if $n - 1 \geq 2$, then we would similarly take account of up and down transitions between levels $m = 1$ and $n - 1$, and so forth. Since for any $j \geq 2$ we have that $\rho_j$ is the total probability of paths making an upward first passage from level $m = 1$ to level $j + 1$, and $\rho_{-j}$ is the total probability of paths making a downward first passage from level $j + 1$ to level $m = 1$, and since runs and short runs statistics add for the concatenation of paths that connect at a turning point, we have that $\rho_j \rho_{-j} g_j^2 = \rho_j \rho_{-j} \kappa^2 g_j^2$ is the unnormalized generating factor for an upward transition from level $m = 1$ to level $j + 1$ followed by a downward transition to level $m = 1$. Accordingly, for each $j \geq 2$, we define

$$\lambda_j = \sum_{\ell=0}^{\infty} (\rho_j \rho_{-j} \kappa^2 g_j^2)^\ell = \frac{1}{1 - \rho_j \rho_{-j} \kappa^2 g_j^2}. \quad (2.16)$$

It follows that the factor $\lambda_j/\lambda_j[1]$ is a probability generating function for a class of paths, including the empty path, starting and ending at the same level 1, where each path besides the empty path makes a positive number of consecutive up and down first passage transitions between levels $m = 1$ and $j + 1$.

Define $M_1 = n$ as the starting level for a downward transition from $n$ to $m = 0$. The path must reach the level $m = 1$ for a first time. Define $M_2$ as the maximum possible level in the remainder of the downward lattice path after the first passage down to level $m = 1$, as long as $M_2 \geq 3$. Here the successive maximum levels $n = M_1 \geq M_2 \geq \cdots \geq M_r$ over the whole future of the path, determined in turn from the points of each of its returns to level $m = 1$ from the previous such maximum, and determined at the first opportunity under this condition, are the future maxima (cf. [9]) of a downward path from level $n \geq 3$ to level $m = 0$. See Figure 2, in which we have $M_1 = 5$, $M_2 = 5$, $M_3 = 4$, $M_4 = 3$; there is no second future maximum of level 4, for example, because there is no return to level 1 between the two successive peaks at level 4. By definition here, we have $M_r \geq 3$, and the downward path goes into a terminal sequence after a return to level 1 from

![Figure 2. Downward First Passage from Level 5 to Level 0: Future Maxima $M_1 = 5$, $M_2 = 5$, $M_3 = 4$, $M_4 = 3$.]
Eventually the path will never rise to level $n$ again but only to lower future maxima at levels $3 \leq m \leq n - 1$; thus the product $\lambda_{n-1}\lambda_{n-2}\ldots\lambda_2$. As noted above the factor $z\eta$ corresponds to the terminal sequence. Therefore we have

$$g_n = \text{const. } z\eta \cdot g_{n-1} \prod_{j=2}^{n-1} \lambda_j, \text{ for all } n \geq 3.$$  (2.17)

for a normalization constant such that $g_n[1] = 1$. By (2.17) we retrieve a closed recurrence for $g_n$. Indeed, we simply have $g_{n+1}/g_n = \text{const. } g_n\lambda_n/g_{n-1}$. Hence,

$$g_{n+1} = \text{const. } g_n^2\lambda_n/g_{n-1}, \text{ for all } n \geq 3.$$  (2.18)

2.2. Closed form for $g_n$. In this section we solve the recurrence (2.18) by employing the Fibonacci recurrences (2.7) for certain parameters $x$ and $\beta$ that carry the generating function variables, but for different initial conditions than the basic ones in (2.7), where these initial conditions will also depend on the generating function variables. Thus, motivated by the idea that a certain sequence $w^*_n$ that satisfies a Fibonacci recurrence $w^*_{n+1} = \beta w^*_n - xw^*_n$, $n \geq 2$, will serve as the denominators of $g_n$, or, by (2.14), $\hat{g} = \kappa g_n/(1 - \xi)g_n/\omega$, we first define

$$x = x(r,y,z) = \xi z^2 r^2; \quad \beta = \beta(r,y,z) = 1 + \xi z^2 \left(1 - r^2(y^2 + z(1 - y)^2z^2)\right),$$  (2.19)

where $\tau$ in the definition of $x$ is defined by (2.12). To explain briefly how $x$ and $\beta$ may be found, we already know from (2.11) that $g_2 = rz^2$. Then one can compute $g_3$ from (2.17), since by (2.5), (2.12), and (2.11), we have $\rho_{2\rho - 2}\kappa^2 = \xi^2/\omega^2$ and thus $\lambda_2 = \frac{\omega^2}{\omega^2 - \xi^2r^2z^4}$. So, by $\eta = (1 - \xi)\tau/\omega$, we have

$$g_3 = z\eta\lambda_2/\lambda_2[1] = \frac{1}{\lambda_2[1]} \frac{z\eta r^2\omega^2}{\omega^2 - \xi^2r^2z^4} = 1 - \xi \frac{\omega\tau rz^3}{\lambda_2[1]} \frac{\omega^2}{\omega^2 - \xi^2r^2z^4}. $$  (2.20)

Thus, using the denominator of $g_3$ we will have guessed that $w^*_3 = \omega^2 - \xi^2r^2z^4$. We also guess that $w^*_2 = \omega$ by the form of $g_2$. To determine $x$ we now define

$$w^*_{n+1} = \beta w^*_n - xw^*_n, \text{ for } n \geq 2, \text{ with } w^*_1 = 1, w^*_2 = \omega,$$  (2.21)

for $\omega$ defined by (2.12). Proceeding one step further via (2.18) to find $g_4$ and therefore $w^*_4$ in its denominator, one can then compute $x$ by applying the commutation relation (2.8) to find $x = \left((w^*_3)^2 - w^*_2w^*_5\right)/\left((w^*_3)^2 - w^*_2w^*_4\right)$. Once we have $x$, we find $\beta$ of (2.19) via (2.21), and we also extend the definition (2.21) to $n = 0$ by solving (2.21) backwards. Thus define

$$w^*_0 = (\beta - \omega)/x,$$  (2.22)

for $x$ and $\beta$ as defined by (2.19). By calculating $g_n$ for small $n$ we glean the following.

**Proposition 2.2.** We have that the following formula is valid for all $n \geq 2$.

$$g_n = C_n\omega r z^n r^{n-2}/w^*_n, \text{ with } C_n = w^*_n[1]/(1 - \xi).$$

To prove this proposition we need two lemmas to follow.

**Lemma 2.3.** Let $w^*_n$ be defined as the solution to (2.21). Then for all $n \geq 1$ we have:

$$(w^*_n)^2 - w^*_{n+1}w^*_n = \xi^2r^2z^4x^{n-2}.$$  (2.23)

**Proof.** By the definition (2.21) and by (2.8) we have:

$$(w^*_n)^2 - w^*_{n+1}w^*_n = -\beta^{-1}x^{n-1}(w^*_3w^*_0 - w^*_2w^*_1).$$  (2.23)

By direct calculation using (2.19) and (2.21)–(2.22) we find $w^*_3w^*_0 - w^*_2w^*_1 = -\xi^2r^2z^4\beta/x$. Hence the lemma follows by substitution of this last formula into (2.23). □
Lemma 2.4. Let \( n \geq 1 \). Then we have the following formula:

\[
\rho_n = \frac{(ph)^n}{w_n^*[1]}.
\]

Here we have \( w_n^*[1] = w_n(\xi, 1) \), as defined by (2.7).

Proof. By the same argument as used for (2.17),

\[
\rho_{n+1} = \frac{(1-p)h}{1-\xi} \cdot \rho_n \cdot \prod_{j=2}^{n} \frac{1}{1-\rho_j \rho_{j-1}}; \quad \text{for all } n \geq 2,
\]

(2.24)

where the factor \((1-p)h/(1-\xi)\) corresponds to the terminal sequence; compare (2.15) and Figure 2. Since by (2.21) we have \( \beta^2 = 1 \), by (2.25) we have \( \rho_1 = ph \), for all \( n \geq 2 \).

Obviously \( \rho_1 \) satisfies the statement of the lemma by \( w_1^* = 1 \). By (2.5) and \( w_2^*[1] = \omega^*[1] = 1-\xi \), we have that also \( \rho_2 = (ph)^2/w_2^+[1] \). Assume by induction that the statement of the proposition holds for all \( 1 \leq m \leq n \) for some \( n \geq 2 \). Then by (2.25) we have

\[
\rho_{n+1} = \frac{\rho_n^2}{\rho_{n-1}} \frac{1}{1-((1-p)/p)n \rho_n^2}; \quad \rho_1 = ph, \quad \text{for all } n \geq 2.
\]

(2.25)

But by Lemma 2.3 we have \( w_n^*[1]^2 - \xi^n = w_{n+1}^*[1]w_{n-1}^*[1] \). Therefore after this substitution and subsequent cancellations in (2.26) the induction step is complete. Since \( \beta^2 = 1 \) we then have \( \lambda_n = \beta \rho_n \), and the definition (2.21), then by (2.20) we have that the proposition holds for \( n = 3 \) as well up to the form of the constant.

Proof of Proposition 2.2. First, since by (2.21) we have \( w_2^* = \omega^* \), when \( n = 2 \) the given formula for \( g_2 \) evaluates to \( C_2 \omega r^2z^2/\omega = rz^2 \), as long as we take \( C_2 = 1 \). But the formula for \( C_2 \) given by the proposition is, by (2.12), indeed equal to \( \omega[1]/(1-\xi) = 1 \). For \( n = 3 \), since indeed \( \omega - \xi^2 r^2 z^4 = \beta \omega - x \cdot 1 = \omega_3 \) by our choice of \( x \) and \( \beta \) and the definition (2.21), then by (2.12) we have that the proposition holds for \( n = 3 \) as well up to the form of the constant.

But because \( \tau[1] = 1 \) and \( \omega[1] = 1-\xi \), the form of the constant \( C_3 \) falls out from the formula \( g_3 = \omega + \tau x^2/3 \) since by definition \( g_3[1] = 1 \).

We now proceed by induction. Since we have verified the cases \( n = 2, 3 \), we assume the statement of the proposition holds with \( m \) in place of \( n \) for all \( 2 \leq m \leq n \), where \( n \geq 3 \). We first compute \( \lambda_n \). By Lemma 2.4 and (2.6) we have \( \rho_n \rho_{-n} - \xi^n/w_n^*[1]^2 \). Therefore by the induction hypothesis we have from the definition (2.16) that \( 1 - 1/\lambda_n \) is given by

\[
1 - \frac{1}{\lambda_n} = \rho_n \rho_{-n} \xi^n = \frac{\xi^n \kappa^2 C_2 \omega^2 r^2 z^{2n} \tau^{2n-4}}{w_n^*[1]^2(w_n^* \xi^n)^2}.
\]

(2.26)

But by the induction hypothesis and (2.12) we have \( C_2 \xi^n \omega^2/w_n^*[1]^2 = 1 \). We also substitute \( \tau^2 = \xi^{-1} z^{-2} x \) to write that \( \xi^n r^2 z^{2n} \tau^{2n-4} = \xi^2 r^2 z^4 x^{n-2} \). Hence we have

\[
\lambda_n = \frac{w_n^*[1]^2}{w_n^*[1]^2 - \xi^2 r^2 z^4 x^{n-2}}.
\]

By Lemma 2.3 the denominator of this last expression for \( \lambda_n \) is simply \( w_{n+1}^*w_{n-1}^* \). Therefore \( \lambda_n = \frac{w_n^*[1]^2}{w_{n+1}^*w_{n-1}^*} \). By (2.18) we then have

\[
g_{n+1} = \text{const.} \cdot \frac{\omega^2 r^2 z^{2n} \tau^{2n-4} w_{n-1}^*}{\omega r z^{n-1} \tau^{n-3}(w_n^* \xi^n)^2} \cdot \frac{(w_n^*)^2}{w_n^*[1]^2} = \text{const.} \cdot \frac{\omega r z^{n+1} \tau^{n-1}}{w_n^*}. \quad (2.27)
\]
Since the normalizing constant must make $g_{n+1}[1] = 1$, and since $\tau[1] = 1$ and $\omega[1] = 1 - \xi$, it is now clear that $g_{n+1}$ is of the desired form and the induction step is complete. 

Even though for the proof Proposition 2.2 we didn’t need a companion Fibonacci sequence $q_n^*$ for $w_n^*$, that satisfies the Fibonacci recurrence (2.7) but with different initial conditions than for the basic sequence $q_n$, we will want such a companion sequence in section ?? to prove a formula for $K_N$ of (2.1). To guess initial values $q_1^*$ and $q_0^*$ we are motivated by the form of $K_N$ in [9, Theorem 1]; note that in [9] short runs are not considered and there $y$ plays the role of the current generating function variable $r$ for runs. Thus we guess that $K_N = \text{const. } r^2 z^2 q_n^*/w_n^*$. Since we can compute directly that $K_1 = r^2 y^2 z^2$, we take $q_1^* = y^2$. Further, from its definition, $K_2 = \text{const. } (\xi r^2 y^2 z^2 + \xi^2 r^2 z^4)/\omega$. From this and direct calculation, assuming $q_2^* = y^2 + \xi z^2 - \xi^2 r^2 y^4 z^2$, because with this definition of $q_n^*$, recalling $w_2^* = \omega$, we indeed have $K_2 = (1 - \xi)r^2 z^2 q_2^*/w_2^*$. Accordingly, for all $n \geq 2$, define

$$q_{n+1}^* = \beta q_n^* - x q_{n-1}^*; \text{ with } q_1^* = y^2, \ q_2^* = y^2 + \xi z^2 - \xi^2 r^2 y^4 z^2.$$ 

Finally define $q_0^*$ by putting $n = 1$ into (2.28). We record the result by direct computation as follows.

$$q_0^* = (\beta q_1^* - q_2^*)/x = \left(y^2 - 1 - \xi r^2 y^2 z^2 (y - 1)^2\right)/r^2.$$ 

Therefore, in summary, for the recurrences $q_n(x, \beta)$ and $w_n(x, \beta)$ of (2.7), we have

$$q_n^* = c_1 q_n(x, \beta) + c_2 w_n(x, \beta), \ c_2 = q_0^*, \ c_1 = y^2 - c_2;$$
$$w_n^* = c_1' q_n(x, \beta) + c_2' w_n(x, \beta), \ c_2' = w_0^*, \ c_1' = 1 - c_2',$$ 

where $w_n^*$ and $q_n^*$ are determined by (2.22) and (2.29) and we substitute $x$ and $\beta$ by (2.19).

We now obtain the analogue of (2.10)(iii) corresponding to the sequences $q_n^*$ and $w_n^*$.

**Lemma 2.5.** For all $n \geq 0$ we have

$$w_n^* q_{n+1}^* - w_{n+1}^* q_n^* = \frac{x^n}{r^2}.$$ 

**Proof.** From (2.30) and (2.9) we have

$$w_n^* = w_0^* w_n + (1 - w_0^*) q_n = w_0^* (q_n - x q_{n-1}) + (1 - w_0^*) q_n = q_n - x w_0^* q_{n-1};$$
$$q_n^* = q_0^* w_n + (y^2 - q_0^*) q_n = q_0^* (q_n - x q_{n-1}) + (y^2 - q_0^*) q_n = y^2 q_n - x q_0^* q_{n-1}.$$ 

It follows by direct substitution from (2.31) that $w_n^* q_{n+1}^* - w_{n+1}^* q_n^*$ is given by $(q_{n+1} - x w_0^* q_n) (y^2 q_n - x q_0^* q_{n-1}) - (q_n - x w_0^* q_n) (y^2 q_{n-1} - x q_0^* q_n)$. First, by expanding this last expression as a sum of 8 products, there are two pairs of terms that cancel, one pair involving the product $q_n q_{n+1}$, the other involving the product $q_{n-1} q_n$. Second, the other 4 terms occur in two pairs such that we now have $w_n^* q_{n+1}^* - w_{n+1}^* q_n^* = (x q_0^* - xy^2 w_0^*) (q_{n+1} q_n - q_0^2 q_{n-1})$. Thus, since by (2.10)(ii) we have $q_{n+1} q_n - q_0^2 = -x^{n-1}$, the statement of the lemma follows for $n \geq 1$ from the direct computation that $q_0^2 - y^2 w_0^* = -1/r^2$. The case $n = 0$ follows by direct computation.

**Proposition 2.6.** Let $w_n^*$ and $q_n^*$, respectively, be defined by (2.21) and (2.28). Then the generating function $K_N$ of (2.1)(ii) has the following formula:

$$K_N = k_N r^2 z^2 q_N^*/w_N^*, \text{ for all } N \geq 1; \text{ with } k_N = 2\xi/\mathbb{P}(E_0 \cap \{1 \leq H \leq N\}).$$
Proof. The idea of the proof parallels the construction of convergents to a continued fraction, [3, Chp. III]. First, the probability generating function \( G_n \) of (2.1) may be written for all \( n \geq 2 \) as follows.

\[
G_n = \begin{cases} 
 z \eta g_{n-1} \kappa g_n, & \text{if } n \geq 3, \\
 \kappa \tau^2 z^4, & \text{if } n = 2.
\end{cases}
\]  

(2.32)

To explain (2.32), we have that the factor \( z \eta \) corresponds to the reverse of the terminal sequence in Figure 2 as an initial factor for a nonnegative excursion, followed by the factor \( g_{n-1} \) to reach level \( n \) for a first time. The factor \( \kappa \eta g_n = g_n \) of (2.14) is the probability generating function of a downward first passage from level \( n \) to level \( m = 0 \). As already argued for \( z \eta, \kappa, \) and \( g_n \), if we condition on a nonpositive excursion of height \( n \) in (2.1) we obtain the same formula (2.32).

