Proof (Corollary 1.1.5). We establish the joint generating function identity in the unweighted case via a direct calculation. Let r, u and z belong to the unit circle. By applying Corollary 1.3.18 with $a = \frac{1}{2}$ we obtain the joint generating function of runs, long runs, and steps by

$$K(\tfrac{1}{2})[ru, 1/u, z] = \tfrac{1}{16}\left(16 - 4z^2 + 4r^2z^2 + r^2z^4 - 2r^2uz^4 + r^2u^2z^4 - S\right),$$

with S given by $S = S_1S_2S_3S_4$, for:

$$S_1 = \sqrt{4 + 2z + 2rz + rz^2 - ruz^2},\ S_2 = \sqrt{4 + 2z - 2rz - rz^2 + ruz^2},$$
$$S_3 = \sqrt{4 - 2z + 2rz - rz^2 + ruz^2},\ S_4 = \sqrt{4 - 2z - 2rz + rz^2 - ruz^2}.$$

On the other hand, with the very same main term S, we have

$$K(\tfrac{1}{2})[u/r, 1/u, rz] = \tfrac{1}{16}\left(16 + 4z^2 - 4r^2z^2 + r^2z^4 - 2r^2uz^4 + r^2u^2z^4 - S\right).$$

The two generating functions differ by $K(\frac{1}{2})[ru, 1/u, z] - K(\frac{1}{2})[u/r, 1/u, rz] = \frac{1}{2}z^2(r^2 - 1)$. The difference is mirrored only in the event that $\mathbf{L} = 2$, when it happens that $\mathbf{R} = 2$ and $\mathbf{U} = 0$. Thus (1.8) holds for $a = \frac{1}{2}$ and $n \geq 2$.

Perhaps the simplest way to obtain (1.8) for $a \neq \frac{1}{2}$ is to apply (1.8) for the case $a = \frac{1}{2}$. Consider an excursion path Γ with $\mathbf{L}(\Gamma) = 2n$ and $\mathbf{L}(\Gamma) - \mathbf{R}(\Gamma) = 2k$. Then $P_a(\Gamma) = \frac{1}{2}a^{2k}(1-a)^{2n-2k-1}$. Here $\mathbf{R}(\Gamma) - 1 = 2n - 2k - 1$ counts the number of turns in the path, so is the exponent of $(1 - a)$ under P_a. Alternatively, $\mathbf{L}(\Gamma) - \mathbf{R}(\Gamma)$ is the total length of long runs minus the number of long runs in Γ, and this gives the exponent of a in $P_a(\Gamma)$. If $2n \geq 4$, then by the first part of the proof there are exactly as many paths Γ with the joint information $\mathbf{L}(\Gamma) = 2n$, $\mathbf{L}(\Gamma) - \mathbf{R}(\Gamma) = 2k$, and $\mathbf{U}(\Gamma) = \ell$, as there are paths Γ' with $\mathbf{L}(\Gamma') = 2n$, $\mathbf{R}(\Gamma') = 2k$, and $\mathbf{U}(\Gamma') = \ell$. Therefore, since for any such path Γ', the probability assigned by the probability measure P_{1-a} yields $P_{1-a}(\Gamma') = \frac{1}{2}a^{2k-1}(1-a)^{2n-2k}$, we have that $aP_{1-a}(\Gamma') = (1-a)P_a(\Gamma)$, $\forall\, \Gamma$ with $\mathbf{L}(\Gamma) \geq 4$. Hence (1.8) holds. □

*****VERIFY $ K(\frac{1}{2}) [ru, 1/u, z] = \frac{1}{16} (16 - 4 z^2 + 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4 - S) $, by using Corollary 1.3.18. We do this in two steps, first showing the square root; and second, squaring to remove the radical sign and obtain a nice factoization.

```
Factor[Expand[1/16 (16 - 4 z^2 + 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) -
    2 (1 - (1 / 2) betaa[1 / 2, r * u, 1 / u, z] - (1 / 2) alphaa[1 / 2, r * u, 1 / u, z])]]
```

$$\sqrt{\left(-z^2\left(1+\frac{1}{4}r^2\left(1-\frac{1}{u}\right)u\,z^2\right)^2+\left(1+z^2\left(\frac{1}{4}-\frac{1}{4}r^2u^2\left(\frac{1}{u^2}+\frac{1}{4}\left(1-\frac{1}{u}\right)^2z^2\right)\right)\right)^2\right)}$$

```
Factor[Expand[(1/16 (16 - 4 z^2 + 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) -
    2 (1 - (1/2) betaa[1/2, r*u, 1/u, z] - (1/2) alphaa[1/2, r*u, 1/u, z]))^2]]
```

$$\frac{1}{256}\left(-4-2z-2rz-rz^2+ruz^2\right)\left(4+2z-2rz-rz^2+ruz^2\right)\left(4-2z+2rz-rz^2+ruz^2\right)\left(-4+2z+2rz-rz^2+ruz^2\right)$$

