Remark 1.3.11. Since $\tau(a,b)[\mathbf{1}] = 1$ for all a and b, one easily checks by Definition 1.3.1 and Lemma1.3.9 that the formulae of Proposition 1.3.10 all yield the evaluation $g_{m,n}[\mathbf{1}] = 1$. The factor $\Pi_{m,n}$ appears in Lemma 1.3.9 the same as it does in the statements *I-II*, so these factors cancel at $\mathbf{1}$.

Remark 1.3.12. All formulae in *III* hold by (1.37) for the homogeneous case. For example, in the statement *III.1*, we have $\frac{a}{b}\Pi_{m,m+\ell} = \ell - (\ell - 1)a$, so there is no dependence on b.

Proof (Proposition 1.3.10). Recall by (1.17) that $g_{m,m+2} = g_{m+2,m} = rz^2$. One can easily check by Definitions 1.3.1 and (1.44) that in each of *I.1* with $j = 1$, and *II.2* with $\ell = 2$, the formulae reduce to rz^2. By (1.24)–(1.25), we must calculate a term $\lambda_{m,n}$ defined by (1.22). The term $\lambda_{m,n}$ is the same in both (1.24) and (1.25), so we only consider $m < n$ in (1.22). The structure of the proof is to first establish (1.58) for $n = m+2$ and to establish the initial cases $n - m = 3$ of the statements *I–II* of the proposition. Following this, an induction step will be established for all cases at once, wherein an inductive step for (1.58) shall be the main stepping stone of the proof.

Thus consider first $n := m + 2$ in (1.22). We consider 4 cases: (i) $n \leq f - 1$; (ii) $m = f - 2$, $n = f$; (iii) $m = f - 1$, $n = f + 1$; (iv) $m \geq f$. We verify by (1.18)–(1.20), Definition 1.3.1, Proposition 1.3.2, Proposition 1.3.6, and direct calculation, that in all cases (i)–(iv), $(*)$ $\lambda_{m,n} = \frac{\omega[a,b]^+_m \omega[a,b]^+_{m+1}}{\omega[a,b]^+_m \omega[a,b]^+_{m+1} - [\overline{w}]_{m,n}}$. Verification of $(*)$ by direct calculation suffices for (1.58), since for the numerator we have by (1.44) that $\overline{w}_{m,n} = \omega[a,b]^+_m$ and $\overline{w}_{m+1,n+1} = \omega[a,b]^+_{m+1}$, and since for the denominator we have by Definition (1.46) that $\overline{w}_{m,n}\overline{w}_{m+1,n+1} - [\overline{w}]_{m,n} = \overline{w}_{m+1,n}\overline{w}_{m,n+1}$.

