Proof (Proposition 1.3.6). By Definition 1.3.5 (I)(1) and by Lemma 1.3.4 we have that statements *4–5* of the proposition hold. Next fix $\ell \geq 2$ and notice that the case $j = 0$ in *1* is similar to the case of statement *2*, the difference being $x(a,b) \neq x_b$. We will first verify *2*. Thus we write, using the Definition 1.3.5 and (1.46), that $[\overline{w}]_{f-\ell,f}$ is given by:

$$w_\ell^*(a)\{\tfrac{1-b}{1-a}w_\ell^*(a) + \tfrac{b-a}{1-a}w_{\ell-1}^*(a)\} - \{\tfrac{1-b}{1-a}w_{\ell+1}^*(a) + \tfrac{b-a}{1-a}w_\ell^*(a)\}w_{\ell-1}^*(a). \tag{1.47}$$

The $w_\ell^*(a)w_{\ell-1}^*(a)$ terms cancel in (1.47). Thus we obtain by (1.47) and Lemma 1.3.4 that
$[\overline{w}]_{f-\ell,f} = \frac{1-b}{1-a}\left(w_\ell^*(a)^2 - w_{\ell+1}^*(a)w_{\ell-1}^*(a)\right) = a^2(1-a)(1-b)r^2z^4x_a^{\ell-2}$. Thus statement *2* is proved.

We now turn to statement *1*. Fix $\ell \geq 2$ and let $j \geq 0$. Denote $[a,b]_0 = (a,b)$ and $[a,b]_j = (b,b)$ for $j \geq 1$. Thus, by Definition 1.3.5(I)(3)–(4) and (1.46),

$$\begin{aligned}[\overline{w}]_{f-\ell,f+j+1} &= \overline{w}_{f-\ell,f+j+1}\{\beta[a,b]_j\overline{w}_{f-\ell+1,f+j+1} - x[a,b]_j\overline{w}_{f-\ell+1,f+j}\} \\ &\quad - \{\beta[a,b]_j\overline{w}_{f-\ell,f+j+1} - x[a,b]_j\overline{w}_{f-\ell,f+j}\}\,\overline{w}_{f-\ell+1,f+j+1}.\end{aligned} \tag{1.48}$$

Now the terms of (1.48) involving $\beta[a,b]_j$ cancel and we obtain from (1.48) and (1.46) that

$$[\overline{w}]_{f-\ell,f+j+1} = x[a,b]_j[\overline{w}]_{f-\ell,f+j}. \tag{1.49}$$

Now put $j = 0$ in (1.49) and conclude by *2* and (1.49) that statement *1* holds for the initial case $j = 1$ for the given fixed $\ell \geq 2$. Now for the same fixed index ℓ, take statement *1* as an induction hypothesis for induction on $j \geq 1$. We have just established this induction hypothesis for $j = 1$. Thus verify by (1.49) again that the induction step holds since $x[a,b]_j = x(b,b) = x_b$ for all $j \geq 1$. Thus statement *1* is proved.

Finally we turn to statement *3*. We note that (1.48)–(1.49) continues to hold by Definition 1.3.5 (I)(3) with $\ell = 1$ as long as $j \geq 1$. Now we compute by (1.44)–(1.45), Definition 1.3.5 (I)(3), and the interlacing bracket definition (1.46) that, since by (1.44), $\overline{w}_{f-1,f+1} = \omega(a,b)$, while by Definition 1.3.5, $\overline{w}_{f,f+2} = w_2^*(b) = \omega(b,b)$,

$$\begin{aligned}[\overline{w}]_{f-1,f+1} &= \omega(a,b)\omega(b,b) - \{\beta(a,b)\omega(a,b) - x(a,b)\cdot 1\}\cdot 1 \\ &= \omega(a,b)[\omega(b,b) - \beta(a,b)] + x(a,b) = b^2(1-a)(1-b)r^2z^4,\end{aligned} \tag{1.50}$$

where at the last step we make a direct calculation based on the definitions in (1.18) and (1.45). Now take statement *3* as an induction hypothesis for induction on $j \geq 1$. By (1.50) have established this induction hypothesis for $j = 1$. Thus verify by (1.49) with $\ell = 1$ and $j \geq 1$ that the induction step holds since $x[a,b]_j = x(b,b) = x_b$ for all $j \geq 1$. So, statement *3* is proved. □