Now the generating function \( K_N \) of (2.1) is written

\[
K_N = \sum_{n=1}^{N} G_n \mathbb{P}(E_0 \cap \{H = n\})/\mathbb{P}(E_0 \cap \{1 \leq H \leq N\}).
\]  

(2.33)

Furthermore, by Lemma 2.4,

\[
\mathbb{P}(E_0 \cap \{H = n\}) = ph\rho_{n-1} - n + (1 - p)h\rho_{n+1} = \frac{2\xi_n}{w_{n-1}^*[1]w_n^*[1]},
\]  

(2.34)

where we wrote \((ph)(ph)^{n-1}(1-p)h\)^\(n\) + ((1-p)h)((1-p)h)^\(n\)-1(ph)^\(n\) = \(2\xi_n\). By Proposition 2.2 and (2.32)–(2.34), we have that \(\mathbb{P}(E_0 \cap \{1 \leq H \leq N\})K_N\) is written for all \(N \geq 2\) as

\[
2\xi \tau^2 y^2 x^2 + \frac{2\xi \tau^2 x^2 z^4}{\omega} + \sum_{n=3}^{N} C_{n-1} C_n \frac{2\xi_n(1-\xi)^2}{w_{n-1}^*[1]w_n^*[1]} w_n^*[1]w_{n-1}^* w_n^*,
\]  

(2.35)

where by (2.12) we applied \(\eta \kappa \omega^2 = (1-\xi)^2 \tau\) to reduce \(z \kappa \omega \tau z^n \kappa \omega \tau z^n \tau^n - 2\) to the form \((1-\xi)^2 \tau^2 x^n - 2\), and we likewise applied \(\kappa = (1-\xi)/\omega\) for the term with \(n = 2\). Since also by Proposition 2.2 we have \(C_{n-1} C_n w_{n-1}^*[1]w_n^*[1] = 1\), we are left with the problem of evaluating

\[
\sum_{n=3}^{N} C_{n-1} C_n \frac{2\xi_n(1-\xi)^2}{w_{n-1}^*[1]w_n^*[1]} w_n^*[1]w_{n-1}^* w_n^* = 2\xi \tau^2 x^2 \sum_{n=3}^{N} \frac{x^{n-1}\tau^{-2}}{w_{n-1}^* w_n^*} \]  

(2.36)

as the main term of (2.35). Here we rearranged some factors using \(\xi_n z^n \tau^n - 2 = x^{n-1}\). Notice that the \(n = 2\) term of (2.35) also conforms to satisfy \(\frac{2\xi x^2 z^4}{\omega} = 2\xi \tau^2 x^2 \frac{z^4}{w_1^* w_2^*}\). Thus by (2.35)–(2.36) we have

\[
\mathbb{P}(E_0 \cap \{1 \leq H \leq N\})K_N = 2\xi \tau^2 y^2 x^2 + 2\xi \tau^2 x^2 \sum_{n=3}^{N} \frac{x^{n-1}\tau^{-2}}{w_{n-1}^* w_n^*}.
\]  

(2.37)

Next we collapse the summation in (2.37). Indeed, we claim that for all \(N \geq 2\) we have

\[
\sum_{n=2}^{N} \frac{x^{n-1}\tau^{-2}}{w_{n-1}^* w_n^*} = \frac{q_{N}^*}{w_n^*} - \frac{q_1^*}{w_1^*}.
\]  

(2.38)

To prove the claim (2.38) by induction, we verify both the basis and the induction step at once. Indeed, by Lemma 2.5 we have

\[
\frac{x^N \tau^{-2}}{w_n^* w_{n+1}^*} = \frac{q_{N+1}^*}{w_{n+1}^*} - \frac{q_N^*}{w_n^*}, \text{ for any } N \geq 0.
\]
Therefore the proof of (2.38) by induction is complete. Finally, notice that the \( n = 1 \) term in (2.37) is simply \( 2\xi^2 y^2 z^2 = 2\xi^2 y^2 z^2 q_1^w \) by \( q_1^w = y^2 \) and \( w_1^* = 1 \). Therefore by (2.37)–(2.38) we have that the proof is complete.

**Remark 2.7.** If \( a = 0 \), so \( h = 1 \), and if also \( p = \frac{1}{2} \), then the statement of Proposition 2.6 becomes

\[
K_N = \frac{1}{2} \mathbb{P}(1 \leq H \leq N)^{-1} 2^{2} \mathbb{E} \mathbb{E} \left[ y^2 z^2 q_N^w \right] = \frac{N+1}{2N^3} 2^2 \mathbb{E} \mathbb{E} \left[ q_N^w \right],
\]

as shown by the symmetric, homogeneous case of [10, Prop. 16.3.17].

3. Joint Generating Function over a Stopped Excursion.

The method of section 2 applies to bona fide excursion statistics. In this section we extend the method by working instead over an excursion attempt that is interrupted by stopping. On the event \( E_c = E_c \cap \{ H \leq N \} \) that the first excursion attempt is stopped, define \( R^o = R^o_N \), \( V^o = V^o_N \), and \( L^o = L^o_N \) as the number of runs, short runs, and steps, respectively, on this stopped excursion. We apply the same convention as described in the paragraph preceding the statement of Theorem 1.1; that is, for the definition of \( R^o \) we shall count a new run on a final step to \( c \) in the transition from epoch \( L - 1 \) to \( L \) if the step just before this final transition is away from the \( j \)-axis, but we do not count a short run for the exit to \( c \) in this situation because it is as though the exit to \( c \) is a long run. Further, even if there is a single step toward the \( j \)-axis after a turn in the path and then the exit to \( c \), we clarify that we do not count a short run on this two–step combined run; it is as though the single step and exit to \( c \) together are in the same direction so make a long run. We now define

\[
G_n^o = \mathbb{E} \left\{ y^R z^{V^o} L^{L^o} \mid E_c \cap \{ H = n \}; \ X_0 = 0; \ X_j \geq 0, \ j = 0,1,\ldots L - 1 \right\},
\]

\[
K_N^o = \mathbb{E} \left\{ y^R z^{V^o} L^c \mid E_c \cap \{ 1 \leq H \leq N \} \right\};
\]

where \( G_n^o \) is defined for all \( 1 \leq n \leq N \). In words we have that \( G_n^o \) is the joint probability generating function of runs, short runs, and steps along all nonnegative stopped excursions of height \( n \). On \( E_c \) the stopped excursion path comes to a jump–off point \((L - 1, X_{L - 1})\) before exiting to state \( c \). Our aim is to compute \( K_N^o \) of (3.1) in Proposition 3.5, and our method is to accomplish this is to first compute \( G_n^o \).

We now focus on \( G_n^o \) for \( n \geq 2 \). For any finite nonnegative stopped nearest neighbor lattice path \( \gamma \) started from level \( m = 0 \), denote \( R^o(\gamma) \) and \( L^c(\gamma) \) respectively as the number of runs and steps along \( \gamma \) with the same convention regarding stopped excursion paths as before. Since the generating function \( G_n^o \) is a normalized infinite sum \( \sum \mu(\gamma) r^{R^o(\gamma)} y^{V^o(\gamma)} z^{L^c(\gamma)} \) such that \( G_n^o[1] = 1 \), where the sum runs over all stopped excursions \( \gamma \) of maximum level \( n \), we may break up the sum in this expression by decomposing the collection of paths \( \gamma \) into those paths that have a jump–off point at level \( k \). This can be formalized in terms of generating functions as follows. For any \( 1 \leq k \leq n \) and \( 1 \leq n \leq N \) define

\[
G_{n,k}^o = \mathbb{E} \left\{ y^R z^{V^o} L^c \mid E_c \cap \{ H = n \}; \ X_j \geq 0, j = 0,\ldots, L - 1; \ X_{L - 1} = k \right\}.
\]

Then define the probability generating function \( g_{n,k}^o \) implicitly by

\[
G_{n,k}^o = \left\{ \begin{array}{ll}
\mathbb{E} \mathbb{E} \left[ \sum \mu(\gamma) r^{R^o(\gamma)} y^{V^o(\gamma)} z^{L^c(\gamma)} \right] & \text{if } 1 \leq k \leq n \text{ and } n \geq 3,
\mathbb{E} \mathbb{E} \left[ \sum \mu(\gamma) r^{R^o(\gamma)} y^{V^o(\gamma)} z^{L^c(\gamma)} \right] & \text{if } 1 \leq k \leq 2 \text{ and } n = 2;
\end{array} \right.
\]

where we recall the definition of \( \eta \) by (2.12). Because the paths that contribute to \( G_{n,k}^o \) must stay above the \( j \)-axis save for the starting point \( m = 0 \) and reach the level \( n \) for a first time, then, for all \( n \geq 3 \), corresponding to this upward first passage we have a factor \( z \eta g_{n-1}^o \) where \( z \eta \) corresponds to
Lemma 3.1. Let $k$ jump–off point at level $n$, the maximum level $n$ has been first achieved, with the condition that this remaining portion achieves levels only in $[1, n]$ until a jump–off point at level $k$.

Let $n \geq 2$. Then for all $3 \leq k \leq n$, we have

$$g_{n,k}^\circ = r \frac{g_n}{\tau g_{k-1}}.$$  

For the case $k = 2$ we have $g_{n,2}^\circ = \kappa g_n/z$. For the case $k = 1$ we have $g_{n,1}^\circ = \kappa g_n$.

Proof. First fix $3 \leq k \leq n$. Consider a stopped nonnegative excursion with maximum level $n$ and jump–off point at level $k$. Denote by $\gamma_{n,k}$ the part of the stopped excursion path starting when the maximum level $n$ is first achieved until the jump–off epoch $L = 1$. We may extend the stopped excursion to a full excursion by appending a path $\gamma_{k,0}$, starting at the jump–off point, such that $\gamma_{k,0}$ goes from level $k$ to $k – 1$ at the first step, its level thereafter stays in $[0, k – 1]$, and $\gamma_{k,0}$ ends by reaching level 0 for a first time. We can denote then the downward first passage from level $n$ to $m = 0$ of the full excursion, starting when the maximum level $n$ is first achieved until the excursion ends, as the concatenation $\gamma_n = \gamma_{n,k}^\circ \gamma_{k,0}$. In Figure 3 we have $\gamma_5 = \gamma_{5,3} \gamma_{3,0}$.

To compute $g_{n,k}^\circ$ of (3.3), we have that there is a one–to–one correspondence between all downward first passage paths $\gamma_n$ from level $n$ to 0 (that stay at levels of $[1, n]$ until a final step) and all possible concatenations $\gamma_{n,k}^\circ \gamma_{k,0}$, where $\gamma_{n,k}^\circ$ stays at levels in $[1, n]$ until level $k$ is achieved, and $\gamma_{k,0}$ is a single step down from $k$ to $k – 1$ concatenated with a downward first passage path from $k – 1$ to 0. That is because any such $\gamma_n$ must attain level $k$ for a last time. Thus we simply divide $\gamma_n$ into the first portion until this last visit to level $k$, calling this portion $\gamma_{n,k}^\circ$, and calling the remaining portion $\gamma_{k,0}$. Assume first that $3 \leq k \leq n$. Define $g_{k,0} = \text{const.} \sum \mu(\gamma_{k,0})z^{R(\gamma_{k,0})}y^{V(\gamma_{k,0})}z^{L(\gamma_{k,0})}$, where the constant is determined to make $g_{k,0}[1] = 1$, the sum runs over all paths $\gamma_{k,0}$ discussed above, and where $R(\gamma_{k,0})$, $V(\gamma_{k,0})$, and $L(\gamma_{k,0})$, respectively, denote the number of runs, short runs, and steps along $\gamma_{k,0}$ and $\mu(\gamma_{k,0})$ is the product of probabilities of steps along the path $\gamma_{k,0}$. We claim that

$$g_{k,0} = rz \eta g_{k-1}, \quad \text{for all } k \geq 3,$$  

(3.4)

where $\eta$ is given by (2.12). Indeed, starting from the jump–off point at level $k$, the sequence $D(DU)^\ell$, for some $\ell \geq 0$, must appear before any path representative of $g_{k-1}$. We already calculated the generating function $z\eta$ of all such sequences. The factor $g_{k-1}$ arises for the remainder of the paths $\gamma_{k,0}$ from the fact that after the sequence $D(DU)^\ell$ the path continues with $DD$ by $k \geq 3$. Therefore (3.4) is verified. Hence, up to reckoning the number of runs, short runs, and steps accounted in $\gamma_{n,k}^\circ$ concatenated with the one–step transition to $c$, and $\gamma_{k,0}$ separately, versus the number of runs, short runs, and steps in the concatenation $\gamma_n = \gamma_{n,k}^\circ \gamma_{k,0}$, by (3.4) and the Markov property we have

$$g_{n,k}^\circ \cdot rz \eta g_{k-1} = rz \kappa g_n, \quad \text{for all } 3 \leq k \leq n.$$  

(3.5)
Here the right side, besides the factor \( rz \), is accounted for by (2.14) since the probability generating function of all paths \( \gamma_n \) is \( \hat{g}_n = \kappa g_n \), for all \( n \geq 2 \). Our reckoning of the factor \( r \) on the right side in (3.5) is as follows. (1) Suppose the step of \( \gamma_{n,k}^\circ \) that comes to the jump–off point at level \( k \) is in the direction of the \( j \)–axis. Then there is a run accounted for in the descent of \( \gamma_{n,k}^\circ \) to this point, while the factor \( z \eta g_{k-1} \) also accounts for a run along the same descent continued from the jump–off point into a path \( \gamma_{k,0} \). Thus we have counted an extra run for this continued descent on the left side of (3.5), so we compensate by adding a factor of \( r \) on the right side. (2) Suppose instead that the step of \( \gamma_{n,k}^\circ \) that comes to the jump–off point at level \( k \) is in the opposite direction of the \( j \)–axis. Then the descent at the beginning of \( \gamma_{k,0} \) from the jump–off point is a fresh descent, so no run is added by the product on the left side of (3.5), but in this case we agreed to add a run for \( \gamma_{n,k}^\circ \) for the transition to the stopping state \( c \). Thus the extra factor of \( r \) again in this situation for the right side of (3.5). Our reckoning for the factor of \( z \) on the right side of (3.5) is that there is a step counted in all cases for the transition to \( c \) so we add a factor of \( z \) on the right side for this step; the factor of \( z \) on the left side is already from (3.4).

Finally, there are no short runs accounted for by the factor \( z \eta g_{k-1} \) in the descent from level \( k \) of the paths \( \gamma_{k,0} \) at the jump–off point due to the initial steps \( DD \) in the path \( \gamma_{k,0} \). Therefore, since we do not add a short run for the transition to \( c \) and since the descent either through or starting fresh from the jump–off point will have no short run associated with it, we have verified (3.5).

From (3.5) we obtain the statement of the lemma for \( k \geq 3 \) since \( \eta = \kappa \tau \) by (2.12). For \( k = 2 \) we must have that \( g_{2,0} = rz^2 \). So we replace the expression \( z \eta g_{k-1} \) from (3.4) on the left side of (3.5) by \( rz^2 \) to obtain \( g_{n,2} = \kappa g_n / z \). For \( k = 1 \), the approach to the jump–off point in \( \gamma_{n,k}^\circ \) must be toward the \( j \)–axis. Therefore \( \gamma_{1,0} \) is simply one step down, and there in no added run for the exit to \( c \) so \( \hat{g}_{n,1} = \hat{g}_n = \kappa g_n \).

Once more, in concert with (3.3), define \( g_n^\circ \) implicitly by

\[
G_n^\circ = \begin{cases} 
  z \eta g_{n-1} g_n^\circ & \text{if } n \geq 3, \\
  rz^2 g_n^\circ & \text{if } n = 2.
\end{cases}
\] (3.6)

In parallel with (3.3), define \( \rho_{n,k}^\circ \) and \( \rho_{-n,-k}^\circ \) for all \( 1 \leq k \leq n \) and \( 1 \leq n \leq N \) by

\[
phcN \cdot \rho_{n-1}^\circ \rho_{n,k}^\circ = \mathbb{P}(E_c \cap \{ \tilde{H} = n \}; \ X_j \geq 0, j = 0, \ldots, L - 1; \ X_{L-1} = k).
\] (3.7)

In words, \( \rho_{n,k}^\circ \) is the total probability of all nearest neighbor lattice paths among paths starting from level \( n \) that stay in levels of \([1, n]\) and and stop at level \( k \). The factor \( c_N \) on the left accounts for the fact that there is a factor \( c_N \) implicit for the transition to \( c \) on the right side of (3.7). We define \( \rho_{-n,-k}^\circ \) analogously by reflection of the paths for \( \rho_{n,k} \): \( \rho_{-n,-k}^\circ \) is the total probability of all nearest neighbor lattice paths among paths starting from level \( -n \) that stay in levels of \([-n, -1]\) and and stop at level \( -k \). By the same argument given for (2.6), where now paths representing \( \rho_{n,k} \) have \( n - k \) more down steps than up steps, we have

\[
\rho_{n,-k}^\circ = (p/(1-p))^{n-k} \rho_{n,k}^\circ, \text{ for all } 1 \leq k \leq n.
\] (3.8)

By the proof of Lemma 3.1 and definition (2.4), we simply have

\[
\rho_{n,k}^\circ (1-p) h_{-k+1} = \rho_{-n}, \text{ for all } 1 \leq k \leq n.
\] (3.9)

### 3.1. **Formula for** \( g_n^\circ \)

Our main novelty is to find a formula for \( g_n^\circ \) as defined by (3.6) that involves an extension of the Fibonacci recurrence (2.7), to include a forcing term. To begin, let \( n \geq 3 \). By
Lemma 3.2. Let notation. Define we introduce the square root of the ratio of down–step to up–step probabilities to simplify the

\[ g_n = C_n^o \sum_{k=1}^{n} \rho_{n,k}^o g_{n,k}^o. \]  

(3.10)

The identity (3.10) arises simply because \( g_{n,k}^o \) is a normalized sum along lattice paths discussed just after (3.3) such that \( g_{n,k}^o[1] = 1 \), so we must multiply \( g_{n,k}^o \) by its normalizing factor, which is \( \rho_{n,k}^o \), and then sum on \( 1 \leq k \leq n \) to cover all possible paths for \( g_n^o \). Thus it follows that \( C_n^o \) equals \( 1/ \sum_{k=1}^{n} \rho_{n,k}^o \) by (3.10) and \( g_n^o[1] = 1 \).