*****VERIFY $ K(\frac{1}{2})[u/r ,1/u, rz]= \frac{1}{16} (16 + 4 z^2 - 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4 - S) $,
again in two steps, as in the previous computation.

```
Factor[Expand[1/16 (16 + 4 z^2 - 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) -
   2 (1 - (1/2) betaa[1/2, u/r, 1/u, r*z] - (1/2) alphaa[1/2, u/r, 1/u, r*z])]]
```

$$\sqrt{\left(-r^2 z^2\left(1+\frac{1}{4}\left(1-\frac{1}{u}\right)u z^2\right)^2+\left(1+r^2 z^2\left(\frac{1}{4}-\frac{1}{4 r^2}u^2\left(\frac{1}{u^2}+\frac{1}{4}r^2\left(1-\frac{1}{u}\right)^2 z^2\right)\right)\right)^2\right)}$$

```
Factor[
 Expand[(1/16 (16 + 4 z^2 - 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) - 2 (1 - (1/2) betaa[1/2, u/r,
       1/u, r*z] - (1/2) alphaa[1/2, u/r, 1/u, r*z]))^2]]
```

$$\frac{1}{256}\left(-4-2z-2rz-rz^2+ruz^2\right)\left(4+2z-2rz-rz^2+ruz^2\right)\left(4-2z+2rz-rz^2+ruz^2\right)\left(-4+2z+2rz-rz^2+ruz^2\right)$$

*****VERIFY that the squares, $ S^2 $, of the two main terms of $ K(\frac{1}{2})[ru, 1/u ,z] $ and $ K(\frac{1}{2})[u/r , 1/u, rz] $ are equal.

```
Factor[Expand[(1/16 (16 - 4 z^2 + 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) -
     2 (1 - (1/2) betaa[1/2, r*u, 1/u, z] - (1/2) alphaa[1/2, r*u, 1/u, z]))^2
   - (1/16 (16 + 4 z^2 - 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) - 2 (1 - (1/2)
        betaa[1/2, u/r, 1/u, r*z] - (1/2) alphaa[1/2, u/r, 1/u, r*z]))^2]]
```

```
0
```

*****VERIFY that $ K(\frac{1}{2})[ru, 1/u ,z] - K(\frac{1}{2})[u/r , 1/u, rz] = \frac{1}{2} z^2 (r^2-1) $.

```
Factor[Expand[1/16 (16 - 4 z^2 + 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) -
    1/16 (16 + 4 z^2 - 4 r^2 z^2 + r^2 z^4 - 2 r^2 u z^4 + r^2 u^2 z^4) - (1 / 2) z^2 (r^2 - 1)]]
```

```
0
```

1.4 Proofs of Theorems 1.1.1 and 1.1.6.

Proof (Theorem 1.1.1.). We fix $t \in \mathbb{R}$. All big oh terms in the proof will refer to the parameter $N \to \infty$ with implied constants depending only on a, b, and t. Since, by [13], for fixed $m > 0$ and $f \sim \eta N \to \infty$, $P(\mathbf{X}_j = 0 \text{ before } \mathbf{X}_j = f | \mathbf{X}_0 = m) \to 1$, we may assume that $\mathbf{X}_0 = 0$. Let

$$r_N := e^{-it(2-a-b)/((1-a)(1-b)N)}, \; y_N := e^{it/((1-a)(1-b)N)}, \; z_N := e^{it/N}. \tag{1.73}$$

Since $\{(1 + \frac{1}{N})X_{N+1}\}$ converges in distribution if and only if $\{X_N\}$ does, by (1.2) it suffices to establish that $E\{e^{it(1+\frac{1}{N})X_{N+1}}\} = \hat{\varphi}(t)$, as $N \to \infty$, It is clear that $a(2-a)z_N h_a[r_N, y_N, z_N] \to 1$ as $N \to \infty$. Therefore , by (1.28), we must show that $\lim_{N\to\infty} g_{0,N}[r_N, y_N, z_N]$ equals the limit in (1.3). By Proposition 1.3.10 *I.1*, we have a formula for $g_{0,N}$, and by Proposition 1.3.7 we have a formula for its denominator $\overline{w}_{0,N}$. The main work is in calculating an asymptotic expression for $\overline{w}_{0,N}[r_N, y_N, z_N]$.