**** Check last step of (1.50)

```
Factor[Simplify[omega[a, b, r, y, z] (omega[b, b, r, y, z] - betaab[a, b, r, y, z]) +
   xab[a, b, r, y, z] - b^2 (1 - a) (1 - b) r^2 z^4]]
```

```
0
```

We turn to the task of obtaining a closed formula for $\overline{w}_{m,n}$. By Definition 1.3.5 (I)(4), given $m = f - \ell < f$, $\overline{w}_{m,f+1}$ and $\overline{w}_{m,f+2}$ form the initial conditions for a recurrence $\overline{w}_{m,f+j+1} := \beta_b \overline{w}_{m,f+j} - x_b \overline{w}_{m,f+j-1}$, $j \geq 2$. Put

18 Gregory J. Morrow

$m = f - \ell$ for some $\ell \geq 1$. We denote the vector of these upward initial conditions across the stratum threshold by the 2×1 vector $\mathbf{W}(\ell)$. Then we define a 2×2 matrix $Q(b)$, and for each $\ell < f$, a 2×1 vector $\mathbf{d} = \mathbf{d}(\ell)$ by

$$Q(b) := \begin{bmatrix} q_1^*(b) & w_1^*(b) \\ q_2^*(b) & w_2^*(b) \end{bmatrix}, \ \mathbf{W}(\ell) := \begin{bmatrix} \overline{w}_{f-\ell,f+1} \\ \overline{w}_{f-\ell,f+2} \end{bmatrix} = Q(b)\mathbf{d}; \ \mathbf{d} := \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix}. \tag{1.51}$$

By Definition 1.3.5 (I)(1)–(3), we can write each term of the right side of the recurrence of (I)(3) using (I)(1)–(2)in terms of $w_\ell^*(a)$ and $w_{\ell+1}^*(a)$ as follows: $\overline{w}_{f-\ell,f+1} = \frac{1-b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{1-a} w_\ell^*(a)$ and $\overline{w}_{f-\ell,f} = w_\ell^*(a)$, so $\overline{w}_{f-\ell,f+2} = \beta(a,b)\left(\frac{1-b}{1-a} w_{\ell+1}^*(a) + \frac{b-a}{1-a} w_\ell^*(a)\right) - x(a,b) w_\ell^*(a)$. We combine terms with the notation $\kappa(a,b) := \left(\frac{b-a}{1-a}\right)\beta(a,b) - x(a,b)$. Thus

$$\mathbf{W}(\ell) = B \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}; \quad B = \begin{bmatrix} \frac{b-a}{1-a} & \frac{1-b}{1-a} \\ \kappa(a,b) & \frac{1-b}{1-a}\beta(a,b) \end{bmatrix}. \tag{1.52}$$

By equating the two expressions for the vector $\mathbf{W}(\ell)$ in (1.51) and (1.52), we recover

$$\mathbf{d}(\ell) = \begin{bmatrix} d_1(\ell) \\ d_2(\ell) \end{bmatrix} = M \begin{bmatrix} w_\ell^*(a) \\ w_{\ell+1}^*(a) \end{bmatrix}; \quad M := Q(b)^{-1} B. \tag{1.53}$$

Here it is clear that the entries of the matrix $M = (\mu_{i,j})$, with $\mu_{i,j} = \mu_{i,j}(a,b)$ $1 \leq i,j \leq 2$, do not depend on ℓ. We note by direct calculation from (1.51) that $\det(Q(b)) = -b^2 z^2$, so we have a straightforward formula for M via (1.51) and (1.53).