******* Define Initial cases (i)-(iv) of \lambda_{m,n} with $ n - m = 2 $. Also define $ [\overline{w}_{m,n} $ in these same cases.

```
rho2upi[a_, b_] := (1 / 2) / (2 - a); rho2downi[a_, b_] := (1 / 2) / (2 - a);
rho2upii[a_, b_] := (1 / 2) / (2 - a);
rho2downii[a_, b_] := (1 / 2) a / (a + b - a * b);
rho2upiii[a_, b_] := (1 / 2) b / (a + b - a * b);
rho2downiii[a_, b_] := (1 / 2) / (2 - b);
rho2upiv[a_, b_] := (1 / 2) / (2 - b); rho2downiv[a_, b_] := (1 / 2) / (2 - b);
lambda2i[a_, b_, r_, y_, z_] := (1 - 4 (1 - a) (1 - a) rho2upi[a, b] × rho2downi[a, b] ×
      k[a, a, r, y, z] × k[a, a, r, y, z] r^2 z^4) ^ (-1);
lambda2ii[a_, b_, r_, y_, z_] := (1 - 4 (1 - a) (1 - b) rho2upii[a, b] × rho2downii[a, b] ×
      k[a, a, r, y, z] × k[a, b, r, y, z] r^2 z^4) ^ (-1);
lambda2iii[a_, b_, r_, y_, z_] :=
  (1 - 4 (1 - a) (1 - b) rho2upiii[a, b] × rho2downiii[a, b] ×
      k[a, b, r, y, z] × k[b, b, r, y, z] r^2 z^4) ^ (-1);
lambda2iv[a_, b_, r_, y_, z_] := (1 - 4 (1 - b) (1 - b) rho2upiv[a, b] × rho2downiv[a, b] ×
      k[b, b, r, y, z] × k[b, b, r, y, z] r^2 z^4) ^ (-1);
wbarbracket2upi[a_, b_, r_, y_, z_] := a^2 r^2 z^4 (1 - a) ^2;
wbarbracket2upii[a_, b_, r_, y_, z_] := a^2 r^2 z^4 (1 - a) (1 - b);
wbarbracket2upiii[a_, b_, r_, y_, z_] := b^2 r^2 z^4 (1 - a) (1 - b);
wbarbracket2upiv[a_, b_, r_, y_, z_] := b^2 r^2 z^4 (1 - b) ^2;
interlace2i[a_, b_, r_, y_, z_] := omega[a, a, r, y, z] × omega[a, a, r, y, z] /
    (omega[a, a, r, y, z] × omega[a, a, r, y, z] - wbarbracket2upi[a, b, r, y, z]);
interlace2ii[a_, b_, r_, y_, z_] := omega[a, a, r, y, z] × omega[a, b, r, y, z] /
    (omega[a, a, r, y, z] × omega[a, b, r, y, z] - wbarbracket2upii[a, b, r, y, z]);
interlace2iii[a_, b_, r_, y_, z_] := omega[a, b, r, y, z] × omega[b, b, r, y, z] /
    (omega[a, b, r, y, z] × omega[b, b, r, y, z] - wbarbracket2upiii[a, b, r, y, z]);
interlace2iv[a_, b_, r_, y_, z_] := omega[b, b, r, y, z] × omega[b, b, r, y, z] /
    (omega[b, b, r, y, z] × omega[b, b, r, y, z] - wbarbracket2upiv[a, b, r, y, z]);
```

****VERIFY IINITIAL CASES of $ \lambda_{m,n} $ for (1.59), via Verification of (*).

```
Factor[Simplify[lambda2i[a, b, r, y, z] - interlace2i[a, b, r, y, z]]]
```

```
0
```

```
Factor[Simplify[lambda2ii[a, b, r, y, z] - interlace2ii[a, b, r, y, z]]]
```

```
0
```

```
Factor[Simplify[lambda2iii[a, b, r, y, z] - interlace2iii[a, b, r, y, z]]]
```

```
0
```

```
Factor[Simplify[lambda2iv[a, b, r, y, z] - interlace2iv[a, b, r, y, z]]]
```

```
0
```

We turn to the initial conditions for *I–II*. There are again four cases to consider. We conform with the notation of (1.21) and (1.23). For the upward cases we write the lower index m and the upper index $m+3$. For the downward cases we write the upper index $m+2$ and the lower index $m-1$. The cases are (I.1) $m=f-2,\ m+3=f+1$; (I.2) $m=f-1,\ m+3=f+2$; (II.1) $m+2=f+1,\ m-1=f-2$; (II.2) $m+2=f,\ m-1=f-3$. We use direct calculation of $g_{m,m+3}$ or $g_{m+2,m-1}$ for the upward and downward cases, respectively. Besides the formulae (1.21) and (1.23), we use $\lambda_{m,m+2}$ given by (1.58), where $\overline{w}_{m+1,m+2}=1$ by definition (1.44). Since the denominator of $\lambda_{m,m+2}$ in each case is $\overline{w}_{m,m+3}=\overline{w}_{m+2,m-1}$ by Lemma 1.3.8, we compute $p_{m,m+3}:=(1/c)g_{m,m+3}\overline{w}_{m,m+3}$ and $p_{m+2,m-1}:=(1/c)g_{m+2,m-1}\overline{w}_{m,m+3}$ in the upwards and downwards cases respectively. Schematically, since $g[\mathbf{1}]=1$, we have $p/p[\mathbf{1}]=g\overline{w}/\overline{w}[\mathbf{1}]=numerator/\overline{w}[\mathbf{1}]$, where *numerator* stands for the stated formula without the denominator $\overline{w}$. By cancellation of the Π–factors as in Remark 1.3.11, we match $p/p[\mathbf{1}]$ with $(numerator/\Pi)/(\overline{w}[\mathbf{1}]/\Pi)$ for verification by direct calculation.