We now make substitutions analogous to (1.72), one for each stratum:

$$\begin{array}{cc} \cos(\theta_1) := \beta_a/\sqrt{4x_a} \ ; & \cos(\theta_2) := \beta_b/\sqrt{4x_b}; \\ \beta_a \pm \alpha_a = \sqrt{4x_a}e^{\pm i\theta_1} \ ; & \beta_b \pm \alpha_b = \sqrt{4x_b}e^{\pm i\theta_2}, \end{array} \quad (1.74)$$

where all functions on the right sides of these expressions are composed with $[r_N, y_N, z_N]$ of (1.73). Here we write $\sqrt{4x_a}$ as a shorthand for the expression $2az\tau_a$; see (1.38). Note that the coefficients in (1.2) have been chosen such that the first order term of the Taylor expansions about $t = 0$ of the substitutions $\cos\theta_j[r_N, y_N, z_N]$, $j = 1, 2$, do in fact vanish in the following:

$$\cos\theta_1 = 1 + \frac{1}{2}\frac{\sigma_1^2 t^2}{(1-b)^2 N^2} + \frac{O(1)}{N^3}; \quad \cos\theta_2 = 1 + \frac{1}{2}\frac{\sigma_2^2 t^2}{(1-a)^2 N^2} + \frac{O(1)}{N^3}, \quad (1.75)$$

where σ_1^2 and σ_2^2 are as defined in the statement of the theorem, and we obtain (1.75) by direct computation. Therefore by (1.75), and by applying the Taylor expansion of $\arccos(u)$ about $u = 1$, we find that θ_1 and θ_2 are both of order $1/N$ as follows:

$$\theta_1 = i\frac{\sigma_1 t}{(1-b)N} + \frac{O(1)}{N^3}; \ \theta_2 = i\frac{\sigma_2 t}{(1-a)N} + \frac{O(1)}{N^3}. \quad (1.76)$$

*****DEFINE Parameters for Proof of Theorem 1.1.1, including Substitions.

```
rN[a_, b_, t_, N_] := E^(-I*t*(2 - a - b) / ((1 - a) (1 - b) N));
yN[a_, b_, t_, N_] := E^(I*t / ((1 - a) (1 - b) N));
zN[a_, b_, t_, N_] := E^(I*t / N);
costheta1[a_, r_, y_, z_] := betaa[a, r, y, z] / (2*a*z*tau[a, a, r, y, z]);
costheta2[b_, r_, y_, z_] := betaa[b, r, y, z] / (2*b*z*tau[b, b, r, y, z]);
sigma1[a_, b_] := Sqrt[a - 2 a*b + b^2];
sigma2[a_, b_] := Sqrt[b - 2 a*b + a^2];
```

*****VERIFY \cos\theta_1= 1+\frac{1}{2}\frac{\sigma_1^{2} t^2}{(1-b)^2N^2}+\frac{O(1)}{N^3}; first formula of (1.75).

```
Collect[Normal[Series[costheta1[a, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]] -
    (1 + sigma1[a, b]^2 t^2 / (2 (1 - b)^2 N^2)), {t, 0, 2}]], N]
```

```
0
```

*****VERIFY \cos\theta_2 =1+ \frac{1}{2}\frac{\sigma_2^{2} t^2}{(1-a)^2N^2} + \frac{O(1)}{N^3}; Second formula of (1.75).

```
Collect[Normal[Series[costheta2[b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]] -
    (1 + sigma2[a, b]^2 t^2 / (2 (1 - a)^2 N^2)), {t, 0, 2}]], N]
```

```
0
```

By Proposition 1.3.7,

$$\overline{w}_{0.N} = d_1(f)q^*_{N-f}(b) + d_2(f)w^*_{N-f}(b). \tag{1.77}$$

We focus first on the coefficients $d_j(f)$, which are written in terms of $w^*_f(a)$ and $w^*_{f+1}(a)$ by (1.53). By (1.13) and (1.41), suppressing dependence on a, $w^*_f = (1-w^*_0)q_f + w^*_0(q_f - xq_{f-1}) = q_f(x,\beta) - w^*_0 xq_{f-1}(x,\beta)$. Thus by (1.13), and (1.74),

$$\begin{aligned} w^*_f(a) &= 2i\alpha_a^{-1}[az\tau_a]^f \left\{\sin f\theta_1 - \sqrt{x_a}w^*_0(a)\sin(f-1)\theta_1\right\} \\ w^*_{f+1}(a) &= 2i\alpha_a^{-1}[az\tau_a]^f \sqrt{x_a}\left\{\sin(f+1)\theta_1 - \sqrt{x_a}w^*_0(a)\sin f\theta_1\right\}; \end{aligned} \tag{1.78}$$

with verification by direct algebra for $q_f(x_a,\beta_a) = 2i\alpha_a^{-1}[az\tau_a]^f\sin(f\theta_1)$, and with $\sqrt{x_a}$ to stand for a factor of $[az\tau_a]$. Next write $f_+ := f+1$, and $f_- := f-1$. Also denote

$$e_j = e_j(f) := \frac{d_j(f)}{\Lambda_1},\ j=1,2,\ \text{for } \Lambda_1 := 2i\alpha_a^{-1}[az\tau_a]^f = (\sin\theta_1)^{-1}[az\tau_a]^{f-1}, \tag{1.79}$$

since $\alpha_a = i\sqrt{4x_a}\sin\theta_1 = 2iaz\tau_a\sin\theta_1$. By (1.78)–(1.79) and direct algebra, we can write expressions for the e_j through (1.53) as follows.