By (2.6), (3.9), and Lemma 2.4 we have for all \( 1 \leq k \leq n \) that

\[ \rho_{n,k}^o = \frac{((1-p)/p)^n \rho_n}{(1-p)h ((1-p)/p)^{k-1} \rho_{k-1}} = ((1-p)h)^{n-k} \frac{w_{k-1}^*}{w_n^*[1]}, \]  

(3.11)

where for \( k = 1 \) we have \( \rho_{n,1}^o = \frac{\rho_n^o}{(1-p)h} \), so the formula (3.11) continues to hold since by (2.22), \( w_0^o[1] = (\beta[1] - \omega[1]) / x[1] = (1 - (1 - \xi)) / \xi = 1 \).

Lemma 3.2. Let \( n \geq 2 \). Then we have

\[ \rho_{n,k}^o g_{n,k}^o = \begin{cases} 
((1-p)h)^{n-k} r z^{n-k+1} r z^{n-k} w_{k-1}^* / w_n^* & \text{if } 2 \leq k \leq n, \\
((1-p)h)^{n-1} r z^n r z^{-2} 1 / w_n^* & \text{if } k = 1.
\end{cases} \]  

(3.12)

Proof. By Proposition 2.2, Lemma 3.1 and (3.11), if \( 3 \leq k \leq n \), then

\[ \rho_{n,k}^o g_{n,k}^o = ((1-p)h)^{n-k} \frac{w_{k-1}^*}{w_n^*[1]} \frac{C_n}{C_{k-1}} \frac{r \omega z^n r z^{-2}}{r z^{k-1} r z^{-3}} \frac{w_{k-1}^*}{w_n^*}, \]

since by the form of \( C_n \) in Proposition 2.2 we have \( \frac{w_{k-1}^*}{w_n^*[1]} \frac{C_n}{C_{k-1}} = 1 \). Further, if \( k = 2 \) and \( n \geq 2 \), by the same calculation, only using

\[ \rho_{n,2}^o g_{n,2}^o = \rho_{n,2}^o \kappa g_n / z = \rho_{n,2}^o C_n \kappa \omega \frac{r z^n r z^{-2}}{z w_n^*} = ((1-p)h)^{n-2} \frac{r z^{n-1} r z^{-2}}{w_n^*}, \]

the formula (3.12) continues to hold with \( k = 2 \) since \( w_1^* = 1 \). Here we used \( \frac{w_1^*}{w_n^*[1]} C_n \kappa \omega = 1 \). Finally, if \( k = 1 \) and \( n \geq 2 \), then \( \rho_{n,1}^o g_{n,1}^o = \rho_{n,1}^o \kappa g_n = \rho_{n,1}^o C_n \kappa \omega \frac{r z^n r z^{-2}}{w_n^*} = ((1-p)h)^{n-1} \frac{r z^{n-1} r z^{-2}}{w_n^*} \). Thus the proof is complete.

We now wish to find a closed formula for \( g_n^o \) as given by (3.10) and Lemma 3.2. For this purpose we introduce the square root of the ratio of down–step to up–step probabilities to simplify the notation. Define

\[ d = \sqrt{(1-p)/p}. \]

In the symmetric case we simply have \( d = 1 \). However, the parameter \( d \) comes into the formula for \( \rho_{n,k}^o g_{n,k}^o \) of Lemma 3.2 as follows. We apply the reformulation that, for all \( k \geq 2 \),

\[ ((1-p)h)^{n-k} r z^{n-k+1} r z^{-k} = r z \left( (1-p)h \sqrt{x/\xi} \right)^{n-k} = r z \left( d \sqrt{x} \right)^{n-k}. \]  

(3.13)
Here, we keep one factor of $z$ by itself because that corresponds to the step to the state $c$. The formulation (3.13) takes a slightly different form when $k = 1$ because of the factor $\tau^{n-2}$ for this case in Lemma 3.2. So in the case $k = 1$ we have the formulation

$$((1 - p)h)^{n-1}r_zn^{-2} = \frac{rz}{\tau}(d\sqrt{x})^{n-1}. \quad (3.14)$$

By (3.10), Lemma 3.2 and (3.13)–(3.14), we have for all $n \geq 2$ that

$$g_n^* = C_n^\infty \sum_{k=1}^{n} \rho_n^k g_n^k = C_n^\infty rz \left( \frac{1}{\tau}(d\sqrt{x})^{n-1} + \sum_{k=2}^{n} (d\sqrt{x})^{n-k} w_{k-1}^* \right) \frac{1}{u_n^*}. \quad (3.15)$$

We now study the whole summation under the parentheses on the right side of (3.15). It is convenient to make a change of index in this expression as follows.

**Definition 3.3.** Define the sequence $u_n^*$, $n \geq 1$, by

$$u_n^* = \frac{1}{\tau}(d\sqrt{x})^{n-1} + \sum_{\ell=1}^{n-1} (d\sqrt{x})^{n-1-\ell} w_{\ell}^*, \quad \text{for all } n \geq 1, \quad (3.16)$$

with $u_1^* = 1/\tau$ and $u_2^* = 1 + d\sqrt{x}/\tau$.

By (3.15) and Definition 3.3 we indeed have that $g_n^* = C_n^\infty rz u_n^*/w_n^*$ for all $n \geq 2$. We now find a recurrence for $u_n^*$ via the following ansatz:

$$u_{n+1}^* = \beta u_n^* - xu_{n-1}^* + A(d\sqrt{x})^{n-1}, \quad \text{with } u_1^* = \frac{1}{\tau}, \; u_2^* = 1 + d\sqrt{x}; \quad \text{for all } n \geq 2. \quad (3.17)$$

**Lemma 3.4.** We have that the sequence $u_n^*$, $n \geq 1$, defined by (3.16) satisfies the recurrence (3.17) with $A = 1 - \sqrt{\frac{\overline{w}}{d}} - \frac{\beta}{\tau} + \frac{(d^2+1)\sqrt{x}}{d}$, where $w_0^*$ is defined by (2.22).

**Proof.** We proceed by induction. Consider the basis $n = 2$ for (3.17) with $A$ defined by the statement of the lemma. By definition (3.16) we have

$$u_3^* = \frac{d^2x}{\tau} + d\sqrt{x}w_1^* + w_2^* = \frac{d^2x}{\tau} + d\sqrt{x} + \omega,$$

because $w_1^* = 1$ and $w_2^* = \omega$. On the other hand we have that the right side of (3.17) at $n = 2$ is given by $\beta u_2^* - xu_1^* + Ad\sqrt{x} = \beta \left( \frac{1 + d\sqrt{x}}{\tau} \right) + Ad\sqrt{x}$. Thus the two sides of (3.17) are equal if and only if $(A - 1)d\sqrt{x} = \omega - \beta \left( \frac{1 + d\sqrt{x}}{\tau} \right) + \frac{(d^2+1)x}{\tau} \iff A = 1 - \sqrt{\frac{\overline{w}}{d}} - \frac{\beta}{\tau} + \frac{(d^2+1)\sqrt{x}}{d}$. But by (2.22) this last statement is equivalent to the value of $A$ given in the statement of the lemma. Thus the basis of induction is verified.

Assume now that (3.17) holds for some $n \geq 2$ with $A$ given by the statement of the lemma. Then by definition (3.16) we have

$$u_{n+2}^* = \frac{(d\sqrt{x})^{n+1}}{\tau} + \sum_{\ell=1}^{n+1} (d\sqrt{x})^{n+1-\ell} w_{\ell}^*$$

$$= u_{n+1}^* + d\sqrt{x} \left( \frac{(d\sqrt{x})^n}{\tau} + \sum_{\ell=1}^{n} (d\sqrt{x})^{n-\ell} w_{\ell}^* \right) = w_{n+1}^* + d\sqrt{x}u_{n+1}^*. \quad (3.18)$$

Now by definition (2.21) write $w_{n+1}^* = \beta u_n^* - xu_{n-1}^*$ and by the induction hypothesis write $u_{n+1}^* = \beta u_n^* - xu_{n-1}^* + A(d\sqrt{x})^{n-1}$. Then by (3.18) we have

$$u_{n+2}^* = \beta (w_n^* + d\sqrt{x}u_n^*) - x (w_{n-1}^* + d\sqrt{x}u_{n-1}^*) + d\sqrt{x}A(d\sqrt{x})^{n-1}. \quad (3.19)$$
Finally, apply definition (3.16) to rewrite \( u_n^* \) and \( u_{n-1}^* \) in (3.19) as sums, and apply the factor \( d\sqrt{x} \) on these terms so as to raise the powers of \( d\sqrt{x} \) by 1 in each of these sums. Hence, by (3.19),

\[
\begin{align*}
    u_{n+2}^* &= \beta \left( u_n^* + \frac{(d\sqrt{x})^n}{\tau} + \sum_{\ell=1}^{n-1} (d\sqrt{x})^{n-\ell} w_\ell^* \right) \\
    - x \left( u_{n-1}^* + \frac{(d\sqrt{x})^{n-1}}{\tau} + \sum_{\ell=1}^{n-2} (d\sqrt{x})^{n-1-\ell} w_\ell^* \right) + A (d\sqrt{x})^n .
\end{align*}
\]

(3.20)

Here, if \( n = 2 \) there is an empty sum evaluating to zero in the second parenthetical term. But obviously the two total sums in parentheses in (3.20) are by the definition (3.16) simply \( u_{n+1}^* \) and \( u_n^* \), respectively, and the term \( A (d\sqrt{x})^n \) is of the correct form for the induction step.

We summarize the calculations of this section as follows.

**Proposition 3.5.** Let \( g_n^\circ \) be defined by (3.1) and (3.6). Let \( u_n^*, \ n \geq 1 \), be given by (3.17) with \( A \) defined by the statement of Lemma 3.4. Then, for all \( n \geq 2 \) we have

\[
g_n^\circ = C_n^\circ \frac{r z u_n^*}{w_n^*}, \quad \text{where } C_n^\circ = 1/\sum_{k=1}^n g_{n,k}^\circ .
\]

Proof. The proposition follows by (3.15), Definition 3.3, and Lemma 3.4, where we noted the evaluation of \( C_n^\circ \) after (3.10) with \( \rho_{n,k}^\circ \) defined by (3.7).

3.2. Calculation of \( K_N^\circ \). Our goal in this section is to calculate \( K_N^\circ \) of definition (3.1). In Proposition 3.5 we focussed on nonnegative stopped excursions to calculate \( g_n^\circ \). However, while the analogue for nonpositive stopped excursions of the probability generating function \( g_{n,k} \) defined by (3.2)–(3.3) will still be the same as given by the formula of Lemma 3.1, the corresponding analogue \( g_n^\circ \) of \( g_n^\circ \) itself will not be as given by Proposition 3.5. Indeed define the analogue \( G_n^\circ \) of \( G_n \) for nonpositive stopped excursions as follows.

\[
G_n^\circ = \mathbb{E} \left( e^{R_n^\circ y V_n^\circ z L_n^\circ} | E_c \cap \{ H = n \}; \ X_0 = 0; \ X_j \leq 0, \ j = 0, 1, \ldots L - 1 \right).
\]

(3.21)

In parallel with (3.6), define \( G_n^\circ \) implicitly for all \( n \geq 2 \) by

\[
G_n^\circ = \begin{cases} 
    z \eta g_{n-1}^\circ z g_{n-\eta}^\circ & \text{if } n \geq 3, \\
    r z g_{n-\eta}^\circ & \text{if } n = 2.
\end{cases}
\]

(3.22)

Then just as in (3.10), we have

\[
g_n^\circ = C_n^\circ \sum_{k=1}^n \rho_{n-k}^\circ \cdot g_{n,k}^\circ,
\]

(3.23)

where \( \rho_{n-k} \) is defined by (3.7)–(3.8). However since the relation (3.8) depends on \( k \) in the asymmetric case \( p \neq \frac{1}{2} \), we have in this case that \( g_n^\circ \neq g_n^\circ \). Hence the formula for \( K_N^\circ \) involves separate contributions form nonnegative and nonpositive stopped excursions as follows:

\[
\begin{aligned}
    &\mathbb{P} \left( E_c \cap \{ 1 \leq H \leq N \} \right) K_N^\circ \text{ is written as:} \\
    &= \sum_{n=1}^N G_n^\circ \mathbb{P} \left( E_c \cap \{ H = n \} \cap \{ X_j \geq 0, j = 0, 1, \ldots L - 1 \} \right) \\
    &\quad + \sum_{n=1}^N G_n^\circ \mathbb{P} \left( E_c \cap \{ H = n \} \cap \{ X_j \leq 0, j = 0, 1, \ldots L - 1 \} \right) = \sum_{n=1}^N \sigma_n + \sum_{n=1}^N \tilde{\sigma}_n
\end{aligned}
\]

(3.24)
We note that the case \( H = 0 \) occurs if and only if there is transition to \( c \) on the first step, which occurs only with very small probability \( c_N = O(1/N^2) \). Therefore we may safely ignore this case. For the calculation of the probabilities involved in the expression \( \sigma_n \) of (3.24) we have

\[
\mathbb{P}(E_c \cap \{H = n\} \cap \{X_j \geq 0, j = 0, 1, \ldots, L - 1\}) = \frac{\rho}{w_{n-1}^* w_n^*} \sum_{k=1}^{n} \rho_{n,k}^\circ. \tag{3.25}
\]

Due to the easy interchange of parameters \( p \) and \( 1 - p \) to handle the contributions for nonpositive stopped excursions from the nonnegative ones, so that \( \sigma_n \) is determined by applying this interchange to a formula for \( \sigma_n \), we focus on the sum \( \sum_{n=1}^{N} \sigma_n \) in (3.24). We first summarize our calculations to this point to find a nice expression for \( \sigma_n \).

**Lemma 3.6.** Denote \( \sigma_n = C_n^\circ \mathbb{P}(E_c \cap \{H = n\} \cap \{X_j \geq 0, j = 0, 1, \ldots, L - 1\}) \). Then for each \( n \geq 2 \) we have

\[
\sigma_n = \frac{\rho}{w_{n-1}^* w_n^*} \sum_{k=1}^{n} \rho_{n,k}^\circ,
\]

where \( u_n^* \) is defined by (3.16).

**Proof.** By definition (3.6), Propositions 2.2 and 3.5, and (3.25), if \( n \geq 3 \), we have that \( \sigma_n/(\rho c_N) \) is written

\[
z \eta \eta_{n-1} g_n^\circ \cdot \rho_{n-1} \sum_{k=1}^{n} \rho_{n,k}^\circ = z \eta \rho_{n-1} C_{n-1} w_z n^{-1} \tau n^{-1} - 1 \cdot \tau z \cdot \frac{u_n^*}{w_{n-1}^* w_n^*} C_n^\circ \sum_{k=1}^{n} \rho_{n,k}^\circ. \tag{3.26}
\]

However by the definition, \( C_n^\circ \sum_{k=1}^{n} \rho_{n,k}^\circ = 1 \). Also, by \( \eta = \tau \kappa \) and Lemma 2.4, we have

\[
\eta \rho_{n-1} C_{n-1} \omega = \frac{(ph)^{n-1} w_{n-1}^* [1]}{w_{n-1}^* [1]} - 1 \cdot \tau \kappa \omega = (ph)^{n-1} \tau,
\]

since \( \kappa \omega/(1 - \xi) = 1 \). Therefore by these cancellations in (3.26) we have

\[
\sigma_n = \frac{\rho}{w_{n-1}^* w_n^*} \sum_{k=1}^{n} \rho_{n,k}^\circ \cdot \tau z \cdot \frac{u_n^*}{w_{n-1}^* w_n^*}.
\]

Finally, as before, write \( z \tau = \sqrt{x/\xi} \) and thus find that \( (ph)^{n-1} z n^{-1} \tau n^{-1} = (\sqrt{x/d})^{n-1} \), so the proof is complete for \( n \geq 3 \). If \( n = 2 \), then we have \( C_2^\circ = rz^2 g_2^\circ \), so again by Proposition 3.5 and (3.25),

\[
\sigma_2 = \frac{\rho}{w_{n-1}^* w_n^*} \sum_{k=1}^{n} \rho_{n,k}^\circ \cdot \tau z \cdot \frac{u_n^*}{w_{n-1}^* w_n^*}.
\]

by \( \rho_1 = ph \) and \( w_1^* = 1 \). Now write \( (ph)^{1} z^{1} \cdot \frac{\sqrt{x/\xi}}{\tau} = \frac{\sqrt{x/d}}{\tau} \), and again we have the form of the statement of the lemma. \( \square \)

Let \( u_n^* \) and \( v_n^* \) be defined respectively by (2.21) and Definition 3.16, where \( u_n^* \) is determined as a recurrence by Lemma 3.4. We wish to find a recurrence for a sequence \( v_n^* \), \( n \geq 1 \), with \( v_1^* = 0 \) such that for all \( N \geq 2 \) we have:

\[
\sum_{n=2}^{N} \frac{(\sqrt{x/d})^{n-1} u_n^*}{w_{n-1}^* w_n^*} = \frac{v_N^*}{w_N^*}. \tag{3.27}
\]
The reason for this desired form is that by Lemma 3.6 the sum \( \sum_{n=2}^{N} \sigma_n \) takes the form of the sum in (3.27) modulo some factors that are constant in \( \sigma \). For \( N = 2 \), since \( u_2^* = 1 + \frac{d\sqrt{x}}{r} \), by (3.27) we therefore want \( \frac{\sqrt{x}}{d} \left( 1 + \frac{d\sqrt{x}}{r} \right) = w_1^* v_2^* = v_0 \). So we take \( v_2^* = \frac{\sqrt{x}}{d} + \frac{r}{x} \). For \( N = 3 \) again by (3.27) we want \( \frac{v_3^*}{w_2^*} + \frac{(x/d)^2 w_1^*}{w_3^*} = \frac{v_3^*}{w_3^*} \iff w_3^* v_3^* = w_1^* v_3^* v_3^* + d\sqrt{x} \). Then, since \( w_2^* = \omega \), \( w_3^* = \beta \omega - x \), and, by (3.16), \( u_3^* = \frac{d\sqrt{x}}{r} + \omega + d\sqrt{x} \), the condition for \( v_3^* \) in the last display becomes \( (\beta \omega - x) \left( \frac{\sqrt{x}}{d} + \frac{r}{x} \right) + \frac{r}{x} \left( \frac{d\sqrt{x}}{r} + \omega + d\sqrt{x} \right) = \omega v_3^* \). Then indeed there are two pairs of terms that cancel on the left of this last equation, leaving all other terms with a factor of \( \omega \). Therefore we want \( v_3^* = \frac{\beta \sqrt{x}}{d} + \frac{\beta x}{r} + \frac{r}{d} \).