***** Statements of Numerators (without constants) for Initial Cases $ n - m = 3 $ for Proposition 1.3.10, part *I.*

```
p3I1[a_, b_, r_, y_, z_] :=
  z * h[a, b, r, y, z] * r * z^2 * omega[a, a, r, y, z] × omega[a, b, r, y, z];
p3I2[a_, b_, r_, y_, z_] := z * h[b, b, r, y, z] * r * z^2 *
   omega[a, b, r, y, z] × omega[b, b, r, y, z];
```

***** Verify Initial case (I.1)

```
Factor[Simplify[(p3I1[a, b, r, y, z] / p3I1[a, b, 1, 1, 1]) -
   (omega[a, a, r, y, z] * r * z^3 * tau[a, b, r, y, z] * (a / (2 - a)) * (1 / a^2))]]
```

```
0
```

**** Verify Initial case (I.2)

```
Factor[Simplify[(p3I2[a, b, r, y, z] / p3I2[a, b, 1, 1, 1]) - (omega[a, b, r, y, z] *
     r * z^3 * (b * tau[b, b, r, y, z]) * (a / (a + b - a * b)) * (1 / (a * b)))]]
```

```
0
```

***** Statements of Numerators (without constants) for Initial Cases $ n - m = 3 $ for Proposition 1.3.10, part *II.*

```
p3II1[a_, b_, r_, y_, z_] :=
  z * h[a, b, r, y, z] * r * z^2 * omega[a, b, r, y, z] × omega[b, b, r, y, z];
p3II2[a_, b_, r_, y_, z_] := z * h[a, a, r, y, z] * r *
   z^2 * omega[a, a, r, y, z] × omega[a, b, r, y, z];
```

**** Verify Initial case (II.1)

```
Factor[Simplify[(p3II1[a, b, r, y, z] / p3II1[a, b, 1, 1, 1]) -
   (omega[b, b, r, y, z] * r * z^3 * tau[a, b, r, y, z] * (a / (2 - b)) * (1 / (a * b)))]]
```

```
0
```

**** Verify Initial case (II.2)

```
Factor[Simplify[(p3II2[a, b, r, y, z] / p3II2[a, b, 1, 1, 1]) -
   (omega[a, b, r, y, z] * r * z^3 * (a * tau[a, a, r, y, z]) * (a / (a + b - a * b)) * (1 / a^2))]]
```

```
0
```

We now proceed by induction on all cases of the proposition at once, where we assume that all statements hold for $g_{m,n}$ and $g_{n,m}$ with $2 \le n-m \le k$, for some $k \ge 3$. By the above we have established all the initial cases, $k=3$,

for this hypothesis; as noted earlier, the case $n-m=2$ is trivial. We now apply the formulae (1.24) and (1.25) to establish an induction step in each of the upward *I.1–2* and downward *II.1–2* cases respectively. We are allowed to use any of the statements of *III* by Remark 1.3.12. Notice that for the range of indices we must now consider, in all cases $n \ge f$ and $m \le f-1$, so by (1.20), $\gamma_m\gamma_n = (1-a)((1-b)$.

Consider first *I.1*. Let first (i) $n+1 = m+k+1$, for some $m \le f-2$ and $n \ge f+1$; there is another subcase (ii) $n=f$, that we handle as a special case by direct calculation below. We rewrite (1.24) for easy reference: $(*)\ g_{m,n+1} = c_1 g_{m,n} g_{m+1,n+1}(g_{m+1,n})^{-1}\lambda_{m,n}$. In the definition (1.22) we have by (1.18)–(1.20) that $k[a,b]_m^+ = k(a,a) = \frac{a(2-a)}{\omega(a,a)}$, $k[a,b]_n^- = k(b,b) = \frac{b(2-b)}{\omega(b,b)}$. Also, by Definition 1.3.1 and Proposition 1.3.2, $4\rho_{m,n}\rho_{n,m} = \frac{(b/a)}{\Pi_{m,n}\Pi_{n,m}} = \frac{ab}{[a\Pi_{m,n}][a\Pi_{n,m}]}$. Therefore by (1.22) and the induction hypothesis *I.1*, for $g_{m,n}$, and *II.1*, for $g_{n,m}$, the expression (†) $1 - 1/\lambda_{m.n}$, is written:

$$\frac{\gamma_m\gamma_n ab}{[a\Pi_{m,n}][a\Pi_{n,m}]}\,\frac{ab(2-a)(2-b)}{\omega(a,a)\omega(b,b)}\,g_{m,n}g_{n,m} = \frac{\gamma_m\gamma_n a^2 r^2 z^{2j+2\ell} b^2\tau^2(a,b)[a^2\tau_a^2]^{\ell-2}[b^2\tau_b^2]^{j-1}}{\overline{w}_{m,n}\overline{w}_{n,m}}. \tag{1.59}$$

Now apply (1.38) and (1.45) to write x_a, $x(a,b) = b^2z^2\tau^2(a,b)$, and x_b, using all but 4 powers of z. So the numerator of the right member of (1.59) is simply the interlacing bracket $[\overline{w}]_{m,n}$ of Proposition 1.3.6 *1*. Thus, after writing $\overline{w}_{n,m} = \overline{w}_{m+1,n+1}$ by Lemma 1.3.8, and applying the bracket definition (1.46), we have established that (†) is given by $\frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1} - \overline{w}_{m,n+1}\overline{w}_{m+1,n}}{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}$, so (1.58) holds. Finally, apply (∗) and the induction hypothesis *I.1* and (1.58). Since the lower index $m+1$ is the same in both the numerator and denominator of the ratio $g_{m+1,n+1}/g_{m+1,n}$, we obtain, by *I.1* for $m+1 \le f-2$, or by *I.2* for $m+1 = f-1$, that $g_{m+1,n+1}/g_{m+1,n} = cz[b\tau(b,b)]\overline{w}_{m+1,n}/\overline{w}_{m+1,n+1}$. Thus, $g_{m,n+1} = c_1 \frac{g_{m,n}g_{m+1,n+1}}{g_{m+1,n}} \frac{\overline{w}_{m,n}\overline{w}_{m+1,n+1}}{\overline{w}_{m,n+1}\overline{w}_{m+1,n}} = cz[b\tau(b,b)][g_{m,n}\overline{w}_{m,n}]/\overline{w}_{m,n+1}$. Hence by plugging in the numerator $p_{m,n} := g_{m,n}\overline{w}_{m,n}$ (ignoring the constants) from the induction hypothesis for *I.1*, the induction step for *I.1*(i), including (1.58), is complete by Remark 1.3.11.

To recapitulate, in general, there are two steps, where for the upward and downward cases we conform to the recurrences (1.24) and (1.25), respectively.

1. Establish (1.58) by showing that the numerator in the analogue of the right hand member of (1.59) gives a bracket $[\overline{w}_{m,n}]$ ($= [\overline{w}_{n,m}]$) from Proposition 1.3.6, for the parameters m, n of $\lambda_{m,n}$.
2. Establish that when the induction hypothesis is applied, the condition (u) $\frac{p_{m,n}p_{m+1,n+1}}{p_{m,n}[\mathbf{1}]p_{m+1,n+1}[\mathbf{1}]} - \frac{p_{m+1,n}p_{m,n+1}}{p_{m+1,n}[\mathbf{1}]p_{m,n+1}[\mathbf{1}]} = 0$, is verified for *I*, and condition (d) $\frac{p_{n,m}p_{n-1,m-1}}{p_{n,m}[\mathbf{1}]p_{n-1,m-1}[\mathbf{1}]} - \frac{p_{n-1,m}p_{n,m-1}}{p_{n-1,m}[\mathbf{1}]p_{n,m-1}[\mathbf{1}]} = 0$, is verified for *II*.