*********** ALGEBRA for (1.78) ***********

LEGEND: s0 = \sin((f-1) \theta_1) , s1 =\sin(f \theta_1) , \sqrt{x_a} = a z \tau , w = w_0^{*}(a) .

******Algebra for First Line of (1.78).

```
Factor[Simplify[Simplify[(a * z * τ)^(f) (s1) -
    w * (a * z * τ)^2 (a * z * τ)^(f - 1) * (s0) - (a * z * τ)^(f) * (s1 - w * (a * z * τ) * s0),
   Element[f, Integers]], Element[x, Positive]]]
```

```
0
```

******** Algebra for Second Line of (1.78).

```
Factor[Simplify[Simplify[(a * z * τ)^(f + 1) (s1) - w * (a * z * τ)^2 (a * z * τ)^(f) * (s0) -
    (a * z * τ)^(f) (a * z * τ) * (s1 - w * (a * z * τ) * s0),
   Element[f, Integers]], Element[x, Positive]]]
```

```
0
```

$$e_j = (\mu_{j,1} - \mu_{j,2} x_a w_0^*(a)) \sin f\theta_1 + \sqrt{x_a}\,[\mu_{j,2} \sin f_+\theta_1 - \mu_{j,1} w_0^*(a) \sin f_-\theta_1]\,. \tag{1.80}$$

************** Algebra for (1.80)

LEGEND: s0=\sin((f-1) \theta_1) ; s1=\sin(f \theta_1); s2=\sin((f+1) \theta_1); A = \mu_{j,1}; B = \mu_{j,2}.

```
Factor[Simplify[A (s1 - Sqrt[x] w * s0) + B * Sqrt[x] (s2 - Sqrt[x] * w * s1) -
   ((A - B * x * w) s1 + Sqrt[x] (B * s2 - A * w * s0)), Element[x, Positive]]]
```

```
0
```

Next we apply the trigonometric identity for the sine of a sum or difference to $\sin f_+\theta_1 = \sin(f+1)\theta_1$ and $\sin f_-\theta_1 = \sin(f-1)\theta_1$ in (1.80). At this point we also introduce some abbreviations to keep the notation a bit compact. Thus write

$$\mathbf{s}_1 := \sin f\theta_1; \quad \mathbf{c}_1 := \cos f\theta_1. \tag{1.81}$$

We rewrite (1.80), with abbreviation $w_0^* = w_0^*(a)$, by collecting terms with a factor $\sqrt{x_a}$. Thus for each $j = 1, 2$,

$$\begin{aligned} e_j = {} & (\mu_{j,1} - \mu_{j,2} x_a w_0^*)\mathbf{s}_1 \\ & + \sqrt{x_a}\left\{\mu_{j,2}(\mathbf{s}_1 \cos\theta_1 + \mathbf{c}_1 \sin\theta_1) - \mu_{j,1} w_0^*(\mathbf{s}_1\cos\theta_1 - \mathbf{c}_1 \sin\theta_1)\right\}. \end{aligned} \tag{1.82}$$

We introduce a book–keeping notation for the coefficient t_j of the variable $\mathbf{x}_j$ in square brackets, within a linear expression $\sum_i t_i \mathbf{x}_i$ in parentheses: $[\mathbf{x}_j]\left(\sum_i t_i \mathbf{x}_i\right) = t_j$. Our method for e_j is to asymptotically expand $[\mathbf{s}_1](e_j)$ and $[\mathbf{c}_1 \sin\theta_1](e_j)$ by (1.82). We will treat $\sin\theta_1$ separately from the asymptotic expansions of the other terms due to the convenient fact that, by (1.76), we have $\sin\theta_1 = \theta_1 + O(N^{-3})$, and this will suffice for our purposes. Note that by (1.76) and (1.81), and $f \sim \eta N$, $\mathbf{s}_1$ and $\mathbf{c}_1$ are both $O(1)$. Further, by direct calculation, $\mu_{i,j}$ are polynomial, and $q_0^*(a)$ and $w_0^*(a)$ only involve negative powers of τ_a, where $\tau_a[\mathbf{1}] = 1$. Thus the Taylor expansions of $[\mathbf{s}_1](e_j)$ and $[\mathbf{c}_1 \sin\theta_1](e_j)$ about $t = 0$ are well behaved.