*****Define \kappa(a,b) and the matrix B of (1.52).

```
kappa[a_, b_, r_, y_, z_] := ((b - a) / (1 - a)) betaab[a, b, r, y, z] - xab[a, b, r, y, z];
b11[a_, b_, r_, y_, z_] := (b - a) / (1 - a);
b12[a_, b_, r_, y_, z_] := (1 - b) / (1 - a);
b21[a_, b_, r_, y_, z_] := kappa[a, b, r, y, z];
b22[a_, b_, r_, y_, z_] := ((1 - b) / (1 - a)) betaab[a, b, r, y, z];
```

**** Determinant of $ Q(b) $.

```
Factor[Simplify[
  q1stara[b, r, y, z] × w2stara[b, r, y, z] - q2stara[b, r, y, z] × w1stara[b, r, y, z]]]
```

$-b^2 z^2$

**** Define entries of $ M := Q(b)^{-1} B $.

```
m11[a_, b_, r_, y_, z_] := (1 / (b^2 z^2))
   (-w2stara[b, r, y, z] × b11[a, b, r, y, z] + w1stara[b, r, y, z] × b21[a, b, r, y, z]);
m12[a_, b_, r_, y_, z_] := (1 / (b^2 z^2))
   (-w2stara[b, r, y, z] × b12[a, b, r, y, z] + w1stara[b, r, y, z] × b22[a, b, r, y, z]);
m21[a_, b_, r_, y_, z_] := (1 / (b^2 z^2))
   (q2stara[b, r, y, z] × b11[a, b, r, y, z] - q1stara[b, r, y, z] × b21[a, b, r, y, z]);
m22[a_, b_, r_, y_, z_] := (1 / (b^2 z^2))
   (q2stara[b, r, y, z] × b12[a, b, r, y, z] - q1stara[b, r, y, z] × b22[a, b, r, y, z]);
```

Proposition 1.3.7. *Let $d_1(\ell)$ and $d_2(\ell)$ be defined by (1.51)–(1.53). Then*

$$\overline{w}_{f-\ell,f+j} = d_1(\ell)q_j^*(b) + d_2(\ell)w_j^*(b), \ \ \ell \geq 1, \ \ j \geq 1. \tag{1.54}$$

Proof (Proposition 1.3.7). Fix $\ell \geq 1$. By Definition 1.3.5 (I)(4), we have: $\overline{w}_{f-\ell,f+j+1} = \beta_b \overline{w}_{f-\ell,f+j} - x_b \overline{w}_{f-\ell,f+j-1}$, $j \geq 2$. But if we denote the right side of (1.54) by v_j, then also $v_{j+1} = \beta_b v_j - x_b v_{j-1}$, $j \geq 2$, because by construction each of $\{q_j^*(b)\}$ and $\{w_j^*(b)\}$ satisfy the same two term recurrence, and the coefficients $d_1(\ell)$ and $d_2(\ell)$ are independent of j. Also by definition (1.51), for any given $\ell \geq 1$, (1.54) holds for $j = 1$ and $j = 2$,that is, $v_j = \overline{w}_{f-\ell,f+j}$, $j = 1, 2$. Hence we have $v_j = \overline{w}_{f-\ell,f+j}$ for all $j \geq 1$. Since ℓ was arbitrary the proof is complete. □

Lemma 1.3.8. *For all $1 \leq m < n$, there holds: $\overline{w}_{m,n} = \overline{w}_{n-1,m-1}$.*

Proof (Lemma 1.3.8). Notice that the lemma holds in the initial cases $n-m = 1, 2$ by (1.44). Also, if $f \leq m < n$ or $1 \leq m < n \leq f$ then the statement holds by Definition 1.3.5 (I)(1) and (II)(1). So consider now $\overline{w}_{f-\ell,f+j}$ for $1 \leq \ell < f$ and $j \geq 1$. Our method is to prove the statement:
$(\mathrm{H})_{\ell,j}$: $\quad \overline{w}_{f-\ell,f+j} = \overline{w}_{f+j-1,f-\ell-1}$,
holds for both the initial cases $\ell = 1$ and $\ell = 2$, and all $j \geq 1$.