For all the remaining cases of the induction steps in *I*–*II*, including the subcase *I.1*(ii), we proceed by direct calculation to check the details of the these 2 Steps. In Step 1 it is implicit that the factors of $\omega(a,b)$ that occur variously by substitution from factors $k(a,b)$ in the formula for $\lambda_{m,n}$, and also from

the numerators of $g_{m,n}$ and $g_{n,m}$, cancel one another in every case. This is borne out in the direct calculations, where the pattern of substitutions from the induction hypothesis is shown. By Remark 1.3.11, conditions (u)–(d) are equivalent to showing, for the ratio $\frac{p_{m+1,n+1}}{p_{m+1,n}} = cz\tau$, in the upward case, or $\frac{p_{n-1,m-1}}{p_{n-1,m}} = cz\tau$, in the downward case, that the factor of $z\tau$ completes the form of the numerator $p_{m,n+1}$ [resp. $p_{n,m-1}$] as one extra factor of the numerator form $p_{m,n}$ [resp. $p_{n,m}$]. Here the factor τ depends on subcases; it is $\tau(a,b)$ in subcases *I.1*(ii), and in *II.1*(ii): $n \ge f+2$, $m = f-1$. We show the pattern of substitutions for (u)–(d) in the direct calculations, [15]. □

****************** VERIFICATION OF BOTH STEPS 1 and 2 for the Proof of Proposition 1.3.10.

****** Exact Formulae for Parameters in Definition of $ 1- 1/\lambda_{m,n} $.

Cases:

I.1 (i): m \le f-2, n \ge f+1

I.1 (ii): m \le f-3, n = f (apply induction hypotheses: III.1.1 for g_{m,n}, and II.2 for g_{n,m}).

I.2: m = f-1, n \ge f+2.

II.1 (i): n \ge f+1, m \le f-2.

II.1 (ii): n \ge f+2, m = f-1 (apply induction hyptheses: III.2.1 for g_{n,m} and I.2 for g_{m,n}).

II.2: n = f, m \le f-3.

```
γm := (1 - a); γn := (1 - b);
kI1iup[a_, b_, r_, y_, z_] := k[a, a, r, y, z];
kI1idown[a_, b_, r_, y_, z_] := k[b, b, r, y, z];
kI1iiup[a_, b_, r_, y_, z_] := k[a, a, r, y, z];
kI1iidown[a_, b_, r_, y_, z_] := k[a, b, r, y, z];
kI2up[a_, b_, r_, y_, z_] := k[a, b, r, y, z];
kI2down[a_, b_, r_, y_, z_] := k[b, b, r, y, z];
kII1iup[a_, b_, r_, y_, z_] := k[a, a, r, y, z];
kII1idown[a_, b_, r_, y_, z_] := k[b, b, r, y, z];
kII1iiup[a_, b_, r_, y_, z_] := k[a, b, r, y, z];
kII1iidown[a_, b_, r_, y_, z_] := k[b, b, r, y, z];
kII2up[a_, b_, r_, y_, z_] := k[a, a, r, y, z];
kII2down[a_, b_, r_, y_, z_] := k[a, b, r, y, z];
Πup[a_, b_, ℓ_, j_] := j + ℓ * (b / a) - (ℓ + j - 1) * b
Πdown[a_, b_, ℓ_, j_] := j + 1 + (ℓ - 1) * (b / a) - (ℓ + j - 1) * b;
rhoI1up[a_, b_, ℓ_, j_] := (1 / 2) * (b / a) / Πup[a, b, ℓ, j];
rhoI1down[a_, b_, ℓ_, j_] := (1 / 2) / Πdown[a, b, ℓ, j];
rhoI2up[a_, b_, ℓ_, j_] := (1 / 2) * (b / a) / Πup[a, b, ℓ, j];
rhoI2down[a_, b_, ℓ_, j_] := (1 / 2) / Πdown[a, b, ℓ, j];
rhoII1up[a_, b_, ℓ_, j_] := (1 / 2) * (b / a) / Πup[a, b, ℓ, j];
rhoII1down[a_, b_, ℓ_, j_] := (1 / 2) / Πdown[a, b, ℓ, j];
rhoII2up[a_, b_, ℓ_, j_] := (1 / 2) * (b / a) / Πup[a, b, ℓ, j];
rhoII2down[a_, b_, ℓ_, j_] := (1 / 2) / Πdown[a, b, ℓ, j];
```