We next find reduced expressions for the terms $q_{N-f}^*(b)$ and $w_{N-f}^*(b)$ of (1.77). The approach is as above, but now with b in place of a, $N - f$ in place of f, and using the second substitution θ_2 in (1.74). Similar to (1.79) we introduce

$$q^* := \frac{q_{N-f}^*(b)}{\Lambda_2}, \; w^* := \frac{w_{N-f}^*(b)}{\Lambda_2}; \; \Lambda_2 := 2i\alpha_b^{-1}[bz\tau_b]^{N-f} = (\sin\theta_2)^{-1}[bz\tau_b]^{N-f-1} \tag{1.83}$$

Similar as for (1.78), by (1.13) and both lines of (1.41) applied in turn, and (1.83),

$$\begin{aligned} q^* &:= y^2 \sin(N-f)\theta_2 - \sqrt{x_b}\, q_0^*(b) \sin(N-f-1)\theta_2, \\ w^* &:= \sin(N-f)\theta_2 - \sqrt{x_b}\, w_0^*(b) \sin(N-f-1)\theta_2. \end{aligned} \tag{1.84}$$

Introduce abbreviations also for the second stratum sines and cosines:

$$\mathbf{s}_2 := \sin(N-f)\theta_2; \quad \mathbf{c}_2 := \cos(N-f)\theta_2. \tag{1.85}$$

We illustrate the book–keeping method by expanding $\sin(N-f-1)\theta_2 = \mathbf{s}_2\cos\theta_2 - \mathbf{c}_2 \sin\theta_2$ to obtain by (1.84),

$$[\mathbf{s}_2]\,(q^*) = y^2 - \sqrt{x_b}\, q_0^*(b)\cos\theta_2; \quad [\mathbf{c}_2 \sin\theta_2]\,(q^*) = \sqrt{x_b}\, q_0^*(b); \tag{1.86}$$

$$[\mathbf{s}_2]\,(w^*) = 1 - \sqrt{x_b}\, w_0^*(b)\cos\theta_2; \quad [\mathbf{c}_2 \sin\theta_2]\,(w^*) = \sqrt{x_b}\, w_0^*(b). \tag{1.87}$$

To handle the asymptotic expansions for the four terms on the right side of (1.77), we expand the coefficients of $\mathbf{s}_1$, $\mathbf{c}_1 \sin\theta_1$, $\mathbf{s}_2$, and $\mathbf{c}_2 \sin\theta_2$ by direct computation and thereby find

$$\frac{\overline{w}_{0,N}}{\Lambda_1 \Lambda_2} = O(N^{-2}) + \left[\left(-(1-a)(1-b) + 2(1-ab)\,\frac{it}{N}\right)\mathbf{s}_1\right]\left[\left(1 + \frac{2}{1-a}\,\frac{it}{N}\right)\mathbf{s}_2\right] + \left[\left(1 - a - \frac{a(b-a)}{1-b}\,\frac{it}{N}\right)\mathbf{s}_1 + a\mathbf{c}_1 \sin\theta_1\right]\left[\left(1 - b - \frac{b(2-a-b)}{1-a}\,\frac{it}{N}\right)\mathbf{s}_2 + b\mathbf{c}_2 \sin\theta_2\right] \quad (1.88)$$

****** Define e_1, with s1= [\mathbf{s}_1] and c1sin := [\mathbf{c}_1 \sin \theta_1] as auxiliary variables.

```
e1[a_, b_, r_, y_, z_, s1_, c1sin_] :=
  (m11[a, b, r, y, z] - m12[a, b, r, y, z] × xa[a, r, y, z] × w0stara[a, r, y, z]) s1 +
   a * z * tau[a, a, r, y, z] (m12[a, b, r, y, z] (s1 * costheta1[a, r, y, z] + c1sin) -
      m11[a, b, r, y, z] × w0stara[a, r, y, z] (s1 * costheta1[a, r, y, z] - c1sin));
```

************ Expand e1[rN, yN, zN]

```
Collect[
 Factor[Collect[Normal[Series[e1[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N],
      s1, c1sin], {t, 0, 1}]], s1]], N]
```

$$-(1 - a - b + a\,b)\;s1 - \frac{s1\,(-2\,\mathrm{i}\,t + 2\,\mathrm{i}\,a\,b\,t)}{N}$$