We first establish $(\mathrm{H})_{\ell,j}$ for $\ell = 1$ and all $j \geq 1$. On the one hand, write $\overline{w}_{f-1,f+j}$ by (1.54) with $\ell = 1$, and on the other hand, write $\overline{w}_{f+j-1,f-2}$ by Definition 1.3.5 (II)(2), as follows.

$$\begin{aligned}\overline{w}_{f-1,f+j} &= d_1(1)q_j^*(b) + d_2(1)w_j^*(b);\\ \overline{w}_{f+j-1,f-2} &= \tfrac{1-a}{1-b}w_{j+1}^*(b) + \tfrac{a-b}{1-b}w_j^*(b).\end{aligned} \tag{1.55}$$

By (1.51), (1.53) and direct calculation, we have that $d_1(1) = \mu_{1,1}w_1^*(a) + \mu_{1,2}w_2^*(a) = -(1-a)(1-b)r^2z^2$, and $d_2(1) = \mu_{2,1}w_1^*(a) + \mu_{2,2}w_2^*(a) = 1$. Therefore, by substitution into (1.55), we find that the two expressions in (1.55) are equal if and only if $(*) \quad -(1-b)^2r^2z^2q_j^*(b) = w_{j+1}^*(b) - w_j^*(b)$. By direct computation we check that $(*)$ is true at both $j = 1$ and $j = 2$. Thus since $\{q_j^*(b)\}$ and $\{w_j^*(b)\}$ each satisfy the same Fibonacci recurrence, $(*)$ holds for all $j \geq 1$.

****VERIFY FORULAE for $ d_1(1) $ and $ d_2(1) $.

```
d1one[a_, b_, r_, y_, z_] :=
  m11[a, b, r, y, z] × w1stara[a, r, y, z] + m12[a, b, r, y, z] × w2stara[a, r, y, z];
d1two[a_, b_, r_, y_, z_] := m21[a, b, r, y, z] × w1stara[a, r, y, z] +
   m22[a, b, r, y, z] × w2stara[a, r, y, z];
```

```
Factor[Simplify[d1one[a, b, r, y, z] - (- (1 - a) (1 - b) r^2 z^2)]]
```

```
0
```

```
Factor[Simplify[d1two[a, b, r, y, z] - (1)]]
```

```
0
```

**** VERIFY (*) for $ j=1 $.

```
Factor[Simplify[
  (- (1 - b) ^2 r^2 z^2) q1stara[b, r, y, z] - (w2stara[b, r, y, z] - w1stara[b, r, y, z])]]
```

```
0
```

**** VERIFY (*) for $ j= 2 $.

```
Factor[Simplify[
  (-(1 - b)^2 r^2 z^2) q2stara[b, r, y, z] - (w3stara[b, r, y, z] - w2stara[b, r, y, z])]]
```

```
0
```

Next we establish that $(\mathrm{H})_{\ell,j}$ holds with $\ell = 2$ and all $j \geq 1$. Write $\overline{w}_{f-2,f+j}$ by (1.54) with $\ell = 2$, and write $\overline{w}_{f+j-1,f-3}$ by Definition 1.3.5 (II)(3), as follows.

$$\begin{aligned} &\text{(i)} \quad \overline{w}_{f-2,f+j} = d_1(2)q_j^*(b) + d_2(2)w_j^*(b); \\ &\text{(ii)} \quad \overline{w}_{f+j-1,f-3} = \beta(b,a)\overline{w}_{f+j-1,f-2} - x(b,a)\overline{w}_{f+j-1,f-1}, \\ &\text{with } \overline{w}_{f+j-1,f-2} = \tfrac{1-a}{1-b}w_{j+1}^*(b) + \tfrac{a-b}{1-b}w_j^*(b); \quad \overline{w}_{f+j-1,f-1} = w_j^*(b). \end{aligned} \tag{1.56}$$