**** Statements of Exact Numerators for Proposition 1.3.10.

```
pI1[a_, b_, ℓ_, j_, r_, y_, z_] :=
  (1 / (2 - a)) omega[a, a, r, y, z] r * z^(j + ℓ) tau[a, b, r, y, z]
   (a * tau[a, a, r, y, z])^(ℓ - 2) (b * tau[b, b, r, y, z])^(j - 1) (a * Πup[a, b, ℓ, j]);
pI2[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (a + b - a * b)) omega[a, b, r, y, z]
   r * z^(j + 1)  (b * tau[b, b, r, y, z])^(j - 1) (a * Πup[a, b, ℓ, j]);
pII1[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (2 - b)) omega[b, b, r, y, z]
   r * z^(j + ℓ) tau[a, b, r, y, z] (a * tau[a, a, r, y, z])^(ℓ - 2)
   (b * tau[b, b, r, y, z])^(j - 1) (a * Πdown[a, b, ℓ, j]);
pII2[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (a + b - a * b)) omega[a, b, r, y, z]
   r * z^(ℓ)  (a * tau[a, a, r, y, z])^(ℓ - 2) (a * Πdown[a, b, ℓ, j]);
pIII1[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (2 - a)) omega[a, a, r, y, z]
   r * z^(ℓ)  (a * tau[a, a, r, y, z])^(ℓ - 2) * (ℓ - (ℓ - 1) a);
pIII11[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (2 - a)) omega[a, a, r, y, z]
   r * z^(ℓ) (a * tau[a, a, r, y, z])^(ℓ - 2) * (ℓ - (ℓ - 1) a);
pIII2[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (2 - b)) omega[b, b, r, y, z]
   r * z^(j) (b * tau[b, b, r, y, z])^(j - 2) (j - (j - 1) b);
pIII21[a_, b_, ℓ_, j_, r_, y_, z_] := (1 / (2 - b)) omega[b, b, r, y, z]
   r * z^(j + 1) (b * tau[b, b, r, y, z])^(j - 1) * (j + 1 - j * b);
```

**** Exact Formulae for Numerators of $ 1- 1/\lambda_{m,n} $ in Right Hand Side of (1.59) under various Cases for the Induction Hypothesis.

Cases:

I.1 (i): m \le f-2, n \ge f+1   (apply induction hypotheses: I.1 for g_{m,n}, and II.1 for g_{n,m}).

I.1 (ii): m \le f-3, n = f   (apply induction hypotheses: III.1.1 for g_{m,n}, and II.2 for g_{n,m}).

I.2:   m = f-1, n \ge f+2   (apply induction hypotheses: I.2 for g_{m,n}, and III.2.1 for g_{n,m}).

II.1 (i): n \ge f+1, m \le f-2 (apply induction hyptheses: II.1 for g_{n,m} and I.1 for g_{m,n}).

II.1 (ii): n \ge f+2, m = f-1   (apply induction hyptheses: III.2.1 for g_{n,m} and I.2 for g_{m,n}).

II.2:   n = f, m \le f-3   (apply induction hyptheses: II.2 for g_{n,m} and III.1,1 for g_{m,n}).

```
numI1i[a_, b_, ℓ_, j_, r_, y_, z_] :=
  γm * γn * 4 rhoI1up[a, b, ℓ, j] × rhoI1down[a, b, ℓ, j] × kI1iup[a, b, r, y, z] ×
   kI1idown[a, b, r, y, z] × pI1[a, b, ℓ, j, r, y, z] × pII1[a, b, ℓ, j, r, y, z];
numI1ii[a_, b_, ℓ_, j_, r_, y_, z_] := γm * γn * 4 rhoI1up[a, b, ℓ, j] ×
   rhoI1down[a, b, ℓ, j] × kI1iiup[a, b, r, y, z] × kI1iidown[a, b, r, y, z] ×
   pIII11[a, b, ℓ, 0, r, y, z] × pII2[a, b, ℓ, 0, r, y, z];
numI2[a_, b_, ℓ_, j_, r_, y_, z_] := γm * γn * 4 rhoI2up[a, b, ℓ, j] ×
   rhoI2down[a, b, ℓ, j] × kI2up[a, b, r, y, z] × kI2down[a, b, r, y, z] ×
   pI2[a, b, 1, j, r, y, z] × pIII21[a, b, 1, j, r, y, z];
numII1i[a_, b_, ℓ_, j_, r_, y_, z_] := γm * γn * 4 rhoII1up[a, b, ℓ, j] ×
   rhoII1down[a, b, ℓ, j] × kII1iup[a, b, r, y, z] × kII1idown[a, b, r, y, z] ×
   pI1[a, b, ℓ, j, r, y, z] × pII1[a, b, ℓ, j, r, y, z];
numII1ii[a_, b_, ℓ_, j_, r_, y_, z_] := γm * γn * 4 rhoII1up[a, b, ℓ, j] ×
   rhoII1down[a, b, ℓ, j] × kII1iiup[a, b, r, y, z] × kII1iidown[a, b, r, y, z] ×
   pIII21[a, b, 1, j, r, y, z] × pI2[a, b, 1, j, r, y, z];
numII2[a_, b_, ℓ_, j_, r_, y_, z_] := γm * γn * 4 rhoII2up[a, b, ℓ, j] ×
   rhoII2down[a, b, ℓ, j] × kII2up[a, b, r, y, z] × kII2down[a, b, r, y, z] ×
   pII2[a, b, ℓ, 0, r, y, z] × pIII11[a, b, ℓ, 0, r, y, z];
```