****** Order 1 term of [s1](e1) :

```
Factor[- (1 - a - b + a b) s1]
```

$$-(-1 + a)\,(-1 + b)\;s1$$

****** Order 1/N term of [s1](e1) :

```
Factor[- s1 (-2 ⅈ t + 2 ⅈ a b t) / N]
```

$$-\frac{2\,\mathrm{i}\,(-1 + a\,b)\;s1\;t}{N}$$

****** Order 1 term of [c1\sin \theta_1](e1) is zero .

```
Factor[0]
```

```
0
```

***CONCLUSION: e_1 = [- (-1+a) (-1+b) $- \frac{2 i (-1+a b) t}{N}$] s1 + O(N^{-2})

************ DEFINE s2qstar := [s2](q^{*}), c2sinqstar :=[\mathbf{c}_2 \sin \theta_2](q^{*}),
s2wstar := [s2](w^{*}) , c2sinwstar :=[\mathbf{c}_2 \sin \theta_2](w^{*}).

```
s2qstar[a_, b_, r_, y_, z_] :=
  y^2 - b * z * tau[b, b, r, y, z] × q0stara[b, r, y, z] × costheta2[b, r, y, z];
c2sinqstar[a_, b_, r_, y_, z_] := b * z * tau[b, b, r, y, z] × q0stara[b, r, y, z]
s2wstar[a_, b_, r_, y_, z_] :=
  1 - b * z * tau[b, b, r, y, z] × w0stara[b, r, y, z] × costheta2[b, r, y, z];
c2sinwstar[a_, b_, r_, y_, z_] := b * z * tau[b, b, r, y, z] × w0stara[b, r, y, z];
```

*********** Expand s2qstar[rN, yN, zN]

```
Collect[Factor[Collect[Normal[Series[
     s2qstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]], {t, 0, 1}]], t]], N]
```

$$1 - \frac{2 i t}{(-1 + a) N}$$

****** Order 1 term of [s2](q^{*}) :

```
Factor[  1  ]
```

```
1
```

****** Order 1/N term of [s2](q^{*}) :

$$\text{Factor}\left[-\frac{2 i t}{(-1 + a) N} \right]$$

$$-\frac{2 i t}{(-1 + a) N}$$

*********** Expand [c2 \sin theta_2](q^{*})[rN, yN, zN]

```
Collect[Factor[Collect[Normal[Series[
     c2sinqstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]], {t, 0, 1}]], t]], N]
```

$$\frac{2 i b t}{(-1 + a) (-1 + b) N}$$

****** By inspection of the above, the order 1 term of [c2 \sin theta_2](q^{*}) is zero.

*** CONCLUSION q^{*} = [1 + $-\frac{2 \, i \, t}{(-1+a) \, N}$] s2 + O(N^{-2})

****** DEFINE e_2, with s1= [\mathbf{s}_1] and c1sin := [\mathbf{c}_1 \sin \theta_1] as auxiliary variables.

```
e2[a_, b_, r_, y_, z_, s1_, c1sin_] :=
  (m21[a, b, r, y, z] - m22[a, b, r, y, z] × xa[a, r, y, z] × w0stara[a, r, y, z]) s1 +
   a * z * tau[a, a, r, y, z] (m22[a, b, r, y, z] (s1 * costheta1[a, r, y, z] + c1sin) -
      m21[a, b, r, y, z] × w0stara[a, r, y, z] (s1 * costheta1[a, r, y, z] - c1sin));
```

*********** Expand e2[rN, yN, zN]

```
Collect[
 Factor[Collect[Normal[Series[e2[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N],
      s1, c1sin], {t, 0, 1}]], s1]], N]
```

$$\frac{1}{-1+b}(-a\,c1sin + a\,b\,c1sin - s1 + a\,s1 + b\,s1 - a\,b\,s1) + \left(i\,a^2\,c1sin\,t - i\,a\,b\,c1sin\,t - i\,a^2\,s1\,t + i\,a\,b\,s1\,t\right) \Big/ ((-1+b)\,N)$$

****** Order 1 term of [s1](e2) :

$$\text{Factor}\left[\frac{-s1 + a\,s1 + b\,s1 - a\,b\,s1}{-1+b}\right]$$

$$-(-1+a)\,s1$$

****** Order 1/N term of [s1](e2) :

$$\text{Factor}\left[\frac{-i\,a^2\,s1\,t + i\,a\,b\,s1\,t}{(-1+b)\,N}\right]$$

$$\frac{i\,a\,(-a+b)\,s1\,t}{(-1+b)\,N}$$

****** Order 1 term of [c1\sin \theta_1](e2) :

$$\text{Factor}\left[\frac{-a\,c1sin + a\,b\,c1sin}{-1+b}\right]$$

$$a\,c1sin$$

*** CONCLUSION: e_2 =[(-(-1+a)+ $\frac{i\,a\,(-a+b)\,t}{(-1+b)\,N}$) s1 + a c1\sin] + O(N^{-2}).

***************Expand [s2](w^{*})[rN, yN, zN]

```
Collect[Factor[Collect[Normal[Series[
    s2wstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]], {t, 0, 2}]], t]], N]
```

$$\left(2-4a+2a^2-2b+4ab-2a^2b\right)/\left(2(-1+a)^2\right)+\left(-4\,i\,b\,t+6\,i\,a\,b\,t-2\,i\,a^2 b\,t+2\,i\,b^2 t-2\,i\,a\,b^2 t\right)/\left(2(-1+a)^2 N\right)+\left(2\,b\,t^2-4\,a\,b\,t^2-7\,b^2 t^2+8\,a\,b^2 t^2+2\,b^3 t^2\right)/\left(2(-1+a)^2 N^2\right)$$