Whereas we define the initial expressions for \( u_n^*, n \geq 1 \), by (3.27), for our proofs we prefer the following definition, leaving the problem to show that (3.27) does indeed follow from it for all \( N \geq 2 \).

**Definition 3.7.** Define a sequence \( v_n^*, n \geq 1 \), by

\[
v_{n+1}^* = \beta v_n^* - x v_{n-1}^* + (\sqrt{x}/d)^n, \quad \text{for } n \geq 2; \quad \text{with } v_1^* = 0, \quad v_2^* = \frac{\sqrt{x}}{d} + \frac{x}{r}. \quad (3.28)
\]

In Lemma 3.8 we will verify that Definition 3.7 works to establish the desired summation identity (3.27). One easily checks that under the Definition 3.7 the desired formula for \( v_3^* \) shown just above indeed comes about by plugging in \( n = 2 \) to the right side of (3.28) as follows: \( \beta v_2^* - x v_1^* + (\sqrt{x}/d)^2 = \beta \left( \frac{\sqrt{x}}{d} + \frac{r}{x} \right) - 0 + \frac{r}{x} = \frac{\beta \sqrt{x}}{d} + \frac{\beta x}{r} + \frac{r}{d} \).

**Lemma 3.8.** Let \( u_n^* \), \( v_n^* \), and \( w_n^* \) be defined respectively by (2.21), Definition 3.3, and Definition 3.7, where \( u_n^* \) is determined as a recurrence by Lemma 3.4. Then (3.27) holds for all \( N \geq 2 \).

Before proving Lemma 3.8, we record solutions for \( u_n^* \) and \( v_n^* \) in terms of the fundamental sequence \( \{q_n, n \geq 0\} \) of (2.7) via generating function manipulations. For the purpose of writing generating functions for the sequences \( u_n^* \) and \( v_n^* \), we extend these sequences to all \( n \geq 0 \). Simply define \( u_0^* \) and \( v_0^* \) by applying the recurrences (3.17) and (3.28), respectively, with \( n = 1 \). We thus obtain by Lemma 3.4 and Definition 3.7, respectively, that

\[
u_0^* = \frac{\beta u_0^* - u_2^* + A}{x} = \frac{1}{x} \left( -\sqrt{x} w_0^* + \frac{\sqrt{x}}{d} \right) = \frac{u_1^* - w_0^*}{d\sqrt{x}};
\]

\[
v_0^* = \frac{\beta v_1^* - v_2^* + \sqrt{x}/d}{x} = -\frac{1}{r} = -u_1^*.
\]

**Lemma 3.9.** Let \( u_n^* \) and \( v_n^* \) be defined respectively by Definition 3.3, and Definition 3.7, where \( u_n^* \) is determined as the recurrence (3.17) with \( A \) given by Lemma 3.4. Also denote \( B = \beta - \frac{d^2 + 1}{d} \sqrt{x} \). Then, with \( q_n \) defined by (2.7), we have for all \( n \geq 1 \) that

\[
u_n^* = xu_1^* q_n - xu_0^* q_{n-1} + \frac{A}{B} \left( -(d\sqrt{x})^{n-1} q_n - \frac{\sqrt{x}}{d} q_{n-1} \right),
\]

\[
u_n^* = xu_1^* q_n - xu_0^* q_{n-1} + \frac{A}{B} \left( -(d\sqrt{x})^{n-1} q_n - \frac{\sqrt{x}}{d} q_{n-1} \right). \quad (3.30)
\]

**Proof.** By a standard generating function manipulation we have by (2.7) that

\[
Q(s) = \sum_{n=0}^{\infty} q_n s^n = \frac{s}{1 - \beta s + xs^2}, \quad (3.31)
\]
[9, Lemma 2]. Denote $U^* = U^*(s) = \sum_{n=0}^{\infty} u_n^* s^n$ and $V^* = V^*(s) = \sum_{n=0}^{\infty} v_n^* s^n$. By standard manipulations under (3.17) and (3.28) we have

\begin{align*}
U^* &= u_0^* + u_1^* s + \beta s (U^* - u_0^*) - x s^2 U^* + \frac{A s^2}{1 - d \sqrt{x}/s}, \\
V^* &= v_0^* + v_1^* s + \beta s (V^* - v_0^*) - x s^2 V^* + \frac{A s^2}{1 - (\sqrt{x}/d)s^2}.
\end{align*}

(3.32)

Now rearrange (3.32) as follows.

\begin{align*}
U^*(1 - \beta s + x s^2) &= u_0^* + (u_1^* - \beta u_0^*) s + \frac{A s^2}{1 - d \sqrt{x}/s}, \\
V^*(1 - \beta s + x s^2) &= v_0^* + (v_1^* - \beta v_0^*) s + \frac{(\sqrt{x}/d)s^2}{1 - (\sqrt{x}/d)s^2}.
\end{align*}

(3.33)

Divide by $1 - \beta s + x s^2$ to bring in $Q = Q(s)$ given by (3.31). So by (3.33) we have

\begin{align*}
U^* &= u_0^* \frac{Q}{s} + (u_1^* - \beta u_0^*) Q + \frac{A s^2}{(1 - d \sqrt{x}/s)(1 - \beta s + x s^2)}; \\
V^* &= v_0^* \frac{Q}{s} + (v_1^* - \beta v_0^*) Q + \frac{(\sqrt{x}/d)s^2}{(1 - (\sqrt{x}/d)s)(1 - \beta s + x s^2)}.
\end{align*}

(3.34)

Next, apply partial fractions to the last terms in the expansions for $U^*$ and $V^*$, as follows:

\begin{align*}
\frac{A s^2}{(1 - d \sqrt{x}/s)(1 - \beta s + x s^2)} &= \frac{A}{B} \left( -\frac{1}{d \sqrt{x}(1 - d \sqrt{x})} + \frac{1}{d \sqrt{x}} \frac{1 + (d \sqrt{x} - \beta)s}{1 - (\sqrt{x}/d)s^2} \right); \\
\frac{(\sqrt{x}/d)s^2}{(1 - (\sqrt{x}/d)s)(1 - \beta s + x s^2)} &= \frac{A}{B} \left( -\frac{1}{1 - (\sqrt{x}/d)s} + \frac{1}{1 + (\sqrt{x}/d - \beta)s} \right).
\end{align*}

(3.35)

Thus, by (3.34)–(3.35) we read off $u_n^*$ and $v_n^*$ for all $n \geq 1$ as follows.

\begin{align*}
u_n^* &= u_0^* q_{n+1} + (u_1^* - \beta u_0^*) q_n + \frac{A}{B} \left( -\frac{1}{d \sqrt{x}} (d \sqrt{x} - \beta) q_n (q_{n+1}) \right) \\
v_n^* &= v_0^* q_{n+1} + (v_1^* - \beta v_0^*) q_n + \frac{A}{B} \left( -\frac{1}{(\sqrt{x}/d)} (\sqrt{x}/d - \beta) q_n (q_{n+1}) \right).
\end{align*}

(3.36)

Finally, rewrite (3.36) by applying the recurrence $q_{n+1} = \beta q_n - \beta q_{n-1}$, $n \geq 1$, wherever $q_{n+1}$ appears. Then there is cancellation of terms $\beta u_0^* q_n$ and $\beta q_n$ for $u_n^*$, and likewise cancellation for $v_n^*$. Also for $v_n^*$ we rewrite one term $-x v_0^* q_{n-1} = xu_1^* q_{n-1}$ by (3.29), and another $v_1^* q_n = 0$ by $v_1^* = 0$. Thus by these simplifications we finally have that (3.30) holds for all $n \geq 1$.

Proof of Lemma 3.8. We proceed by induction. We have by (3.17) that

\begin{equation*}
\frac{(\sqrt{x}/d) u_n^*}{w_1^* u_2} = \frac{\sqrt{x}/d + x/\tau}{\omega}.
\end{equation*}

and by (3.28) this is the same as $v_2^*/w_2^*$, so have that the lemma holds for $N = 2$. As the induction step, it now suffices to show that for any $n \geq 2$ we have

\begin{equation*}
\frac{v_{n+1}^*}{w_{n+1}^*} - \frac{v_n^*}{w_n^*} = \frac{(\sqrt{x}/d)^n u_{n+1}^*}{w_n^* u_{n+1}^*} \iff w_n^* v_n^* - v_n^* u_{n+1}^* = (\sqrt{x}/d)^n u_{n+1}^*.
\end{equation*}

(3.37)

We will verify (3.37) by direct calculation using Lemma 3.9, and also $w_n^* = q_n - xu_0^* q_{n-1}$ from (2.31). We first work to reduce $w_n^* v_n^* - v_n^* u_{n+1}^*$ of (3.37). We substitute the slightly reduced expression

\begin{align*}
v_n^* &= xu_1^* q_{n-1} + \frac{1}{B} \left( -\frac{1}{(\sqrt{x}/d)} + \frac{1}{B} \frac{1}{(\sqrt{x}/d)^n} \right) q_{n-1} - \frac{1}{B} (\sqrt{x}/d)^n \\
&= \frac{\sqrt{x}}{B} q_n + \frac{B^* - x}{B} q_{n-1} - \frac{1}{B} (\sqrt{x}/d)^n.
\end{align*}

(3.38)
where we combined the terms involving \( q_{n-1} \) from Lemma 3.9, and apply the expression (2.31) for \( w_n^* \). Thus we have that \( w_n^*v_{n+1} - w_{n+1}^*v_n^* \) is written by

\[
(q_n - xw_0^*q_{n-1}) \left( \frac{\sqrt{x}}{dB} q_{n+1} + \left( xu_1^* - \frac{x}{B} \right) q_n - \frac{1}{B} (\sqrt{x}/d)^{n+1} \right) \\
- (q_{n+1} - xw_0^*q_n) \left( \frac{\sqrt{x}}{dB} q_n + \left( xu_1^* - \frac{x}{B} \right) q_{n-1} - \frac{1}{B} (\sqrt{x}/d)^n \right) .
\]

(3.39)

We account for a total of 12 products in the expansion of (3.39) as a sum of products. There are 2 pairs of products whose sum is

\[
I = \left( xu_1^* - \frac{x}{B} + \frac{x\sqrt{x}w_0^*}{dB} \right) (q_n^2 - q_{n+1}q_{n-1}) = \left( u_1^*- \frac{1}{B} + \frac{\sqrt{x}w_0^*}{dB} \right) x^n,
\]

where we applied \( q_n^2 - q_{n+1}q_{n-1} = x^{n-1} \) by (2.10)(ii). There is a pair of products involving \( q_nq_{n+1} \) which cancel in opposite signs, and there is a pair of products involving \( q_{n-1}q_n \) which likewise cancel. Finally, there are 4 terms each involving \( \frac{1}{B}(\sqrt{x}/d)^n \) whose sum is as follows:

\[
II = \frac{1}{B}(\sqrt{x}/d)^n \left( q_n - \left( xu_0^* + \frac{\sqrt{x}}{d} \right) q_n + \frac{x\sqrt{x}w_0^*}{d} q_{n-1} \right),
\]

where we combined two terms involving \( q_n \). Finally apply \( q_{n+1} = \beta q_n - xq_{n-1} \) to rewrite \( II \), and therefore obtain that

\[
w_n^*v_{n+1} - w_{n+1}^*v_n^* \text{ is rewritten as: } I + II = \left( u_1^* - \frac{1}{B} + \frac{\sqrt{x}w_0^*}{dB} \right) x^n + \frac{(\sqrt{x}/d)^n}{B} \left( (\beta - xu_0^* - \frac{\sqrt{x}}{d}) q_n + \left( \frac{x\sqrt{x}w_0^*}{d} - x \right) q_{n-1} \right). \]

Finally, apply \( u_1^* = 1/\tau \) and the definition of \( A \) in Lemma 3.4 we have that the coefficient of \( x^n \) in the first term of this last expression is \( u_1^* - \frac{1}{B} + \frac{\sqrt{x}w_0^*}{dB} = -A/B \), so \( I + II \) has a leading term \( -A/Bx^n \), which by Lemma 3.9 matches the “pure power” term of \( (\sqrt{x}/d)^n u_{n+1}^* \). For the remainder of the verification of (3.37), by Lemma 3.9 we write

\[
(\sqrt{x}/d)^n u_{n+1}^* + \frac{A}{B} x^n = (\sqrt{x}/d)^n \left( u_1^*q_{n+1} - xu_0^*q_n + \frac{A}{B} \left( q_{n+1} - \frac{\sqrt{x}}{d} q_n \right) \right). 
\]

Now rewrite \( q_{n+1} = \beta q_n - xq_{n-1} \) in this last display. So we have \( (\sqrt{x}/d)^n u_{n+1}^* + \frac{A}{B} x^n = (\sqrt{x}/d)^n \left( (\beta u_1^* - xu_0^* + \frac{A}{B} (\beta - \frac{\sqrt{x}}{d}) q_n - \frac{xu_0^* - \frac{\sqrt{x}}{d}}{d} q_{n-1} \right). \)

Therefore to complete the proof it suffices to show that the coefficients of \( q_n \) and \( q_{n-1} \), respectively, match in \( I + II \) and this last expression. Therefore we wish to verify that

\[
\begin{align*}
(\text{i}) \quad & \quad \beta u_1^* - xu_0^* + \frac{A}{B} (\beta - \frac{\sqrt{x}}{d}) = \frac{1}{B} \left( \beta - xu_0^* - \frac{\sqrt{x}}{d} \right) ; \\
(\text{ii}) \quad & \quad -x(u_1^* + \frac{A}{B}) = \frac{1}{B} \left( \frac{x\sqrt{x}w_0^*}{d} - x \right). \tag{3.40}
\end{align*}
\]

We start with the easier (3.40)(ii). First, there is a common factor of \( x \) that we can drop from both sides. Second, write \( u_1^* = 1/\tau \). So (3.40)(ii) is verified if, after writing all expressions with a denominator \( B \), we have \( B/\tau + A = 1 - \sqrt{x}w_0^*/d \), and by the definition of \( B = \beta - d^2/d \sqrt{x} \), this last condition follows easily by the definition of \( A \) in Lemma 3.4.

We turn finally to verification of (3.40)(i). By Lemma 3.4 and the definition of \( u_0^* \), we recall extends the recurrence for \( u_n^* \) to all \( n \geq 0 \), we have that \( \beta u_1^* - xu_0^* = u_2^* - A \). Therefore, after transposing the term \( u_1^* \) thus formed to the right side of (3.40) followed by writing both sides with a common denominator \( B \) we must show that \( \left( \beta - \frac{\sqrt{x}}{d} - B \right) A = \beta - xu_0^* - \frac{\sqrt{x}}{d} - Bu_2^* \). But \( \beta - \frac{\sqrt{x}}{d} - B = d/\sqrt{x} \), so after dividing both sides by \( d/\sqrt{x} \), and writing the definition of \( u_2^* = 1 + d/\sqrt{\tau} \), we must show \( A = \frac{\beta}{d\sqrt{x}} - \frac{\sqrt{x}w_0^*}{d} - \frac{1}{d^2} - \left( \beta - \frac{d^2 + 1}{d \sqrt{x}} \right) \left( \frac{1}{d \sqrt{x}} + \frac{1}{\tau} \right) \). In this last equation we have cancellation on the right side in the form \( \frac{\beta}{d \sqrt{x}} - \frac{\beta}{d \sqrt{x}} \) and \( \frac{1}{d^2} + \frac{1}{d \sqrt{x}} \), and the other terms add to the
definition of \( A \) in Lemma 3.4. Therefore we have verified (3.40)(i)–(ii), so the proof of the induction step (3.37) is complete.

We summarize the result of Lemma 3.8 as follows. Denote the solution to \( v_n^* \) of Lemma 3.9 by \( \tilde{v}_n^* \) when \( d \) is replaced by \( 1/d \), that is when the roles of \( p \) and \( 1 - p \) are interchanged. Since \( x, \beta, \) and \( \tau \), and also \( u_1^* = 1/\tau \) and \( B = \beta - (d + 1/d)\sqrt{x} \) do not change under this procedure, then it is a simple matter to transform the formula of Lemma 3.9 as follows:

\[
\tilde{v}_n^* = xu_1^*q_{n-1} + \frac{1}{B} \left( -(d\sqrt{x})^n + d\sqrt{x}q_n - xq_{n-1} \right). \tag{3.41}
\]

**Proposition 3.10.** Let \( v_n^* \) and \( \tilde{v}_n^* \) be given by Lemma 3.9 and (3.41), respectively. Then, for all \( N \geq 2 \), we have, with \( K_N^* = hc_N/\mathbb{P} (E_e \cap \{ 1 \leq H \leq N \}) \), that

\[
K_N^* = k_N^* \left( r^2yz^2 + \frac{\sqrt{2}\beta}{\tau} \frac{pv_N^*}{w_N^*} (1 - p)\tilde{v}_N^* \right).
\]

**Proof.** The proof follows by the representation (3.24), Lemma 3.6, and Lemma 3.8 after substituting the formula for \( \sigma_n \) and also by determining \( \tilde{\sigma}_n \) via the interchange of \( p \) and \( 1 - p \) in the formula for \( \sigma_n \), including the trivial cases \( \sigma_1 = phc_Nr^2yz^2 \) and \( \tilde{\sigma}_1 = (1 - p)hc_Nr^2yz^2 \). \( \square \)

4. PROOFS OF THEOREM 1.1, COROLLARY 1.4, AND THEOREM 4.4

In this section we first set up how we will attack the calculation of the limiting Fourier transforms in the statements of Theorem 1.1. The indicated transforms of this theorem, written as expectations or conditional expectations under a limit as \( N \to \infty \), may be rewritten by applying certain substitutions \( r = r(s,t,N), \ y = y(s,t,N), \) and \( z = z(s,t,N) \) in either \( K_N \) of (2.1) or \( K_N^* \) of (3.1). We introduce these substitutions as follows.

\[
r(s,t,N) = e^{i(s-3t)/N}; \quad y(s,t,N) = e^{2i(t-s)/N}; \quad z(s,t,N) = e^{it/N}. \tag{4.1}
\]

The origin of these expressions follow easily in the context of Theorem 1.1, since to be explicit in case (b), we have

\[
\frac{i}{N} (s \mathcal{Y}^\circ + t \mathcal{Z}^\circ) = \frac{is}{N} (\mathcal{R}^\circ - 2\mathcal{V}^\circ) + \frac{it}{N} (\mathcal{L}^\circ - 3\mathcal{R}^\circ + 2\mathcal{V}^\circ).
\]

However, by the Markov property, we may calculate

\[
\mathbb{E} \{ e^{i\frac{1}{N} (s \mathcal{Y}^\circ + t \mathcal{Z}^\circ)} | \mathcal{O}^\circ \} = \mathbb{E} \{ e^{i\frac{u}{R_N^\circ} (\mathcal{R}^\circ - 2\mathcal{V}^\circ) + \frac{v}{L_N^\circ} (\mathcal{L}^\circ - 3\mathcal{R}^\circ + 2\mathcal{V}^\circ)} | E_e \cap \{ 1 \leq H \leq N \} \}, \tag{4.2}
\]

where \( \mathcal{R}^\circ = R_N^\circ \), etc., are defined at the beginning of section 3. Thus, for example, after rewriting the right side of (4.2) in terms of \( K_N^* \) of (3.1), the meaning of the statement of Theorem 1.1(b) is that

\[
\lim_{N \to \infty} \mathbb{E} \{ e^{i\frac{1}{N} (s \mathcal{Y}^\circ + t \mathcal{Z}^\circ)} | \mathcal{O}^\circ \} = \lim_{N \to \infty} K_N^* (r(s,t,N), y(s,t,N), z(s,t,N)). \tag{4.3}
\]

The expressions (4.1) will be applied for all cases of Theorem 1.1.