By (1.51) and (1.53) we directly verify that $d_1(2) = -(1-a)(1-b)r^2z^2\beta(b,a)$; $d_2(2) = \beta(b,a) - x(b,a)$. To verify that the expressions (i) and (ii) in (1.56) are equal, we substitute, $d_1(2)$ and $d_2(2)$ and obtain, after a little algebra in which $x(b,a)x_j^*(b)$ cancels on the two sides, the condition
$(**)\quad -(1-b)^2r^2z^2\beta(b,a)q_j^*(b) = \beta(b,a)\left(w_{j+1}^*(b) - w_j^*(b)\right), \ \forall\, j \geq 1$. But obviously $(**)$ is equivalent to the condition $(*)$ that was verified in the previous paragraph. Hence the two expressions in (1.56) are equal for all $j \geq 1$, so $(\mathrm{H})_{\ell,j}$ holds also at $\ell = 2$ for all $j \geq 1$.

****VERIFY FORULAE for $ d_1(2) $ and $ d_2(2) $.

```
d2one[a_, b_, r_, y_, z_] :=
  m11[a, b, r, y, z] × w2stara[a, r, y, z] + m12[a, b, r, y, z] × w3stara[a, r, y, z];
d2two[a_, b_, r_, y_, z_] := m21[a, b, r, y, z] × w2stara[a, r, y, z] +
   m22[a, b, r, y, z] × w3stara[a, r, y, z];
```

```
Factor[Simplify[d2one[a, b, r, y, z] - (-(1 - a) (1 - b) r^2 z^2 betaab[b, a, r, y, z])]]
```

```
0
```

```
Factor[Simplify[d2two[a, b, r, y, z] - ( betaab[b, a, r, y, z] - xab[b, a, r, y, z])]]
```

```
0
```

Finally, fix any $j \geq 1$. We appeal to (1.53) and (1.54) and to Definition 1.3.5 (II)(4), to obtain, for any $\ell \geq 3$,

$$\overline{w}_{f-\ell,f+j} = \left(\mu_{1,1}w_\ell^*(a) + \mu_{1,2}w_{\ell+1}^*(a)\right) q_j^*(b) + \left(\mu_{2,1}w_\ell^*(a) + \mu_{2,2}w_{\ell+1}^*(a)\right) w_j^*(b); \quad (1.57)$$
$$\overline{w}_{f+j-1,f-\ell-1} = \beta_a \overline{w}_{f+j-1,f-\ell} - x_a \overline{w}_{f+j-1,f-\ell+1}.$$

Write $u_\ell := \overline{w}_{f-\ell,f+j}$ for the first line of (1.57). Since with j fixed, u_ℓ is a linear combination of two successive terms of the sequence $\{w_\ell^*(a)\}$, it follows that,$\{u_\ell,\ \ell \geq 2\}$ itself satisfies the recursion $u_{\ell+1} = \beta_a u_\ell - x_a u_{\ell-1}$. But also $\{v_\ell := \overline{w}_{f+j-1,f-\ell-1}, \ell \geq 2\}$, in the second line of (1.57), satisfies the same recurrence in ℓ. Moreover, we proved that $(\mathrm{H})_{\ell,j}$ holds for $\ell = 1$ and $\ell = 2$, so for the given j, we have $u_1 = v_1$, and $u_2 = v_2$. Therefore we have $u_\ell = v_\ell$ for all $\ell \geq 1$. Thus by (1.57), $(\mathrm{H})_{\ell,j}$ is proved for all $\ell \geq 1$ with the given j. Since $j \geq 1$ was arbitrary, $(\mathrm{H})_{\ell,j}$ is true for all $\ell, j \geq 1$. □