****** Order 1 term of [s2](w^{*}) :

```
Factor[  (2 - 4 a + 2 a^2 - 2 b + 4 a b - 2 a^2 b) / (2 (-1 + a)^2) ]
```

$$1-b$$

****** Order 1/N term of [s2](w^{*}) :

```
Factor[  (-4 i b t + 6 i a b t - 2 i a^2 b t + 2 i b^2 t - 2 i a b^2 t) / (2 (-1 + a)^2 N) ]
```

$$-\frac{i\,b\,(-2+a+b)\,t}{(-1+a)\,N}$$

************** Expand [c2 \sin theta_2](w^{*})[rN, yN, zN]

```
Collect[Factor[Collect[Normal[Series[
    c2sinwstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]], {t, 0, 1}]], t]], N]
```

$$b+\frac{b\,(-2\,i\,t+i\,a\,t+i\,b\,t)}{(-1+a)\,N}$$

******* Order 1 term of [c2 \sin theta_2](w^{*})

```
Factor[ b ]
```

$$b$$

*** CONCLUSION w^{*} = [(1- b – $\frac{i\,b\,(-2+a+b)\,t}{(-1+a)\,N}$) s2 + b c2sin] + O(N^{-2}).

Since, by (1.76), $\sin\theta_1$ and $\sin\theta_2$ are of order $1/N$, observe that the two terms of order 1 on the right hand side of (1.88) are of form $\pm(1-a)(1-b)$ and therefore cancel. Also, since $\sin\theta_j = \theta_j + O(N^{-3})$ for θ_j are given by (1.76), we substitute these relations into (1.88) and collect the order $1/N$ terms to find by direct asymptotics that:

$$\frac{\overline{w}_{0,N}}{\Lambda_1\Lambda_2} = \left\{a\sigma_1\mathbf{c}_1\mathbf{s}_2 + b\sigma_2\mathbf{s}_1\mathbf{c}_2 + (b-a)^2\mathbf{s}_1\mathbf{s}_2\right\}\frac{it}{N} + O(N^{-2}). \tag{1.89}$$

We check the whole procedure for (1.89) by direct asymptotical analysis of a formula for $e_1q^* + e_2w^*$, with $\sin\theta_j$ replaced by the order $1/N$ term of (1.76).

************** DEFINE AUXILIARY Variables to handle the \sin \theta_{ j }, by substituting main terms of (1.76)

```
sintheta1asymp[a_, b_, t_, sig1_, N_] := I * sig1 * t / ((1 - b) N);
sintheta2asymp[a_, b_, t_, sig2_, N_] := I * sig2 * t / ((1 - a) N);
```

************** CALCULATION of \overline{w}_{0,N}/(\Lambda_1\Lambda_2) from (1.88)

```
Collect[
 Normal[Series[(- (1 - a) (1 - b) + 2 (1 - a * b) I * t / N) (1 + (2 / (1 - a)) I * t / N) s1 * s2 +
    ((1 - a - (a (b - a) / (1 - b)) I * t / N) s1 + a * c1 * sintheta1asymp[a, b, t, sig1, N])
     ((1 - b - (b (2 - a - b) / (1 - a)) I * t / N) s2 +
       b * c2 * sintheta2asymp[a, b, t, sig2, N]), {t, 0, 1}]], N]
```

```
1
— ⅈ (a² s1 s2 - 2 a b s1 s2 + b² s1 s2 + a c1 s2 sig1 + b c2 s1 sig2) t
N
```

************** AUTOMATIC CALCULATION of \overline{w}_{0,N}/(\Lambda_1\Lambda_2)

```
Collect[
 Factor[Normal[Series[e1[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1, c1 *
        sintheta1asymp[a, b, t, sig1, N]] * (s2 * s2qstar[a, b, rN[a, b, t, N],
          yN[a, b, t, N], zN[a, b, t, N]] + c2 * sintheta2asymp[a, b, t, sig2, N] *
         c2sinqstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]]) +
     e2[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1,
       c1 * sintheta1asymp[a, b, t, sig1, N]] *
      (s2 * s2wstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]] +
        c2 * sintheta2asymp[a, b, t, sig2, N] * c2sinwstar[a, b,
          rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]]), {t, 0, 1}]]], N]
```

```
1
— ⅈ (a² s1 s2 - 2 a b s1 s2 + b² s1 s2 + a c1 s2 sig1 + b c2 s1 sig2) t
N
```