**** Formulae from Proposition 1.3.6.

```
bracketwbar1[a_, b_, ℓ_, j_, r_, y_, z_] := a^2 r^2 z^4 (1 - a)
   (1 - b) xa[a, r, y, z]^(ℓ - 2) × xab[a, b, r, y, z] × xa[b, r, y, z]^(j - 1);
bracketwbar2[a_, b_, ℓ_, j_, r_, y_, z_] :=
  a^2 r^2 z^4 (1 - a) (1 - b) xa[a, r, y, z]^(ℓ - 2) ;
bracketwbar3[a_, b_, ℓ_, j_, r_, y_, z_] :=
  b^2 r^2 z^4 (1 - a) (1 - b) xa[b, r, y, z]^(j - 1);
```

**** VERIFY STEP 1 of Induction Proof for Proposition 1.3.10.

**** Verify Induction Step 1. Case I.1(i).

```
Factor[Simplify[Simplify[numI1i[a, b, ℓ, j, r, y, z] / bracketwbar1[a, b, ℓ, j, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** Verify Induction Step 1. Case I.1(ii).

```
Factor[Simplify[Simplify[numI1ii[a, b, ℓ, 0, r, y, z] / bracketwbar2[a, b, ℓ, 0, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** Verify Induction Step 1. Case I.2.

```
Factor[Simplify[Simplify[numI2[a, b, 1, j, r, y, z] / bracketwbar3[a, b, 1, j, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** Verify Induction Step 1. Case II.1(i).

```
Factor[Simplify[Simplify[numII1i[a, b, ℓ, j, r, y, z] / bracketwbar1[a, b, ℓ, j, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** Verify Induction Step 1. Case II.1(ii).

```
Factor[Simplify[Simplify[numII1ii[a, b, 1, j, r, y, z] / bracketwbar3[a, b, 1, j, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** Verify Induction Step 1. Case II.2.

```
Factor[Simplify[Simplify[numII2[a, b, ℓ, 0, r, y, z] / bracketwbar2[a, b, ℓ, 0, r, y, z],
   Element[ℓ, Integers]] , Element[j, Integers]]]
```

```
1
```

**** VERIFY STEP 2 of Induction Proof for Proposition 1.3.10.

**** Step 2. Case I.1(i)(a) m \le f-3, n \ge f+1. p_{m+1,n+1} / p_{m+1,n} = c z \tau(b,b) from I.1.

```
Factor[Simplify[(pI1[a, b, ℓ, j, r, y, z] / pI1[a, b, ℓ, j, 1, 1, 1])
     (pI1[a, b, ℓ - 1, j + 1, r, y, z] / pI1[a, b, ℓ - 1, j + 1, 1, 1, 1]) -
   (pI1[a, b, ℓ - 1, j, r, y, z] / pI1[a, b, ℓ - 1, j, 1, 1, 1])
     (pI1[a, b, ℓ, j + 1, r, y, z] / pI1[a, b, ℓ, j + 1, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case I.1(i)(b) m = f-2, n \ge f+1. p_{m+1,n+1} / p_{m+1,n} = c z \tau(b,b), from I.2

```
Factor[Simplify[(pI1[a, b, 2, 1, r, y, z] / pI1[a, b, 2, 1, 1, 1, 1])
     (pI2[a, b, 1, 2, r, y, z] / pI2[a, b, 1, 2, 1, 1, 1]) -
   (pI2[a, b, 1, 1, r, y, z] / pI2[a, b, 1, 1, 1, 1, 1])
     (pI1[a, b, 2, 2, r, y, z] / pI1[a, b, 2, 2, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case I.1(ii): m \le f-3, n =f p_{m+1,n+1} / p_{m+1,n} = c z \tau(a,b) from I.1 and III.1.1.