Lemma 1.3.9. *The following identities hold.*

1. $\overline{w}_{f-\ell,f+j}[\mathbf{1}] = a^\ell b^{j-1} \Pi_{f-\ell,f+j},\ \forall\ \ell \geq 1,\ j \geq 1.$

2. $\overline{w}_{f+j,f-\ell}[\mathbf{1}] = a^{\ell-1} b^j \Pi_{f+j,f-\ell},\ \forall\ \ell \geq 2,\ j \geq 0.$

3. $q_\ell^*(a)[\mathbf{1}] = \ell a^{\ell-1}, \quad w_\ell^*(a)[\mathbf{1}] = a^{\ell-1}[\ell - (\ell-1)a];\ \forall\ \ell \geq 1.$

Proof. At $(r,y,z) = \mathbf{1}$ we have $\beta_a = 2a$ and $x_a = a^2$. Thus $\alpha = 0$ in (1.13). Therefore by (1.13), $q_\ell^*(a)[\mathbf{1}] = \lim_{\alpha\to 0} \frac{2^{-\ell}}{\alpha}\{(2a+\alpha)^\ell - (2a-\alpha)^\ell\} = \ell a^{\ell-1}$. Thus, by the second formula of (1.13), we obtain $w_\ell^*(a)[\mathbf{1}]$ by $x_a[\mathbf{1}] = a^2$, so *3* is proved. Now apply (1.54), also at $(r,y,z) = \mathbf{1}$. By (1.53) and direct calculation, $d_1(\ell)[\mathbf{1}] = -(1-a)(1-b)\ell a^{\ell-1}$, and $d_2(\ell)[\mathbf{1}] = a^{\ell-1}[\ell - (\ell-1)a]$. Now plug in $q_j^*(b)[\mathbf{1}]$ and $w_j^*(b)[\mathbf{1}]$ from *3*, into (1.54) to obtain formula *1* from Proposition 1.3.7 after direct simplification. The proof of *2* follows from *1* and Lemma 1.3.8, in view of Definition 1.3.1. □

****DEFINE q_n(x,\beta) and w_n(x,\beta) via (1.13).

```
alpha[x_, β_] := Sqrt[β^2 - 4 x];
q[x_, β_, n_] := (2^(-n) / alpha[x, β]) ((β + alpha[x, β])^n - (β - alpha[x, β])^n);
w[x_, β_, n_] := q[x, β, n] - x * q[x, β, n - 1];
```

```
Factor[Simplify[{q[x, β, 2], w[x, β, 2]}]]
```

```
{β, -x + β}
```

```
Simplify[xa[a, 1, 1, 1]]
```

a^2

****DEFINE q_{\ell}^{*}(a)[\mathbf{1}] , w_{\ell}^{*}(a)[\mathbf{1}], d_1(\ell)[\mathbf{1}], and d_2(\ell)[\mathbf{1}] .

```
qstara111[a_, ℓ_] := ℓ * a^(ℓ - 1);
wstara111[a_, ℓ_] := a^(ℓ - 1) (ℓ - (ℓ - 1) a);
d1eval[a_, b_, ℓ_] :=
  m11[a, b, 1, 1, 1] × wstara111[a, ℓ] + m12[a, b, 1, 1, 1] × wstara111[a, ℓ + 1];
d2eval[a_, b_, ℓ_] := m21[a, b, 1, 1, 1] × wstara111[a, ℓ] +
   m22[a, b, 1, 1, 1] × wstara111[a, ℓ + 1];
```