Now plug (1.89) into the formula for $g_{0,N}$ in Proposition 1.3.10, *I.1*, apply Proposition 1.3.2, *1* to rewrite $\Pi_{0,N}$, and recall Λ_j in (1.79) and (1.83) to write

$$g_{0,N} = \frac{\omega_a \tau(a,b) r z^2}{a(2-a)\tau_a} \left(\frac{\sin\theta_1 \sin\theta_2 [(N-f)a + fb - (N-1)ab]}{[a\sigma_1 \mathbf{c}_1 \mathbf{s}_2 + b\sigma_2 \mathbf{s}_1 \mathbf{c}_2 + (b-a)^2 \mathbf{s}_1 \mathbf{s}_2] \frac{it}{N} + O(N^{-2})} \right). \tag{1.90}$$

Finally, to find the limit as $N \to \infty$ of the expression (1.90), we substitute (1.76) into the definitions (1.81) and (1.85), and again employ $\sin\theta_j \sim \theta_j$. We note: $\lim_{N\to\infty} \omega_a [a(2-a)]^{-1} r_N z_N^2 \tau(a,b) \tau_a^{-1} = 1$, since $\omega_a[\mathbf{1}] = a(2-a)$ and $\tau(a,b)[\mathbf{1}] = 1$. Since by assumption $f \sim \eta N$, we have $[(N-f)a + fb - (N-1)ab] \sim N[(1-\eta)a + \eta b - ab]$, and since by (1.76), $\theta_1\theta_2 \sim i^2 \frac{\sigma_1\sigma_2}{(1-a)(1-b)} t^2 N^{-2}$, by (1.90) we obtain, as $N \to \infty$,

$$g_{0,N} \sim \frac{i^2 t^2}{N} \frac{\sigma_1 \sigma_2}{(1-a)(1-b)} \frac{(1-\eta)a + \eta b - ab}{[a\sigma_1 \mathbf{c}_1 \mathbf{s}_2 + b\sigma_2 \mathbf{s}_1 \mathbf{c}_2 + (b-a)^2 \mathbf{s}_1 \mathbf{s}_2] \frac{it}{N}}.$$

Here we use implicitly that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$, so that by (1.76), (1.81) and (1.85), and by definition of κ_1 and κ_2, $\mathbf{s}_j \sim i\sinh(\kappa_j t)$, $j = 1,2$, and $\mathbf{c}_j \sim \cosh(\kappa_j t)$, $j = 1,2$. Thus we obtain, $\lim_{N\to\infty} g_{0,N}[r, s_N, t_N] = \hat{\varphi}(t)$, for $\hat{\varphi}(t)$ given by (1.3). □

Proof (Corollary 1.1.2). We now assume that $a = b$ and consider the random variable $sY_{1,N} + tY_{2,N}$ defined by (1.4) in place of tX_N in the proof of Theorem 1.1.1. By the definition (1.4) we write $sY_{1,N} + tY_{2,N} = \frac{1}{N}\left(t\mathcal{L}'_N + \frac{(1-a)s-(2-a)t}{(1-a)} \mathcal{R}'_N + \frac{t-s}{(1-a)} \mathcal{V}'_N \right)$. Accordingly, define

32 Gregory J. Morrow

$$r_{s,t,N} := e^{i((1-a)s-(2-a)t)/((1-a)N)},\ y_{s,t,N} := e^{i(t-s)/((1-a)N)},\ z_{s,t,N} := e^{it/N}. \tag{1.91}$$

It suffices to prove that, for each fixed pair of real numbers $s, t \in \mathbb{R}$, $\lim_{N\to\infty} g_{0,N}(a,a)[r_{s,t,N}, y_{s,t,N}, z_{s,t,N}]$ exists and is given by the right side of (1.5). Define $\theta = \theta_{s,t,N}$ via $\cos\theta = \beta_a/\sqrt{4x_a}$, where the functions β_a and x_a are composed with the complex exponential terms in (1.91). It follows by making a direct calculation that

$$\cos\theta = 1 + \frac{1}{2}\frac{(1-a)s^2+at^2}{N^2} + \frac{O(1)}{N^3};\ \theta = i\frac{\sqrt{(1-a)s^2+at^2}}{N} + \frac{O(1)}{N^3}. \tag{1.92}$$

****** DEFINE r_{s,t,N}, y_{s,t,N}, z_{s,t,N}, and costhetaa := \cos \theta, and also proposed thetaa := \theta[r_{s,t,N}, y_{s,t,N}, z_{s,t,N}].