Our method to obtain the limiting joint characteristic functions of Theorem 1.1 rests on a trigonometric substitution that goes back to \([5, \text{p. } 352]\). The motivation for this is that by Propositions 2.6 and 3.10 we will ultimately apply the closed formula (2.9) for \( q_N \), which lends itself nicely to a trigonometric formulation. Introduce \( \theta \) by

\[
\beta = \sqrt{4x} \cos \theta, \quad \alpha = \sqrt{\beta^2 - 4x}; \quad \beta \pm \alpha = \sqrt{4x} (\cos \theta \pm i \sin \theta) = \sqrt{4x} e^{\pm i \theta}. \tag{4.4}
\]

We apply (2.19), (4.1), and (4.4), after composing \( x \) and \( \beta \) with (4.1), to find \( \cos(\theta) \) as a function of \( s, t, \) and \( N \). In the following we understand without additional notation that the composition
with (4.1) has been taken. We apply direct computation to expand this composition for \( \cos(\theta) \) using the Mathematica, [12], command Series[\( \cdots \), \( \{N,\text{Infinity},3\} \)] to find
\[
\cos \theta = 1 + \frac{a^2 + 4b^2 + (s^2 + t^2)/2}{2N^2} + O(1/N^3), \quad \text{as } N \to \infty. \tag{4.5}
\]
It follows that, by choosing a branch of \( \theta \) so that \( |e^{iN\theta}| > 1 \) for large \( N \), we have
\[
\theta = -i\sqrt{s^2 + (s^2 + t^2)/2}/N + O(1/N^2) = -\frac{-i\varphi(s,t)}{N} + O(1/N^2), \quad \text{as } N \to \infty. \tag{4.6}
\]
Lemma 4.1. Let \( x \) and \( \beta \) be defined by (2.19) and let \( q_N = q_N(x, \beta) \) as defined by (2.7). Then
\[
q_N = \frac{2i}{\alpha}(\sqrt{x})^N \sin N\theta. \tag{4.7}
\]
Proof. By (2.9) and (4.4), we have \( q_N = \frac{2^{-N}}{\alpha}(\sqrt{x})^N(e^{iN\theta} - e^{-iN\theta}) \), which reduces to the stated form.

We start with the proof of statement (b) of Theorem 1.1 as it depends on Proposition 3.10. The proofs of statements (a) and (c) that come afterward depend only on the development of Section 2.

Proof of Theorem 1.1(b). We must show that the limit on the right side of (4.3) equals the limit asserted in statement (b) of the theorem. By Proposition 3.10 and (4.3) we want to calculate the unnormalized term \( r^2 y z^2 + r^2 z^2 \frac{pu_N + (1-p)\bar{v}_N}{w_N} \), where \( r, y, \) and \( z \) are substituted by (4.1) and where \( p = \frac{1}{2} + \frac{b}{N} \). It is easy to see that under (4.1) the coefficient \( r^2 z^2/\tau = 1 + O(1/N) \) and likewise \( r^2 y z^2 = 1 + O(1/N) \). Therefore it suffices to establish an asymptotic expression for \( (pu_N + (1-p)\bar{v}_N)/w_N \) that is of order \( N \). Our method is to simply write the formulae for \( w_N \), \( v_N \), and \( \bar{v}_N \) from (2.31), Lemma 3.9, and (3.41) into expressions involving the terms \( q_N \) and \( \sqrt{x}q_{N-1} \), with certain coefficients that we will evaluate asymptotically by direct calculation using Mathematica, similar to our computation for (4.5). The reason for the factor \( \sqrt{x} \) on \( q_{N-1} \) is that by Lemma 4.1 both \( q_N \) and \( \sqrt{x}q_{N-1} \) have a common factor \( (\sqrt{x})^N \). By Lemma 3.9 and \( u_0 = 1/\tau \), the formula for \( v_N^* \) is rewritten by
\[
v_N^* = \frac{1}{B} \left( -\frac{\sqrt{x}}{d} N + \frac{(\sqrt{x})/d}{\sqrt{x}B/\tau - \sqrt{x}} \left( \sqrt{x}q_{N-1} \right) \right), \tag{4.8}
\]
where \( B = \beta - (d+1)/d\sqrt{x} \). The formula for \( \bar{v}_N^* \) is the expression (4.7) with \( d \) replaced by \( 1/d \). By Lemma 4.1 and (4.7), since by (2.31) we have \( w_N^* = q_N - \sqrt{x}u_0^*(\sqrt{x}q_{N-1}) \), there is a common factor of \( (\sqrt{x})^N \) in both the numerator and denominator of each of \( v_N^*/w_N^* \) and \( \bar{v}_N^*/w_N^* \). To evaluate each of these fractions asymptotically, again under the composition with (4.1), we have as \( N \to \infty \),
\[
B = \frac{a^2 + (s^2 + t^2)/2}{2N^2} + O(1/N^3); \quad d_1 = 1 - \frac{2b}{N} + O(1/N^2); \quad \sqrt{x} = \frac{1}{2} + \frac{i(s+t)}{4N} + O(1/N^2);
\]
\[
\sqrt{x}B/\tau - \sqrt{x} = -\frac{1}{2} + \frac{i(s-t)}{4N} + O(1/N^2); \quad \sqrt{x}u_0^* = \frac{1}{2} + \frac{i(s-t)}{4N} + O(1/N^2);
\]
\[
\sqrt{x}/d = \frac{1}{2} + \frac{4b+i(s+t)}{4N} + O(1/N^2); \quad d/\sqrt{x} = \frac{1}{2} + \frac{-4b+i(s+t)}{4N} + O(1/N^2). \tag{4.9}
\]
To asymptotically evaluate the fraction \( (pu_N + (1-p)\bar{v}_N)/w_N \), we first focus on \( v_N^*/w_N^* \). It follows from Lemma 4.1, (4.7), and (2.31), after dividing both numerator and denominator of \( v_N^*/w_N^* \) by \( \frac{2i}{\alpha}(\sqrt{x})^N \), that
\[
\frac{v_N^*}{w_N^*} = \frac{1 - \frac{a}{2N}(1/d)^N + \frac{1}{\sqrt{x}}\sin N\theta + (\sqrt{x}/d)\sin(N-1)\theta}{\sin N\theta - \sqrt{x}u_0^*\sin(N-1)\theta} = \frac{1}{\frac{B}{\delta_N}}. \tag{4.10}
\]
where \( \nu_N = \frac{\alpha}{2\pi}(\sqrt{x})^{-N}Bv_N^* \) and \( \delta_N = \frac{\alpha}{2\pi}(\sqrt{x})^{-N}w_N^* \). The factor \( 1/B \) is of order \( N^2 \). We examine the remaining fraction \( \nu_N/\delta_N \) in the product of (4.9). By (4.8)–(4.9) we have that

\[
\frac{\nu_N}{\delta_N} = -\left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) \sin N\theta - \left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) \sin(N-1)\theta + O\left(\frac{1}{N^2}\right)
\]

(4.10)

where we use implicitly that \( \sin N\theta \) and \( \sin(N-1)\theta \) are each \( O(1) \) as \( N \to \infty \) by (4.6). We claim that this numerator \( \nu_N \) is of order \( 1/N \) and the denominator \( \delta_N \) is of order \( 1 \). Since by the substitution (4.4) and (4.6) we have \( \frac{\alpha}{2\pi} = \sqrt{x}\sin \theta = \frac{\theta}{2} + O(1/N^2) \) is of order \( 1/N \), while \( (1/d)^N \sim e^{2b} \), we must show how the order \( 1 \) contributions in the remaining terms of the numerator \( \nu_N \) in fact cancel. We apply the angle addition formula for the sine to write

\[
\sin(N-1)\theta = (\cos \theta) \sin N\theta - (\sin \theta) \cos N\theta,
\]

so that the sine terms in the numerator \( \nu_N \) are written

\[
\left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) \sin N\theta - \left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) \sin(N-1)\theta
\]

\[
= \frac{b}{N} \sin N\theta + \left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) (1 - \cos \theta) \sin N\theta + \left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) (\sin \theta) \cos N\theta.
\]

(4.11)

Now \( \sin N\theta \) and \( \cos N\theta \) are each of order \( 1 \) and by (4.5) we have \( 1 - \cos \theta = O(1/N^2) \), and finally, by (4.6), \( \sin \theta = \theta + O(\theta^3) = -i\frac{e^{2b}}{N} + O(1/N^2) \), as \( N \to \infty \). Therefore by (4.11) the numerator of (4.10) becomes

\[
\nu_N = \frac{\theta}{2}\left(-e^{2b} + \cos N\theta\right) + \frac{b}{N} \sin N\theta + O(1/N^2).
\]

(4.12)

We also have, by replacing \( d \) by \( 1/d \) in our calculation of \( \nu_N \) of (4.9)–(4.10), by way of (4.8), that

\[
\tilde{\nu}_N = \frac{\alpha}{2\pi}(\sqrt{x})^{-N}B\tilde{v}_N^* \text{ takes the form}
\]

\[
\tilde{\nu}_N = -\frac{\alpha}{2\pi}d^N + \left(\frac{1}{2} + \frac{4b+i(s+t)}{4N}\right) \sin N\theta - \left(\frac{1}{2} + \frac{i(s+t)f}{4N}\right) \sin(N-1)\theta + O\left(\frac{1}{N^2}\right).
\]

(4.13)

Therefore, by comparing \( \nu_N \) of (4.10) with \( \tilde{\nu}_N \) of (4.13), we find by a wholly similar analysis upon replacing \( b \) by \( -b \) in (4.11)–(4.12) that

\[
\tilde{\nu}_N = \frac{\theta}{2}\left(-e^{-2b} + \cos N\theta\right) - \frac{b}{N} \sin N\theta + O(1/N^2).
\]

(4.14)

The denominator \( \delta_N = \sin N\theta - \left(\frac{1}{2} + \frac{i(3t-s)}{4N}\right) \sin(N-1)\theta + O(1/N^2) \) is easily treated by expanding

\[
\sin(N-1)\theta \text{ as before so that}
\]

\[
\delta_N = \left(1 - \left(\frac{1}{2} + \frac{i(3t-s)}{4N}\right) \cos \theta\right) \sin N\theta + \left(\frac{1}{2} + \frac{i(3t-s)}{4N}\right) (\sin \theta) \cos N\theta + O\left(\frac{1}{N^2}\right)
\]

(4.15)

where we used (4.5)–(4.6). Finally, we have by (4.12), (4.14), and (4.15) that

\[
\frac{\nu_N^*}{w_N^*} = \frac{1}{B} \nu_N^* + \frac{1}{B} \tilde{\nu}_N^* = \frac{1}{B} \nu_N^* + \frac{1}{B} \tilde{\nu}_N^* = \frac{1}{B} \nu_N^* + \frac{1}{B} \tilde{\nu}_N^* + O\left(\frac{1}{N^2}\right),
\]

(4.16)

where we note that the sum of the terms \( \frac{1}{2}\frac{b}{N} \sin N\theta \) and \( -\frac{1}{2}\frac{b}{N} \sin N\theta \) cancel in finding the last equality. Here the higher order expansion of \( \delta_N \) in (4.15) was not applied; however it will be used in the proof Theorem 1.1(a). Now put \( p = \frac{1}{2} + \frac{b}{N} \) and compute

\[
\frac{\nu_N^* + (1-p)\tilde{\nu}_N^*}{w_N^*} = \frac{1}{B} \nu_N + \frac{1}{B} \tilde{\nu}_N = \frac{1}{B} \nu_N + \frac{1}{B} \tilde{\nu}_N + O(1),
\]

(4.17)

where the \( O(1) \) error term comes about from \( \frac{1}{B} \frac{b}{N} (\nu_N - \tilde{\nu}_N)/\delta_N \), since \( \frac{1}{B} \) is of order \( N^2 \) so that by (4.12) and (4.14) each of \( \nu_N/B \) and \( \tilde{\nu}_N/B \) are of order \( N^2 \frac{1}{N} = N \). Here by (4.6) we asymptotically
evaluate that $\delta_N \sim \frac{1}{2} \sin(-i\varphi(s,t)) = \frac{1}{2} \sin \varphi(s,t)$. Note that also $\cos N\theta \sim \cosh \varphi(s,t)$, while $\theta$ is given by (4.6) and $B$ is given by (4.8), so that finally by (4.16)–(4.17) we have, as $N \to \infty$

$$\frac{p\nu_N^s + (1-p)\nu_N^w}{w_N^s} \sim \frac{1}{B} \frac{1}{\delta_N} \frac{1}{\nu_N^s} \sim \frac{2\varphi(s,t) (\cosh \varphi(s,t) - \cosh(2b))}{(a^2 + (s^2 + t^2)/2) \sinh \varphi(s,t)} N. \quad (4.18)$$

Therefore by Proposition 3.10 and (4.18) we have that $K_N^2(r, y, z)/k_N^2$ is asymptotic to the right side of (4.18). Hence by $K_N^2[1] = 1$ we have, after setting $s = 0$ and $t = 0$, that the normalization constant $k_N^2$ in Proposition 3.10 satisfies

$$1/k_N^2 = \frac{\mathbb{P}(E_c \cap \{1 \leq H \leq N\})}{h_{cN}} \sim \frac{2c (\cosh(c) - \cosh(2b))}{a^2 \sinh(c)} N, \quad \text{as } N \to \infty. \quad (4.19)$$

It follows by (4.19) and the definitions of $h$ and $c_N$ that

$$\mathbb{P}(E_c \cap \{1 \leq H \leq N\}) \sim \frac{c (\cosh(c) - \cosh(2b))}{\sinh(c)} \frac{1}{N}, \quad \text{as } N \to \infty. \quad (4.20)$$

Furthermore, the limit of the right side of (4.3) is given as in the statement of Theorem 1.1(b). \qed

**Proof of Theorem 1.1(a).** We have that the number of excursions $\mathcal{M}_N$ until the last visit epoch $\mathcal{L}_N$ is a geometric variable with distribution $\mathbb{P}(\mathcal{M}_N = \ell) = \pi_0^\ell(1 - \pi_0)$, $\ell = 0, 1, \ldots$, where we denote $\pi_0 = \mathbb{P}(E_0 \cap \{1 \leq H \leq N\})$. Therefore by (2.1), we have that the joint probability generating function $\mathbb{E}(r^{\mathcal{M}_N} y^{2N} z^{\mathcal{L}_N} w^{\mathcal{M}_N})$ is given by

$$(1 - \pi_0)^{\infty} (\pi_0 u)^\ell \mathbb{E}(r^{\mathcal{M}_N} y^{2N} z^{\mathcal{L}_N} | \mathcal{E}_0 \cap \{H \leq N\})^\ell = \frac{1 - \pi_0}{1 - \pi_0 u K_N(r, y, z)}, \quad (4.21)$$

since $\mathbb{E}(r^{\mathcal{M}_N} y^{2N} z^{\mathcal{L}_N} | \mathcal{M}_N = \ell) = \mathbb{E}(r^{\mathcal{M}_N} y^{2N} z^{\mathcal{L}_N} | \mathcal{E}_0 \cap \{H \leq N\})^\ell$ by the Markov property. Now we substitute (4.1) into (4.21), and, since we include the statistic $\frac{1}{2} \mathcal{M}_N$ in both the statistics $\mathcal{Y}_N$ and $\mathcal{Z}_N$, we also substitute

$$u = u(s, t, N) = e^{\pi \mathrm{i}(s+t)}. \quad (4.22)$$

Thus by Proposition 2.6, since the normalization constant $k_N$ of $K_N$ satisfies $\pi_0 k_N = 2\xi$, we have

$$\mathbb{E}(e^{\frac{1}{N} \ell(s\mathcal{Y}_N + t\mathcal{Z}_N)}) = \frac{1 - \pi_0}{1 - \pi_0 u \cdot K_N(r, y, z)} = \frac{(1 - \pi_0)w_N^s}{w_N^s - 2\xi ur^2 z^2 q_N^s}, \quad (4.23)$$

where it is understood that in this last expression $r, y, z, u, w_N^s$, and $q_N^s$ are all composed with (4.1) and (4.22). Now apply (2.31) to write $q_N^s = y^2 q_N - \sqrt{x} q_N^0 (\sqrt{x} q_{N-1})$. We evaluate the coefficients of $q_N$ and $\sqrt{x} q_{N-1}$ for the term $2\xi ur^2 z^2 q_N^s$ asymptotically as follows. We have, as $N \to \infty$,

$$2\xi ur^2 z^2 y^2 = \frac{1}{2} + \frac{i(t-3s)}{4N} + O(\frac{1}{N^2}); \quad 2\xi ur^2 z^2 \sqrt{x} q_N^0 = \frac{-i(s-t)}{N} + O(\frac{1}{N^2}). \quad (4.24)$$