****Check Foormula for d_1(\ell)[\mathbf{1}] :

```
Factor[Simplify[d1eval[a, b, ℓ]]]
```

$-(-1+a)\, a^{-1+\ell}\, (-1+b)\, \ell$

****Check Foormula for d_2(\ell)[\mathbf{1}] :

```
Factor[Simplify[d2eval[a, b, ℓ]]]
```

$-a^{-1+\ell}\, (-a-\ell+a\,\ell)$

****VERIFY Forrmula *1* in statement of Lemma 1.3.9:

```
Factor[Simplify[d1eval[a, b, ℓ] × qstara111[b, j] +
   d2eval[a, b, ℓ] × wstara111[b, j] - (a^(ℓ - 1) b^(j - 1) (j a + ℓ b - (ℓ + j - 1) a b))]]
```

```
0
```

****VERIFY Forrmula *2* in statement of Lemma 1.3.9.

```
Factor[
 Simplify[d1eval[a, b, ℓ - 1] × qstara111[b, j + 1] + d2eval[a, b, ℓ - 1] × wstara111[b, j + 1] -
   (a^(ℓ - 2) b^(j) ((j + 1) a + (ℓ - 1) b - (ℓ + j - 1) a b))]]
```

```
0
```

1.3.4 Closed formula for $g_{m,n}$.

Proposition 1.3.10. *We have the following formulae for $\{g_{m,n}\}$.*
I. The formulae for upward between-strata cases, $j \geq 1$ and $\ell \geq 2$:

1. $g_{f-\ell,f+j} = \frac{\omega(a,a)}{2-a} r z^{j+\ell} \tau(a,b)[a\tau(a,a)]^{\ell-2}[b\tau(b,b)]^{j-1} \left(a\Pi_{f-\ell,f+j}/\overline{w}_{f-\ell,f+j}\right)$,
2. $g_{f-1,f+j} = \frac{\omega(a,b)}{a+b-ab} r z^{j+1}[b\tau(b,b)]^{j-1} \left(a\Pi_{f-1,f+j}/\overline{w}_{f-1,f+j}\right)$;

II. The formulae for downward between-strata cases, $j \geq 1$ and $\ell \geq 2$:

1. $g_{f+j,f-\ell} = \frac{\omega(b,b)}{2-b} r z^{j+\ell} \tau(a,b)[a\tau(a,a)]^{\ell-2}[b\tau(b,b)]^{j-1} \left(a\Pi_{f+j,f-\ell}/\overline{w}_{f+j,f-\ell}\right)$,
2. $g_{f,f-\ell} = \frac{\omega(a,b)}{a+b-ab} r z^{\ell}[a\tau(a,a)]^{\ell-2} \left(a\Pi_{f,f-\ell}/\overline{w}_{f,f-\ell}\right)$;

III. The formulae for within stratum cases:

1. $g_{m,m+\ell} = g_{m+\ell,m} = \frac{\omega(a,a)}{2-a} r z^{\ell}[a\tau(a,a)]^{\ell-2} \left(\frac{a}{b}\Pi_{m,m+\ell}/w^*_{\ell}(a)\right)$, $m < m+\ell \leq f-1$;

   1.1 $g_{f-\ell,f} = \frac{\omega(a,a)}{2-a} r z^{\ell}[a\tau(a,a)]^{\ell-2} \left(\frac{a}{b}\Pi_{f-\ell,f}/w^*_{\ell}(a)\right)$, $\ell \geq 1$;

2. $g_{m,m+j} = g_{m+j,m} = \frac{\omega(b,b)}{2-b} r z^{j}[b\tau(b,b)]^{j-2} \left(\Pi_{m,m+j}/w^*_{j}(b)\right)$, $f \leq m < m+j$;

   2.1 $g_{f+j,f-1} = \frac{\omega(b,b)}{2-b} r z^{j+1}[b\tau(b,b)]^{j-1} \left(\Pi_{f+j,f-1}/w^*_{j+1}(b)\right)$, $j \geq 1$.

Furthermore, the following identity holds for all $n \geq m+2$, where $\lambda_{m,n}$ is defined by (1.22).

$$\lambda_{m,n} = \frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}{\overline{w}_{m,n+1}\overline{w}_{m+1,n}}. \tag{1.58}$$