```
Factor[Simplify[(pIII11[a, b, ℓ, 0, r, y, z] / pIII11[a, b, ℓ, 0, 1, 1, 1])
    (pI1[a, b, ℓ - 1, 1, r, y, z] / pI1[a, b, ℓ - 1, 1, 1, 1, 1]) -
   (pIII11[a, b, ℓ - 1, 0, r, y, z] / pIII11[a, b, ℓ - 1, 0, 1, 1, 1])
    (pI1[a, b, ℓ, 1, r, y, z] / pI1[a, b, ℓ, 1, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case I.2: m = f-1, n \ge f+2 p_{m+1,n+1} / p_{m+1,n} = c z \tau(b,b) from III.2.

```
Factor[Simplify[(pI2[a, b, 1, j, r, y, z] / pI2[a, b, 1, j, 1, 1, 1])
    (pIII2[a, b, 0, j + 1, r, y, z] / pIII2[a, b, 0, j + 1, 1, 1, 1]) -
   (pIII2[a, b, 0, j, r, y, z] / pIII2[a, b, 0, j, 1, 1, 1])
    (pI2[a, b, 1, j + 1, r, y, z] / pI2[a, b, 1, j + 1, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case II.1(i)(a): n \ge f+2, m \le f-2. p_{n-1,m-1} / p_{n-1,m} = c z \tau(a,a) from II.1. [\tau(a,b) cancels]

```
Factor[Simplify[(pII1[a, b, ℓ, j, r, y, z] / pII1[a, b, ℓ, j, 1, 1, 1])
    (pII1[a, b, ℓ + 1, j - 1, r, y, z] / pII1[a, b, ℓ + 1, j - 1, 1, 1, 1]) -
   (pII1[a, b, ℓ, j - 1, r, y, z] / pII1[a, b, ℓ, j - 1, 1, 1, 1])
    (pII1[a, b, ℓ + 1, j, r, y, z] / pII1[a, b, ℓ + 1, j, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case II.1(i)(b): n = f+1, m \le f-2. p_{n-1,m-1} / p_{n-1,m} = c z \tau(a,a) from II.2.

```
Factor[Simplify[(pII1[a, b, ℓ, 1, r, y, z] / pII1[a, b, ℓ, 1, 1, 1, 1])
    (pII2[a, b, ℓ + 1, 0, r, y, z] / pII2[a, b, ℓ + 1, 0, 1, 1, 1]) -
   (pII2[a, b, ℓ, 0, r, y, z] / pII2[a, b, ℓ, 0, 1, 1, 1])
    (pII1[a, b, ℓ + 1, 1, r, y, z] / pII1[a, b, ℓ + 1, 1, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case II.1(ii): n \ge f+2, m = f-1. p_{n-1,m-1} / p_{n-1,m} = c z \tau(a,b) from II.1 and III.2.1.

```
Factor[Simplify[(pIII21[a, b, 1, j, r, y, z] / pIII21[a, b, 1, j, 1, 1, 1])
    (pII1[a, b, 2, j - 1, r, y, z] / pII1[a, b, 2, j - 1, 1, 1, 1]) -
   (pIII21[a, b, 1, j - 1, r, y, z] / pIII21[a, b, 1, j - 1, 1, 1, 1])
    (pII1[a, b, 2, j, r, y, z] / pII1[a, b, 2, j, 1, 1, 1])]]
```

```
0
```

**** Step 2. Case II.2: n = f, m \le f-3 . p_{n-1,m-1} / p_{n-1,m} = c z \tau(a,a) from III.1 .