As in the proof of Theorem 1.1(b) write $\delta_N = \frac{2}{\pi}(\sqrt{x})^{-N} w_N$, and introduce $\epsilon_N = \frac{2}{\pi}(\sqrt{x})^{-N} q_N$. We divide both the numerator and denominator of the last ratio in (4.23) by $\frac{2\xi}{\alpha}(\sqrt{x})^N$ to write

$$\mathbb{E}(e^{\frac{1}{N} \ell(s\mathcal{Y}_N + t\mathcal{Z}_N)}) = \frac{(1 - \pi_0)\delta_N}{\delta_N - 2\xi ur^2 z^2 \epsilon_N}. \quad (4.25)$$

Now by (4.24) and Lemma 4.1 we have that

$$2\xi ur^2 z^2 \epsilon_N = \left(\frac{1}{2} + \frac{i(t-3s)}{4N}\right) \sin N\theta + \frac{i(s-t)}{N} \sin(N-1)\theta + O(\frac{1}{N^2})$$

$$= \left(\frac{1}{2} + \frac{i(s-3t)}{4N}\right) \sin N\theta + O(\frac{1}{N^2}), \quad (4.26)$$
where the last equality follows by expanding \( \sin(N - 1)\theta \) as before and using that \( \cos \theta \) stands as 1 to the order required, and again \( \sin \theta = O(1/N) \). From the expressions (4.15) and (4.26) we see that there is a cancellation of the \( \left( \frac{1}{2} + i(s-3\ell)/4N \right) \sin N\theta \) terms such that

\[
\delta_N - 2\xi w^2\epsilon_N = \frac{1}{2} (\sin \theta) \cos N\theta + O(1/N^2).
\]

Therefore by (4.25) and the result of (4.15) for the numerator we have

\[
\mathbb{E}\{e^{i\frac{\pi}{N}(s\epsilon N + t\bar{Z}N)}\} = \frac{(1 - \pi_0)\left(\frac{1}{2} \sin N\theta + O(\frac{1}{N})\right)}{\frac{1}{2} (\sin \theta) \cos N\theta + O(1/N^2)}.
\]

(4.27)

Now plug in (4.6) to (4.27) and multiply the top and bottom of the fraction by \( N \) to find that

\[
\lim_{N \to \infty} \mathbb{E}\{e^{i\frac{\pi}{N}(s\epsilon N + t\bar{Z}N)}\} = \lim_{N \to \infty} \frac{N(1 - \pi_0) \sinh \varphi(s,t)}{\varphi(s,t) \cosh \varphi(s,t)}.
\]

(4.28)

By setting \( s = t = 0 \) we conclude by (4.28) that

\[
\lim_{N \to \infty} N(1 - \pi_0)\frac{\tanh(c)}{c} = 1; \quad \text{that is, } \pi_0 \sim \frac{c}{N \tanh(c)} \quad \text{as } N \to \infty.
\]

(4.29)

Therefore by (4.28)–(4.29) the proof of Theorem 1.1(a) is complete. \( \square \)

**Discussion.** Since \( \mathbb{P}(E_0) + \mathbb{P}(E_c) + \mathbb{P}(E_N) = 1 \), we have by (4.20) and (4.29) that

\[
\mathbb{P}(E_N) \sim \frac{1}{N} \frac{c \cdot \cosh(2b)}{\sinh(c)}, \quad \text{as } N \to \infty.
\]

(4.30)

**Proof of Theorem 1.1(c).** The event \( E_N \) consists of paths that start from \( m = 0 \) and either stay strictly positive after the starting point until they reach level \( N + 1 \) or else stay strictly negative after the starting point and reach level \( -(N + 1) \). The joint probability generating function of runs, short runs, and steps for all paths that exit one of the boundaries \( \pm (N + 1) \) on the first excursion attempt is \( z\eta g_N \), as discussed just after (2.32). Therefore by the Markov property, we have, under the substitution (4.1), that

\[
\mathbb{E}\left\{e^{i\frac{\pi}{N}(s\epsilon N + t\bar{Z}N)}|\Omega'\right\} = z\eta g_N.
\]

(4.31)

Recall that by Proposition 2.2 we have \( g_n = C_n \omega r z^{n-2}/w_n^s \), where \( C_n = w_n^s[1]/(1 - \xi) \). Therefore, by \( \eta = \kappa \tau, \kappa = 1 - \xi \), and \( \tau = \sqrt{x}/\sqrt{r} \), we have

\[
z\eta g_N = (1 - \xi)C_N \frac{r_z^N + 1\tau^{N-1}}{w_N^s} = \frac{(1 - \xi)C_N r_z(\sqrt{x})^N}{(\sqrt{x})^N \tau w_N^s}.
\]

(4.32)

Now apply (2.31) as before to find \( w_N^s = q_N - \sqrt{x}w_0^s(\sqrt{x}q_{N-1}) \), where we have both the coefficient \( \sqrt{x}w_0^s = \frac{1}{2} + O(1/N) \) from (4.8), and also by direct computation \( rz/\tau = 1 + O(1/N) \), as \( N \to \infty \). Therefore by Lemma 4.1 we have from (4.32), after dividing top and bottom of the fraction by \( \frac{2\xi^N}{\sqrt{x}} \), that

\[
z\eta g_N = \frac{(1 - \xi)C_N}{(\sqrt{x})^N \sin N\theta - \frac{1}{2} \sin(N - 1)\theta + O(N)}.
\]

(4.33)

Therefore, since by (4.6) we have \( \frac{\eta}{\sqrt{x}} = \frac{1}{2} \sin \theta \sim \frac{1}{2} \theta \sim -\frac{i\varphi(s,t)}{2N} \), as \( N \to \infty \), and since by (4.15) we have \( \sin \theta = \frac{1}{2} \sin(N - 1)\theta \sim \frac{1}{2} \sin N\theta \sim -\frac{1}{2} \sin \varphi(s,t) \), as \( N \to \infty \), we have by (4.31)–(4.33) that

\[
\lim_{N \to \infty} \mathbb{E}\{e^{i\frac{\pi}{N}(s\epsilon N + t\bar{Z}N)}|\Omega'\} = \lim_{N \to \infty} \frac{(1 - \xi)C_N \varphi(s,t)}{N(\sqrt{x})^N \sinh \varphi(s,t)}.
\]

(4.34)
By setting \( s = t = 0 \) in (4.34) we have
\[
\lim_{N \to \infty} \frac{(1 - \xi)C_N}{N(\sqrt{N})^N} = \frac{\sinh(c)}{c}.
\] (4.35)
Therefore by (4.34)–(4.35) the proof of Theorem 1.1(c) is complete. \( \Box \)

**Proof of Theorem 1.1(d).** Write \( \pi' = \mathbb{P}(E_N) \) and recall that \( \pi_0 = \mathbb{P}(E_0 \cap \{1 \leq H \leq N\}) \) is asymptotically evaluated by (4.29). Thus calculate by (4.29) and (4.30) that, as \( N \to \infty \),
\[
\mathbb{P}(\Omega') = \pi' + \pi_0 \pi' + \pi_0^2 \pi' + \cdots = \frac{\pi'}{1 - \pi_0} \sim \frac{c \cosh(2b) \tanh(c)}{\sinh(c)} = \frac{\cosh(2b)}{\cosh(c)};
\] (4.36)
that is, \( \lim_{N \to \infty} \mathbb{P}(\Omega') = \frac{\cosh(2b)}{\cosh(c)}. \)

Now denote the limits of Theorem 1.1, parts (a)–(c), respectively as \( T(s, t), U(s, t), \) and \( V(s, t) \). Then, by independence between \( (Y_N, Z_N) \) with \( (Y_N^\circ, 1 \leq N \leq Y_N^\circ \cdot 1_{\Omega'}, Z_N^\circ, 1 \leq N \leq Z_N^\circ \cdot 1_{\Omega'}) \), we have by (4.36) and \( \mathbb{P}(\Omega^0) = 1 - \mathbb{P}(\Omega') \) that \( \lim_{N \to \infty} \mathbb{E}\{e^{i\frac{k}{2}(sY_N + tZ_N)}\} \) is written by
\[
T(s, t) \lim_{N \to \infty} \left( \mathbb{P}(\Omega^0) \mathbb{E}\{e^{i\frac{k}{2}(sY_N + tZ_N)}\} + \mathbb{P}(\Omega') \mathbb{E}\{e^{i\frac{k}{2}(sY_N + tZ_N)}\} \right) = T(s, t) \left( \frac{\cosh(c) - \cosh(2b)}{\cosh(c)} U(s, t) + \frac{\cosh(2b)}{\cosh(c)} V(s, t) \right).
\] (4.37)

With a bit of algebra after substituting the expressions for \( T(s, t), U(s, t), \) and \( V(s, t) \) from the statement of Theorem 1.1(a)–(c) into the right side of (4.37), we obtain that the limit of (d) is \( \left( \frac{a^2 \cosh \varphi - \cosh(2b)}{a^2 + (a^2 + t^2)} + \cosh(2b) \right) \) / \( \cosh \varphi = \frac{2a^2 \cosh \varphi + (a^2 + t^2) \cosh(2b)}{2(a^2 + s^2 + t^2) \cosh \varphi}. \) \( \Box \)

### 4.1 Examples for Theorem 1.1

In this section we compute the limiting measure for a linear combination of the normalized pair of statistics of Theorem 1.1. Consider first the linear combination \( \frac{1}{N}(k_1 Y_N^\circ + k_2 Z_N^\circ) \) given \( \Omega^0 \), where for simplicity we take \( k_1^2 + k_2^2 = 2 \). Then by Theorem 1.1(b), after writing \( \cosh(2b) = 1 - 1 + \cosh(2b) \), and separating terms we have that
\[
\lim_{N \to \infty} \mathbb{E}\{e^{i\frac{k}{2}(k_1 Y_N^\circ + k_2 Z_N^\circ)} \mid \Omega^0\} = \frac{C_{a,b} \sqrt{c^2 + t^2}}{a^2 + t^2} \left( \frac{\tanh \frac{1}{2} \sqrt{c^2 + t^2} + \frac{1 - \cosh(2b)}{\sinh(c^2 + t^2)} \right). \] (4.38)

Here we have applied the trigonometric identity for \( \tanh \frac{\varphi}{2} \) of Remark 1.2 and \( C_{a,b} \) is defined in Theorem 1.1. By the uniqueness and continuity theorems, [2, Sect. 26], there is a unique probability measure \( \mu_{a,b} \) such that its characteristic function \( \tilde{\mu}_{a,b}(t) = \int_{-\infty}^{\infty} e^{itx} \mu_{a,b}(dx) \) is given by (4.38). Recall the following well known Mittag–Leffler expansions:
\[
\frac{\tanh(u)}{u} = \sum_{n=0}^{\infty} \frac{8}{(2n + 1)^2 \pi^2 + 4u^2}; \quad \frac{u}{\sinh(u)} = \sum_{n=0}^{\infty} \frac{(-1)^n u^2}{n^2 \pi^2 + u^2}.
\] (4.39)

Therefore by (4.38)–(4.39) we have that
\[
\tilde{\mu}_{a,b}(t) = \frac{C_{a,b}}{a^2 + t^2} \left( \sum_{n=0}^{\infty} \frac{4(c^2 + t^2)}{(2n + 1)^2 \pi^2 + c^2 + t^2} + (1 - \cosh(2b)) \sum_{n=-\infty}^{\infty} \frac{(-1)^n (c^2 + t^2)}{n^2 \pi^2 + c^2 + t^2} \right).
\] (4.40)

Now define
\[
s_k(x) = \frac{4b^2 e^{-a|x|} + \frac{k^2 \pi^2}{\sqrt{c^2 + k^2 \pi^2}} e^{-|x|\sqrt{c^2 + k^2 \pi^2}}}{4b^2 + k^2 \pi^2}, \quad \text{for all } -\infty < x < \infty, \ k \in \mathbb{Z}.
\] (4.41)

Here \( s_k(x) \) has been chosen such that, by direct calculation,
\[
\tilde{s}_k(t) = \int_{-\infty}^{\infty} e^{itx} s_k(x) \, dx = \frac{2(c^2 + t^2)}{(a^2 + t^2)(k^2 \pi^2 + c^2 + t^2)}.
\] (4.42)
By the monotone convergence theorem and (4.42) with \( t = 0 \), we have that \( \sum_{k=0}^{\infty} s_k(x) \) is integrable on \( \mathbb{R} \). Hence \( \int_{-\infty}^{\infty} e^{itx} \sum_{n=0}^{\infty} s_{2n+1}(x) \, dx = \sum_{n=0}^{\infty} \hat{s}_{2n+1}(x) \, dx \). By the dominated convergence theorem we justify also the term by term calculation of \( \int_{-\infty}^{\infty} e^{itx} \sum_{n=-\infty}^{\infty} (-1)^n s_n(x) \, dx \).

**Example 4.2.** Let \( s_k(x) \) be defined by (4.41). Define \( f_{a,b}(x) \) as follows.

\[
f_{a,b}(x) = 2C_{a,b} \sum_{n=0}^{\infty} s_{2n+1}(x) + \frac{1}{2} C_{a,b}(1 - \cosh(2b)) \sum_{n=-\infty}^{\infty} (-1)^n s_n(x).
\]

By (4.41)–(4.42) and the discussion following these displays we have shown that indeed the characteristic function \( \hat{f}_{a,b}(t) = \int_{-\infty}^{\infty} e^{itx} f_{a,b}(x) \, dx \) is given by the right side of (4.40) and so by the form (4.38) of Theorem 1.1(b). Therefore the probability measure \( \mu_{a,b} \) is absolutely continuous and satisfies \( \mu_{a,b}(dx) = f_{a,b}(x) \, dx \). In case \( b = 0 \), the formula (4.43) reduces to \( f_{a,0}(x) = 2C_{a,0} \sum_{n=0}^{\infty} \frac{e^{-|x|\sqrt{a^2 + (2n+1)^2 \pi^2}}}{\sqrt{a^2 + (2n+1)^2 \pi^2}} \). If in addition \( a = 0 \), then we simply obtain \( f_{0,0}(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{e^{-|2n+1|\pi|x|}}{2n+1} = \frac{1}{\pi} \arctanh(e^{-\pi|x|}) \). It is a curious fact that the density \( g(x) \) on the half-line defined by \( g(x) = f_{0,0}(x/2) \), \( x > 0 \), is its own inverse. Hence there is a logarithmic singularity at \( x = 0 \) for the green curve \( f_{0,0} \) in Figure 4. Let \( \mu \) be the probability measure with the limiting characteristic function \( \hat{\mu}(t) = \frac{e^{-|\tanh(c)|t}}{\sqrt{c^2+t^2}} \) of (1.3)(a) with \( s = t \). By direct calculation we have that

\[
\int_{-\infty}^{\infty} e^{itx} \frac{e^{-|x|\sqrt{(2n+1)^2 \pi^2 + 4c^2}}}{\sqrt{(2n+1)^2 \pi^2 + 4c^2}} \, dx = \frac{4}{(2n+1)^2 \pi^2 + 4(c^2+t^2)}. \]

Therefore by the dominated convergence theorem and (4.39) we have that

\[
\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} f_{2c,0}(x/2) \, dx; \quad \text{that is, } \mu(dx) = \frac{1}{2} f_{2c,0}(x/2) \, dx. \tag{4.44}
\]

**4.1.1. Inversion via the residue theorem for Remark 1.3.** We follow up on Remark 1.3 by applying the residue theorem to calculate the probability density of the measure corresponding to the limiting characteristic function \( h(t) = \frac{e^{-2c}}{a^2+t^2} + \frac{t^2 \cosh(2b)}{a^2+t^2} \) of the linear combination \( \frac{1}{N} (k_1 Y_N + k_2 Z_N) \)
with \( k_1^2 + k_2^2 = 2 \) in Theorem 1.1(d). Fix \( c > 0 \) and the real variable \( x > 0 \). We may assume \( c > 0 \) because the case \( c = 0 \) is handled classically; see the statement of [9, Cor. 2(iii)]. Let \( R > R_0 = \min\{2c, 2c^2\} \) and let \( M \) be a positive integer so large that \( 2\pi M > R \). By our choices, since \( 4M \) is even and \( 2(n+1) \) is odd, for all integers \( n \geq 0 \) we will have \( (4M)^2 - (2n+1)^2 > 4M \). So we have \( (n+\frac{1}{2})^2 - c^2 \neq (2\pi M)^2 \) and therefore none of the poles of \( h(t) \) will meet the horizontal line \( \Re(z) = 2\pi M \). We take the positively oriented, closed rectangular contour \( \Gamma \) in the complex plane with vertices \( \pm R + 0i \) and \( \pm R + 2\pi Mi \). As usual we construct the integrals along the four oriented sides:

\[
I_0 = \int_{-R}^{R} e^{itx}h(t) \, dt, \quad I_1 = \int_{-2\pi M}^{2\pi M} e^{i(R+iy)x}h(R+iy) \, idy, \tag{4.45}
\]

\[
I_2 = \int_{R}^{-R} e^{i(t+2\pi iM)x}h(t+2\pi Mi) \, dt, \quad I_3 = \int_{2\pi M}^{0} e^{i(-R+iy)x}h(-R+iy) \, idy.
\]

Thus \( \int_{\Gamma} e^{itx}h(z) \, dz = I_0 + I_1 + I_2 + I_3 \). To successfully apply the residue theorem we want to show that \( \lim_{R \to \infty} I_j = 0 \) for each \( j = 1, 2, 3 \). To accomplish this, we first observe that on all parts of the contour \( \Gamma \) that lie strictly above the real axis we have that

\[
|\sqrt{c^2 + z^2 - z}| \leq c^2/|z|, \quad z \in \Gamma \setminus \Re.
\]  

(4.46)

The estimate (4.46) is equivalent to \(|\sqrt{1 + c^2/2^2} - 1| \leq c^2/|z|^2\), where, by our choice of \( R > 2c^2 \) we have \( c^2/|z|^2 < \frac{1}{4} \), for all \( z \in \Gamma \setminus \Re \). Therefore it suffices to show that \(|\sqrt{1 + \epsilon - 1}| \leq |\epsilon|\) for all complex \( \epsilon \) with \( |\epsilon| \leq \frac{1}{2} \). This is a simple matter that one can derive from \( |\epsilon^n - 1| \leq \frac{1}{2} |\epsilon| \), if \( |\epsilon| \leq \frac{1}{2} \), and \(|\log(1 + \epsilon)| \leq \frac{3}{2} |\epsilon|\), if \( |\epsilon| \leq \frac{1}{2} \), where \( \epsilon, v \in \mathbb{C} \). Therefore by substituting \( v = \frac{1}{2} \log(1 + \epsilon) \) we obtain that (4.46) holds.

If we replace \( h(t) \) by its first term \( a^2/(a^2 + t^2) \), then by a standard result for the residue theorem, [8, Prop. 4.3.4], we have that the integrals in (4.45) converge to zero as \( R \to \infty \). By standard estimates, using \( \lim_{R \to \infty} 2Re^{-xR} = 0 \) and \( \int_0^\infty e^{-xy}dy = 1/x \), where \( x > 0 \) is fixed for the calculation, to finish showing \( \lim_{R \to \infty} I_j = 0 \) for each \( j = 1, 2, 3 \), it suffices to show that (1) for each \( \epsilon > 0 \) there exists \( R_0 > 0 \) such that \(|\sech(c^2 + z^2)| < \epsilon\) for all \( z \) in the vertical sides of the contour \( \Gamma \) and all \( R > R_0 \), and (2) there are constants \( C \) and \( R_1 \) such that \(|\sech(c^2 + (t + 2\pi Mi)^2)| \leq C \), for all \( -R \leq t \leq R \) and all \( R > R_1 \). Consider first the estimate (2). Put \( z = t + 2\pi Mi \) and \( w = \sqrt{c^2 + t + 2\pi Mi} \). Put also \( \delta = w - z \). We have \( \cosh(w) - \cosh(z) = 2 \sinh((w + z)/2) \sinh((w - z)/2) = 2 \sinh(z + \delta/2) \sinh(\delta/2) \). Also write \( \sinh(z + \delta/2) = \cosh(\delta/2) \sinh(z) + \cosh(z) \sinh(\delta/2) = \cosh(\delta/2) \sinh(t) + \cosh(t) \sinh(\delta/2) \). Therefore \( \cosh(w) - \cosh(z) = 2 \sinh(\delta/2) \cosh(t) + \sinh(\delta) \sinh(t) \). By (4.46) we have \(|\delta| \leq c^2/|z|\), for all \( R > R_0 \). Choose \( R_1 > R_0 \) so large that, if \(|\delta| < c^2/R_1 \), then \(|\sinh(\delta)| < 2|\delta| \) and also \( c^2/R_1 < 1/4 \). Thus, by \( \cosh(z) = \cosh(t) \), we have \(|\cosh(w)| \geq (1 - \frac{1}{8}) \cosh(t) - \frac{1}{2} \sinh(t) \geq \frac{3}{8} \cosh(t) \), because \( t \) is real so \( \cosh(t) \) is real and positive and \(|\sinh(t)| \leq \cosh(t) \). Since finally \( \cosh(t) \geq 1 \), for real \( t \), the statement (2) has been verified.

To verify the statement (1) we work with the vertical portion of the contour \( \Gamma \) along \( \Re(z) = R \); the same analysis works for the other vertical portion. Now we put \( z = R + iy \) and \( w = \sqrt{c^2 + z^2} \). By the identity for \( \cosh(w) - \cosh(z) \) of the previous paragraph, for all \( R > R_1 \) we have \(|\cosh(w)| \geq \frac{7}{8} |\cosh(z)| - \frac{2}{3} |\sinh(z)| \geq \frac{1}{2} \frac{3}{8} R - \frac{11}{8} e^{-R} \). Since this lower bound tends to \( \infty \) as \( R \to \infty \), we have that \( \sech(w) \) converges to zero uniformly for \( z \) belonging to the vertical portions of \( \Gamma \). Thus (1) is verified.

Clearly \( \int_{-\infty}^{\infty} |h(t)| \, dt < \infty \). Therefore the measure determined by the characteristic function \( h(t) \) has a bounded density \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}h(t) \, dt \); [4, Thm. 3.3.5]. Since \( h(t) \) is an even
function of real $t$, we have that $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} h(t) \, dt$ and $f(x)$ is an even function of $x \in \mathbb{R}$. Therefore, by (1) and (2), since $\int_{-\infty}^{\infty} e^{itx} h(t) \, dt$ is absolutely convergent, by the residue theorem we have that the sum of residues of $e^{izx}h(z)$ at poles inside $\Gamma$ converges as $R \to \infty$, and we have $\int_{-\infty}^{\infty} e^{itx} h(t) \, dt = 2\pi i \cdot \text{Res}(e^{izx}h(z), ai) + 2\pi i \sum_{n=0}^{\infty} \text{Res}(e^{izx}h(z), zn)$, for all $x > 0$, where $zn = \sqrt{((n + \frac{1}{2})\pi)i^2 - c^2}$ is an enumeration of all the poles of $e^{izx}h(z)$ in the upper half-plane besides the one at $z = ai$. By direct calculation, Res$(e^{izx}h(z), ai) = 0$ and

$$\text{Res}(e^{izx}h(z), zn) = \frac{-i\pi(-1)^n(2n+1)\cosh(2b)}{16b^2 + (2n+1)^2\pi^2} \sqrt{4c^2 + (2n+1)^2\pi^2} e^{-\frac{1}{2}x\sqrt{4c^2 + (2n+1)^2\pi^2}}. \quad (4.47)$$

Hence by the residue theorem and (4.47) we have that the density $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)\cosh(2b)}{16b^2 + (2n+1)^2\pi^2} \sqrt{4c^2 + (2n+1)^2\pi^2} e^{-\frac{1}{2}x\sqrt{4c^2 + (2n+1)^2\pi^2}}, \text{ for all } x \neq 0. \quad (4.48)$$

By a similar construction as shown above for case (d) of Remark 1.3, the probability density $g(x)$ whose characteristic function is $\frac{\sinh(c)}{c^2 + \pi^2}$ is given by

$$g(x) = \frac{\sinh(c)}{c} \sum_{n=0}^{\infty} \frac{(-1)^nn^2\pi^2}{\sqrt{c^2 + n^2\pi^2}} e^{-|x|\sqrt{c^2 + n^2\pi^2}}, \text{ for all } x \neq 0. \quad (4.49)$$

4.2. Limiting Univariate Laplace transforms with scaling by $N^2$.

**Proof of Corollary 1.4.** We focus first on statement (b) of the corollary. We start with the runs statistic and then turn to the other two statistics. For all cases in which we study a runs statistic alone we set $r = r(\lambda, N) = e^{-\lambda/N^2}$, and $y = z = 1$. Define $\beta$ and $x$ composed with these substitutions according to (2.19). We define $\cos \theta$ again by (4.4) so that $\cos \theta$ is a function of $\lambda \geq 0$ and $N \geq 1$. By direct computation we have

$$\cos \theta = 1 + \frac{c^2 + \lambda}{2N^2} + O(1/N^4), \text{ as } N \to \infty. \quad (4.50)$$

We choose a branch of $\theta$ so that $|e^{i\lambda / N^2}| > 1$ for large $N$, so by (4.50) we have

$$\theta = -i\sqrt{c^2 + \lambda} = O(1/N^3), \text{ as } N \to \infty. \quad (4.51)$$

By direct computation we rewrite (4.8) in the current context as follows:

$$B = \frac{a^2 + 1}{2N^2} + O(1/N^4); \quad d = 1 - \frac{2b}{N} + O(1/N^2); \quad \sqrt{x} = \frac{1}{2} + O(1/N^2); \quad \sqrt{x}/\tau - \sqrt{x} = -\sqrt{x} + O(1/N^2); \quad \sqrt{x}w_0 = \sqrt{x} + O(1/N^2);$$

$$\sqrt{x}/d = \frac{1}{2} + \frac{b}{N} + O(1/N^2); \quad d/\sqrt{x} = \frac{1}{2} - \frac{b}{N} + O(1/N^2). \quad (4.52)$$

Recall the formula (4.7) for $v_N^*$. By (2.31) we have $w_N^* = q_N - \sqrt{x}w_0^*(\sqrt{x}q_{N-1})$. We now again compute an asymptotic expression for $v_N^*/w_N^* = \frac{1}{B}(\nu_N/\delta_N)$ of (4.9), this time under (4.50)–(4.52). We apply the angle addition formula for the sine terms as in (4.11) and obtain again a cancellation of order 1 terms for the sine terms $(\sqrt{x}/d)\sin N\theta + (\sqrt{x}B/\tau - \sqrt{x})\sin(N-1)\theta$ of the numerator $v_N$, now under (4.52), as follows: $(\frac{1}{2} + \frac{b}{N})\sin N\theta - \frac{1}{2}\sin(N-1)\theta + O(1/N^2) = \frac{1}{2}\sin \theta \cos N\theta + \frac{b}{N}\sin N\theta + O(1/N^2)$. Therefore, just as in the analysis that yields (4.12) and (4.14), these equations continue to hold
verbatim in the current context since $\theta$ of (4.51) is still of order $1/N$. Hence by (4.12) and (4.14) we have

$$p\nu_N + (1 - p)\tilde{\nu}_N = \frac{\theta}{2} (\cos N\theta - \cosh(2b)) + O(1/N^2),$$

(4.53)

where again the contributions of $+\frac{b}{N} \sin N\theta$ and $-\frac{b}{N} \sin N\theta$ cancel in the linear combination $p\nu_N + (1 - p)\tilde{\nu}_N$ to order $1/N^2$. The denominator $\delta_N = \sin N\theta - \sqrt{x}w_0^* \sin(N - 1)\theta$ of (4.9) is simply given, again by the angle addition formula for the sine and (4.52), by $\delta_N = (1 - \frac{1}{2} \cos \theta) \sin N\theta + \frac{1}{2} (\sin \theta) \cos N\theta + O(\frac{1}{N^2})$, so that by (4.50)–(4.51),

$$\delta_N = \frac{1}{2} \sin N\theta + \frac{1}{2}(\sin \theta) \cos N\theta + O(\frac{1}{N^2}).$$

(4.54)

Therefore by Proposition 3.10, (4.53), this expression for $\delta_N$, and the asymptotical expansions of $\theta$ and $B$ in (4.51)–(4.52), we have that $\lim_{N \to \infty} E\{e^{-\lambda \frac{1}{N^2} \bar{R}_N^2} \mid \Omega^o\}$ is given as

$$\lim_{N \to \infty} k_N^o N B^{-1} \frac{p\nu_N + (1 - p)\tilde{\nu}_N}{\delta_N} = C_{a,b} \frac{\sqrt{c^2 + \lambda} \left( \cosh \sqrt{c^2 + \lambda} - \cosh(2b) \right)}{(a + \lambda) \sinh \sqrt{c^2 + \lambda}},$$

(4.55)

since by (4.19) we have $\lim_{N \to \infty} Nk_N^o = C_{a,b}$. Therefore the proof of part (b) of the corollary is complete for the case of the runs statistic.

We now briefly discuss the case of the other two statistics for part (b) of the corollary. For the short runs statistic we put $r = 1$, $y = y(\lambda, N) = e^{-2\lambda/N^2}$, and $z = 1$. The reason for the factor of 2 in the exponent of $y$ is that we want to compute $\lim_{N \to \infty} E\{e^{-\lambda \frac{1}{N^2} \bar{R}_N^2} \mid \Omega^o\} = \lim_{N \to \infty} K_N^o(1, e^{-2\lambda/N^2}, 1)$, where $K_N^o$ is defined by (3.1). Even though now, for example, $\beta, x, \tau,$ and $w_0^*$ composed with these values of $r, y$ and $z$ are no longer the same as for the case of the runs statistic, it turns out that there is only a difference starting from the order $1/N^2$ term in the expansions of these quantities. Also the value of $B$ matches the case of the runs statistic through order $1/N^2$, and only differs starting from order $1/N^4$. In fact (4.52) continues to hold verbatim for the short runs statistic. This is also true of the steps statistic under the substitutions $r = y = 1$ and $z = e^{-\frac{1}{2} \lambda/N^2}$. Moreover in all three cases of statistics, with the appropriate values of $r, y$ and $z$ according to the case, we have that (4.50)–(4.51) hold. Therefore, by the same lines of proof as for the case of runs, the proof of part (b) is complete.

We turn to the proof of statement (a). For the runs statistic, by the proof of part (a) of Theorem 1.1, by rewriting (2.43) with $u = 1$ and $w_N^*$ and $q_N^*$ composed with the appropriate expressions for $r, y$, and $z$, depending on the case, we have that $\lim_{N \to \infty} E\{e^{-\lambda R/N^2}\}$ is computed as

$$\lim_{N \to \infty} \frac{1 - \pi_0}{1 - \pi_0 K_N} = \lim_{N \to \infty} \frac{(1 - \pi_0)w_N^*}{w_N^* - 2\xi r^2 z^2 \epsilon_N} = \lim_{N \to \infty} \frac{(1 - \pi_0)\delta_N}{\delta_N - 2\xi r^2 z^2 \epsilon_N};$$

(4.56)

with $\epsilon_N = \frac{\sqrt{x}}{2} (\sqrt{x})^{-N} q_N^*$. Now of course by (2.31) we have $q_N^* = y^2 q_N - \sqrt{x}q_N^* (\sqrt{x}q_N - 1)$. But in every case of the three types of statistics of the corollary, instead of (4.24) we now have

$$2\xi w^2 z^2 y^2 = \frac{1}{2} + O(1/N^2); \quad 2\xi r^2 z^2 \sqrt{x} q_N^* = O(1/N^2).$$

(4.57)

Therefore by (4.54) and (4.57) we have that the denominator the last limit in (4.56), namely $\delta_N - 2\xi r^2 z^2 \epsilon_N$, is given by

$$\frac{1}{2} \sin N\theta + \frac{1}{2}(\sin \theta) \cos N\theta - \frac{1}{2} \sin \theta + O(\frac{1}{N^2}) = \frac{1}{2} (\sin \theta) \cos N\theta + O(\frac{1}{N^2}).$$

(4.58)

Now plug in $\delta_N \sim \frac{1}{2} \sin N\theta$ in the numerator and $\delta_N - 2\xi r^2 z^2 \epsilon_N \sim \frac{1}{2} \sin \theta \cosh N\theta$ for the denominator of (4.56). Here, as shown in (4.29), $(1 - \pi_0)$ has order $1/N$ to match the order $1/N$ of $\sin \theta$. Therefore by way of (4.51) and (4.58) plugged into (4.56), the proof of part (a) is complete.
The proof of part (c) follows exactly as in Theorem 1.1(c) since we merely take the limit as 
\( N \to \infty \) in (4.33), where the denominator of that display is by (4.52) asymptotically 
\( \frac{1}{2} \sin N\theta \sim -i \sinh \sqrt{c^2 + \lambda} \). The proof of part (d) follows by algebra as in the proof of Theorem 1.1(d). \( \square \)

4.3. Applications. Recall the extended Markov chain process \( \{\bar{X}_j\} \) of section 1 wherein the state \( c \) is identified with the state \( m = 0 \) for the purpose of modifying the original chain \( \{X_j\} \). In the symmetric case, the absolute value process \( \{|X_j|\} \) models the length of queue in a discrete queuing process with catastrophes. Let \( \mathcal{N} \geq 0 \) denote the number of times that the state \( c \) is reached until the extended process terminates at one of the boundaries \( \pm (N + 1) \), and call \( \mathcal{N} \) the number of catastrophic cycles. Note that \( 1 + \mathcal{N} \) is a standard geometric random variable with success probability \( \mathbb{P}(\Omega^c) \), where, for the asymptotically symmetric case, \( \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) \) is given by (4.36). Denote \( \tilde{\mathcal{L}}_\nu, \tilde{\mathcal{R}}_\nu, \tilde{\mathcal{V}}_\nu, \) and \( \tilde{\mathcal{M}}_\nu \), respectively, as the number of steps, runs, short runs, and excursions of the absolute value process \( \{|X_j|\} \) during the \( \nu \)th catastrophic cycle. Denote also \( \bar{\mathcal{L}}', \bar{\mathcal{R}}', \bar{\mathcal{V}}', \) and \( \bar{\mathcal{M}}' \), respectively, as the number of steps, runs, short runs, and excursions of the absolute value process after the last catastrophic cycle, including further possible excursions as well as the meander (escape to boundary) section of \( \{\bar{X}_j\} \). Then \( \bar{L} = \bar{\mathcal{L}}' + \sum_{\nu=1}^{\mathcal{N}} \bar{\mathcal{L}}_\nu, \bar{R} = \bar{\mathcal{R}}' + \sum_{\nu=1}^{\mathcal{N}} \bar{\mathcal{R}}_\nu, \)
\( \bar{V} = \bar{V}' + \sum_{\nu=1}^{\mathcal{N}} \bar{\mathcal{V}}_\nu, \) and \( \bar{M} = \bar{\mathcal{M}}' + \sum_{\nu=1}^{\mathcal{N}} \bar{\mathcal{M}}_\nu \), respectively, denote the total number of steps, runs, short runs and excursions during the whole process \( \{|\bar{X}_j|\} \).

**Corollary 4.3.** Let \( p = \frac{1}{2} + \frac{c}{\sqrt{2}} \) and denote \( \bar{Y} = \bar{\mathcal{L}}' - 2\bar{V} + \frac{\bar{c}}{2} \bar{M} \), and \( \bar{Z} = \bar{\mathcal{L}} - 3\bar{R} + 2\bar{V} + \frac{3\bar{c}}{2} \bar{M} \). Denote \( \varphi(s,t) = \sqrt{a^2 + 4b^2 + (s^2 + t^2)/2} \). Then we have the following limiting joint characteristic function under scaling of \( \bar{Y} \) and \( \bar{Z} \) by \( N \):
\[
\lim_{N \to \infty} \mathbb{E}\left\{e^{i\frac{\bar{Y}}{N}}(s\bar{Y} + t\bar{Z})\right\} = \frac{(2a^2 + s^2 + t^2) \cosh(2b)}{(s^2 + t^2) \cosh \varphi(s,t) + 2a^2 \cosh(2b)}.
\]

**Proof.** As in the proof of part (d) of Theorem 1.1 denote the limits of Theorem 1.1, parts (a)–(c), respectively, as \( T(s,t), U(s,t), \) and \( V(s,t) \). Then \( \mathbb{P}(\mathcal{N} = \ell) = \mathbb{P}(\Omega^c)\mathbb{P}(\Omega^c)^\ell, \ell = 0,1,2,\ldots \). Also, denoting \( \bar{\mathcal{L}} = \bar{\mathcal{L}}_1, \bar{\mathcal{R}} = \bar{\mathcal{R}}_1, \) etc., corresponding to the first possible catastrophic cycle \( (\nu = 1) \), by Theorem 1.1 (a)–(b) we have
\[
\lim_{N \to \infty} \mathbb{E}\left\{e^{i\frac{\bar{Y}}{N}}(s(\bar{\mathcal{L}}' - 2\bar{V} + \frac{\bar{c}}{2} \bar{M}) + t(\bar{\mathcal{L}} - 3\bar{R} + 2\bar{V} + \frac{3\bar{c}}{2} \bar{M}))\right\} = T(s,t)U(s,t).
\]
Further, the corresponding limiting characteristic function with primes on the variables in this last display, corresponding to the process remaining after the last catastrophic cycle, is \( T(s,t)V(s,t) \). Therefore \( \lim_{N \to \infty} \mathbb{E}\left\{e^{i\frac{\bar{Y}}{N}}(s\bar{Y} + t\bar{Z})\right\} \) is given by
\[
T(s,t)V(s,t) \lim_{N \to \infty} \sum_{\ell=0}^{\infty} \mathbb{P}(\Omega^c)^\ell (T(s,t)U(s,t)\mathbb{P}(\Omega^c))^\ell = \lim_{N \to \infty} \frac{\mathbb{P}(\Omega')T(s,t)V(s,t)}{1 - \mathbb{P}(\Omega^c)T(s,t)U(s,t)}.
\]
The proof follows after some algebra since by (4.36) we have \( \lim_{N \to \infty} \mathbb{P}(\Omega') = \frac{\cosh(2b)}{\cosh(c)} \). \( \square \)

Our last result concerns the joint distribution of the level \( X_{L-1} \) at the jump–off point and the height \( H \) of the path \( \{(j, X_j)\} \) given \( E_c \). Of course the given event \( E_c \) has probability of order \( O(1/N) \), but we may obviously reframe the following result to mean the limiting joint distribution of jump–off level and height under scaling by \( N \) along the last excursion attempt given the macroscopic event \( \Omega^c \) that \( \{X_j\} \) terminates at \( c \).
Theorem 4.4. Let \( p = \frac{1}{2} + \frac{b}{N} \), and denote \( c = \sqrt{a^2 + 4b^2} \). Let \( \{\mu_N\} \) be the sequence of measures on \([-1, 1] \times [0, 1]\) where \( \mu_N \) is the conditional joint distribution of the pair \( (\frac{1}{N}X_{L-1}, \frac{1}{N}H) \) given \( E_c \). Then \( \{\mu_N\} \) converges weakly to \( \mu \), where \( \mu \) is absolutely continuous with respect to planar Lebesgue measure with joint probability density: \( f(x, y) = k_{a,b}e^{2bx} \sinh(c|x|)/\sinh^2(cy) \), if \( 0 < |x| < y < 1 \), and \( f(x, y) = 0 \), otherwise, with \( k_{a,b} = \frac{a^2 \sinh(c)}{2(\cosh(c) - \cosh(2b))} \).

Proof. Consider first the definition (3.7)(i). By Lemma 2.4 and (3.11) we have, for all \( 1 \leq k \leq n \leq N \),

\[
\mathbb{P}(E_c)\mathbb{P}(X_{L-1} = k, H = n | E_c) = c_N ph \left( \frac{(ph)^{n-1} ((1 - p)h)^{n-k} w_k^{*}[1]}{w_{n-1}[1]} \right). \tag{4.59}
\]

Now use that \( \beta[1] = 1 \), \( x[1] = \xi \), and \( w_0^*[1] = 1 \). Thus by (2.31), we have

\[
w_n^*[1] = q_n - \sqrt{\xi} \left( \sqrt{\xi} q_{n-1} \right), \tag{4.60}
\]

where \( q_n \) and \( q_{n-1} \) are determined by Lemma 4.1 with \( x = \xi \) and \( \beta = 1 \). Now we rewrite the right side of (4.59) using \((ph)^{n}((1-p)h)^{k-1} = \sqrt{\xi}^{n-k} \sqrt{\xi}^{k-1} \), wherein \((1-p)h = d \sqrt{\xi} \) and \( ph/\sqrt{\xi} = 1/d \).

\[
c_N ph \left( \frac{(ph)^{n-1} ((1 - p)h)^{n-k} w_k^{*}[1]}{w_{n-1}[1]} \right) = \frac{\sqrt{\xi}^{n-1} \sqrt{\xi}^{n-k} w_k^{*}[1]}{w_{n-1}[1]} \frac{w_{n-1}[1]}{w^*_n[1]} d^k (\sqrt{\xi})^{k-1}. \tag{4.61}
\]

Now plug in (4.60) and apply Lemma 4.1 with \( x = \xi \) to rewrite \( q_n \) and \( \sqrt{\xi} q_{n-1} \). Also denote \( S(n, \theta) = \sin n\theta - \sqrt{\xi} \sin (n-1)\theta \). Therefore, after canceling common powers of \( \sqrt{\xi} \) in numerator and denominator, we have \( \sqrt{\xi}^{n-k}/w_n^*[1] = \frac{\alpha}{2i} (1/S(n, \theta)) \). Thereby obtain that the product \( \sqrt{\xi}^{n-k} w_k^{*}[1] / w_{n-1}[1] \) on the right side of (4.61) is written

\[
\frac{\alpha}{2i} \cdot \frac{1}{S(n-1, \theta)} \cdot \frac{1}{S(n, \theta)} \cdot \frac{S(k - 1, \theta)}{d^k}. \tag{4.62}
\]

Also we write \( \theta = \theta[1] = -\frac{\theta}{N} + O(1/N^2) \) by (4.6). Further, \( \sqrt{\xi} = \frac{1}{2} + O(1/N^2) \). Therefore, for example, by the angle addition formula for sine, we have \( \sin n\theta - \sqrt{\xi} \sin (n-1)\theta = \frac{1}{2} \sin n\theta + e_N \), where \( e_N \) varies from line to line but \( e_N = O(1/N) \) with a constant implied by the big oh that is uniform in \( 1 \leq k \leq n \leq N \). Hence (4.62) is rewritten as

\[
\frac{\sin \theta}{2i} \cdot \frac{1}{2 \sin(n-1)\theta + e_N} \cdot \frac{1}{2 \sin n\theta + e_N} \cdot \frac{\frac{1}{2} \sin(k - 1)\theta + e_N}{d^k}. \tag{4.63}
\]

If we now replace \( k \) by \(-k\) in the left side of (4.59), still for \( 1 \leq k \leq n \leq N \), then the right side of (4.59) changes only by replacing \( p \) by \(-1 - p\). Then in place of (4.61)–(4.63) we have the same forms except with \( 1/d \) in place of \( d \). Since \( (1/d)^{k} = d^{-k} \) for \( k \geq 1 \), then by setting \( k = \left| xN \right| \), which we do to calculate \( \lim_{N \to \infty} N^2 \mathbb{P}(X_{L-1} = \left| xN \right|, H = \left| yN \right| | E_c) \) in (4.65), by (4.8) we will find

\[
\lim_{N \to \infty} 1/d^{N} \mathbb{P}(X_{L-1} = \left| xN \right|, H = \left| yN \right| | E_c) = e^{2bx} \tag{4.64}
\]

arising from the denominator of the last fraction in (4.63) no matter whether \( 0 < x < 1 \) or \(-1 < x < 0\).

Now let \( \psi(x, y) \) be a continuous function on \([-1, 1] \times [0, 1]\) and let \( \lambda \) denote planar Lebesgue measure. We wish to determine the limit of \( \int \psi \, d\lambda_{N} \). We introduce the step function \( f_{N}(x, y) \) on \([-1, 1] \times [0, 1]\) defined by the step-sets \( S_{k,n} = \left[ \frac{k-1}{N}, \frac{k}{N} \right] \times \left[ \frac{n-1}{N}, \frac{n}{N} \right] \), for all \( -(N-1) \leq k \leq N \) and \( 1 \leq n \leq N \), and values \( f_{N}(x, y) = N^2 \mathbb{P}(X_{L-1} = \left| xN \right|, H = \left| yN \right| | E_c) \) on \( S_{\left| xN \right|, \left| yN \right|} \), for all
\(-1 \leq x < 1 \) and \(0 \leq y < 1\). Let \(0 < \epsilon < 1\). By continuity of \(\psi\) and the definitions of the distribution \(\mu_N\) and step function \(f_N\), we have that

\[
\lim_{N \to \infty} \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi(x, y) \, d\mu_N = \lim_{N \to \infty} \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi(x, y) f_N(x, y) \, d\lambda. \tag{4.64}
\]

Further, by tracking down the relations (4.59) and (4.61)–(4.63), on the given region \(\{1 > y \geq |x| \geq \epsilon\}\), we have by way of \(\theta \sim -\frac{i\epsilon}{N}\), as \(N \to \infty\), that

\[
\lim_{N \to \infty} f_N(x, y) = \lim_{N \to \infty} N^2 c_N \frac{1}{\mathbb{P}(E_c)} \frac{1}{2N} \int_{d|x|<\epsilon} \frac{1}{2} \sin((|yN| - 1)\theta + \epsilon_N) \left(\frac{1}{2} \sin((|yN| + \epsilon_N) \frac{1}{2} \sin(|yN| + \epsilon_N) \right) = k_{\alpha, \beta} e^{2\alpha x} \sinh(c|x|) \sinh^2(cy), \tag{4.65}
\]

where we referred to the asymptotics of \(1/\mathbb{P}(E_c) \sim \frac{\sinh(c)}{\epsilon(c\cosh(c) - \cosh(2\theta))} N \) from (4.20), and \(\frac{\alpha}{\beta} = \frac{\sin\theta}{2} \sim -\frac{i\epsilon}{2N}\). Notice that by the uniform error \(e_N\) in the denominators of (4.63) with \(n = |yN|\) and \(\theta\) independent of \(y\), we have \(f_N\) is bounded on \(\{1 > y \geq |x| \geq \epsilon\}\). Therefore, by the Lebesgue bounded convergence theorem we have by (4.65) that

\[
\lim_{N \to \infty} \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi(x, y) f_N(x, y) \, d\lambda = k_{\alpha, \beta} \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi(x, y) e^{2\alpha x} \sinh(cx) \sinh^2(cy) \, d\lambda. \tag{4.66}
\]

By (4.64)–(4.66), we have \(\lim_{N \to \infty} \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi \, d\mu_N = \int_{\{1 > y \geq |x| \geq \epsilon\}} \psi \cdot f \, d\lambda\). To complete the proof, it suffices to show: (1) \(\mu_N(\{|x| \leq \epsilon\}) \leq cB_N\), where \(B_N\) is a bounded function of \(N\) alone, and (2) \(\lim_{\epsilon \to 0} \int_{\{0 < |x| \leq \epsilon, y \geq \epsilon\}} f(x, y) \, d\lambda = 0\). Condition (2) follows easily by direct integration since

\[
\int_{\{0 < |x| \leq \epsilon, y \geq \epsilon\}} f(x, y) \, dy = k_{\alpha, \beta} e^{2\alpha x} \sinh(c|x|) \int_{\{1 \geq |x| \geq \epsilon\}} \csc^2(cy) \, dy \quad \text{where} \quad \int_{\{1 \geq |x| \geq \epsilon\}} \csc^2(cy) \, dy = \frac{1}{c} (\coth(c|1|) - \coth(c)).
\]

Thus condition (2) is obviously verified by \(\lim_{\epsilon \to 0} \int_{\{0 < |x| \leq \epsilon, y \geq \epsilon\}} f(x, y) \, d\lambda = 0\).

To verify condition (1) we focus on estimating \(\mu_N(\{0 < x \leq \epsilon\})\); the \(\mu_N\)-measure of \(\{-\epsilon \leq x < 0\}\) will be handled in the same way. By (4.59)–(4.61) write \(\mu_N(\{0 < x \leq \epsilon\}) = \mathbb{P}(1 \leq X_{L-1} \leq \epsilon N \mid E_c)\) as

\[
\mu_N(\{0 < x \leq \epsilon\}) = \frac{c_N}{\mathbb{P}(E_c)} \sum_{k=1}^{\epsilon N} \sum_{n=0}^{N} (\sqrt{\xi})^{n-1} \left(\frac{\sqrt{\xi}}{w_{n-1}}\right)^{w_{n-1-1}} \frac{w_{k-1}^*}{w_{n}^*} \frac{w_n^*}{w_{n+1}^*} \frac{w_{n-1}^*}{d_k(\sqrt{\xi})^{k-1}}. \tag{4.67}
\]

Now, since again \(x[1] = \xi, \tau[1] = 1, w_0^*[1] = 1, \) and \(q_0^*[1] = 0\), we have the identity

\[
\sum_{n=k}^{N} \frac{\xi^{n-1}[1]}{w_n^*[1]} = \frac{q_n^*[1]}{w_n^*[1]} - \frac{q_{k-1}^*[1]}{w_{k-1}^*[1]}, \quad \text{for all} \quad 1 \leq k \leq N,
\]

that follows from Lemma 2.5 as in the derivation of (2.38). Therefore, by (4.67) we have

\[
\mu_N(\{0 < x \leq \epsilon\}) = \frac{c_N \sqrt{\xi}}{\mathbb{P}(E_c)} \sum_{k=1}^{\epsilon N} \left(\frac{q_n^*[1]}{w_n^*[1]} - \frac{q_{k-1}^*[1]}{w_{k-1}^*[1]}\right) \frac{w_{k-1}^*}{d_k(\sqrt{\xi})^{k-1}}. \tag{4.68}
\]

Now apply the identity \(q_n^*[1] = q_N(\xi, 1)\) from (2.31), together with (4.60), and also apply Lemma 4.1 to rewrite the sum on the right side of (4.68) as follows

\[
\frac{2i}{\alpha} \sum_{k=1}^{\epsilon N} \left(\frac{S(k-1, \theta) \sin N\theta}{S(N, \theta)} - \sin(k-1)\theta\right) \, d^{-k}, \tag{4.69}
\]

where the factor \(\frac{2i}{\alpha}\) comes about for the individual terms \(w_{k-1}^*[1]/(\sqrt{\xi})^{k-1}\) and \(q_{k-1}^*[1]/(\sqrt{\xi})^{k-1}\). Now there will be a cancellation of order under the summation sign in (4.69) because \(\sin N\theta/S(N, \theta) \sim 2\), while \(S(k-1, \theta) \sim \frac{1}{2} \sin(k-1)\theta\), as \(N \to \infty\). To obtain a precise estimation we again apply the
sine angle addition formula to write \( \sin N\theta - \sqrt{\xi} \sin(N-1)\theta = \frac{1}{2} \sin N\theta + \frac{1}{2} \sin \theta \cosh N\theta + E_N \), where here and in the following we have a generic error \( E_N = O(1/N^2) \) wherein the constant implied by the \( O(1/N^2) \) is uniform over all \( 1 \leq k \leq N \). Therefore, after substituting \( \theta = -ic/N + E_N \), we have \( \sin N\theta/S(N, \theta) = 2 - 2\csc(c)/N + E_N \). Similarly, we have \( S(k-1, \theta) = \frac{1}{2} \sin (k-1)\theta + \frac{1}{2} \sin \theta \cos (k-1)\theta + E_N \). Therefore we have precisely that
\[
\frac{S(k-1, \theta) \sin N\theta}{S(N, \theta)} - \sin (k-1)\theta \tag{4.70}
\]
Therefore after canceling the \( \sin (k-1)\theta \) terms in (4.70) we have that
\[
\frac{S(k-1, \theta) \sin N\theta}{S(N, \theta)} - \sin (k-1)\theta = \frac{\sin \theta \cos (k-1)\theta - c \coth(c) \sin (k-1)\theta}{N} + E_N \tag{4.71}
\]
Now substitute for \( \theta = \theta[1] = -ic/N + E_N \) in (4.71), plug the result of (4.71) into (4.69), and in turn plug the expression (4.69) in for the sum of (4.68) to finally deduce that \( \mu_N(\{0 < x \leq \epsilon\}) \) equals
\[
\frac{c_N \sqrt{\xi}}{P(E_c)} \frac{2i}{\alpha} \sum_{k=1}^{N} \left( -\frac{ic}{N} \left( \cosh \frac{(k-1)c}{N} - \coth(c) \sinh \frac{(k-1)c}{N} \right) + E_N \right) d^{-k}. \tag{4.72}
\]
Now \( \frac{2i}{\alpha} = \frac{2}{\sin \theta} \sim \frac{2a}{c} N \), and by (4.20) and the definition of \( c_N = \frac{1}{2}a^2/N^2 \), we have \( c_N \sqrt{\xi}/P(E_c) \sim \frac{k_{a,b} \gamma}{2(c N)} \). Therefore, since \( d^{-k} \) is bounded in \( 1 \leq k \leq \epsilon N \) as is the numerator of the fraction under the sum, we have by (4.72) that \( \mu_N(\{0 < x \leq \epsilon\}) = \frac{1}{N} \sum_{k=1}^{N} \epsilon N^2 E_N = \epsilon N^2 E_N \). By the same steps we have \( \mu_N(\{-\epsilon \leq x < 0\}) = \epsilon N^2 E_N \). Since by construction the term \( N^2 E_N \) is a bounded function of \( N \), independent of \( \epsilon \), we have verified condition (1), and therefore the proof is complete. □

References


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