```
rstN[a_, s_, t_, N_] := E^(I ((1 - a) s - (2 - a) t) / ((1 - a) N));
ystN[a_, s_, t_, N_] := E^(I (t - s) / ((1 - a) N));
zstN[a_, s_, t_, N_] := E^(I * t / N);
costhetaa[a_, r_, y_, z_] := betaa[a, r, y, z] / (2 * a * z * tau[a, a, r, y, z]);
Clear[thetaa]
thetaa[a_, s_, t_, N_] := I * Sqrt[(1 - a) s^2 + a * t^2] / N;
```

**** VERIFY first expression in (1.92) for \cos \theta [r_{s,t,N}, y_{s,t,N}, z_{s,t,N}]

```
Collect[Normal[
  Series[Normal[Series[costhetaa[a, rstN[a, s, t, N], ystN[a, s, t, N], zN[a, s, t, N]] -
      (1 + (1 / 2) ((1 - a) s^2 + a * t^2) / N^2), {t, 0, 2}]], {s, 0, 2}]], N]
```

$$\frac{\frac{3\,i\,a\,s^2\,t}{2\,(-1+a)} - \frac{2\,i\,a^2\,s^2\,t}{-1+a} + \frac{i\,a^3\,s^2\,t}{2\,(-1+a)} + \frac{1}{2}\,i\,a^2\,s\,t^2}{N^3} + \frac{-\frac{a\,s^2\,t^2}{-1+a} + \frac{17\,a^2\,s^2\,t^2}{4\,(-1+a)} - \frac{5\,a^3\,s^2\,t^2}{2\,(-1+a)}}{N^4}$$

**** VERIFY second expression in (1.92) for \theta [r_{s,t,N}, y_{s,t,N}, z_{s,t,N}]

```
Collect[Normal[
  Series[Normal[Series[Cos[thetaa[a, s, t, N]] - (1 + (1 / 2) ((1 - a) s^2 + a * t^2) / N^2),
     {t, 0, 3}]], {s, 0, 3}]], N]
```

$$\frac{s^2\,\left(a\,t^2 - a^2\,t^2\right)}{12\,N^4}$$

*** CONCLUSION: thetaa[a, r, s, t, N] yields the correct expansion of \cos \theta [r_{s,t,N}, y_{s,t,N}, z_{s,t,N}] through order \$N^{-3}\$,
so there is no non-zero term of order \$N^{-2}\$ in the expansion of \theta [r_{s,t,N}, y_{s,t,N}, z_{s,t,N}] .

Since the model is homogeneous, we need only apply the first line of (1.78) with $f := N$ to obtain

$$w_N^*(a) = (\sqrt{x_a}\sin\theta)^{-1}[az\tau_a]^N \{\sin N\theta - \sqrt{x_a}w_0^*(a)\sin(N-1)\theta\}. \quad (1.93)$$

Expand $\sin(N-1)\theta = \mathbf{s}\cos\theta - \mathbf{c}\sin\theta$, for $\mathbf{s} := \sin N\theta$ and $\mathbf{c} := \cos N\theta$. Put $\Lambda := (\sin\theta)^{-1}[az\tau_a]^{N-1}$. After direct calculation we find $1 - \sqrt{x_a}w_0^*(a) = 1 - a + O(N^{-1})$. Therefore by (1.93) we have

$$\frac{w_N^*(a)}{\Lambda} = \mathbf{s} - \sqrt{x_a}w_0^*(a)(\mathbf{s}\cos\theta - \mathbf{c}\sin\theta) = (1-a)\mathbf{s} + \frac{O(1)}{N^1}. \quad (1.94)$$

Note that there is no cancellation of the order 1 term in (1.94) Now plug (1.94) into (1.37) to obtain

$$g_{0,N} = \frac{\omega_a}{a(2-a)} r z \tau_a^{-1} \frac{(N-(N-1)a)\sin\theta}{(1-a)\mathbf{s} + O(N^{-1})}.$$

Finally apply the asymptotic expression for θ in (1.92) and let $N \to \infty$. □

******** EXPAND $ (1- \sqrt{x_b} w_0^{*}(b)) [r_{s,t,N}, y_{s,t,N}, z_{s,t,N}] $ to pass from (1.93) to (1.94).

```
Collect[Normal[
  Series[Normal[Series[1 - a * zN[a, s, t, N] * tau[a, a, rstN[a, s, t, N], ystN[a, s, t, N],
        zN[a, s, t, N]] × w0stara[a, rstN[a, s, t, N],
        ystN[a, s, t, N], zN[a, s, t, N]], {t, 0, 1}]], {s, 0, 1}]], N]
```

$$1 - a + \frac{i a s - i a^2 s - 2 i a t + i a^2 t}{N} + \frac{3 a^2 s t - 2 a^3 s t}{N^2}$$

Proof (Theorem 1.1.6). By the same reasoning given at the outset of the proof of Theorem 1.1.1, we may assume that $\mathbf{X}_0 = 0$. By the fact that the absolute value process starts afresh at the end of each excursion, we have that $1 + \mathcal{M}_N$ is a standard geometric random variable with success probability $P(\mathbf{H} \geq N)$. Thus

$$P(\mathcal{M}_N = \nu) = [P(\mathbf{H} < N)]^\nu \, P(\mathbf{H} \geq N), \quad \nu = 0, 1, 2, \ldots. \tag{1.95}$$

Let $\mathbf{L}_N$, $\mathbf{R}_N$, and $\mathbf{V}_N$, respectively, be random variables for the number of steps, runs, and short runs, in an excursion, given that the height of the excursion is at most $N-1$. Therefore, in distribution, we may write:

$$\mathcal{R}_N = \textstyle\sum_{\nu=0}^{\mathcal{M}_N} \mathbf{R}^{(\nu)}, \quad \mathcal{V}_N = \textstyle\sum_{\nu=0}^{\mathcal{M}_N} \mathbf{V}^{(\nu)}, \quad \mathcal{L}_N = \textstyle\sum_{\nu=0}^{\mathcal{M}_N} \mathbf{L}^{(\nu)}, \tag{1.96}$$

where $\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \ldots$; $\mathbf{V}^{(1)}, \mathbf{V}^{(2)}, \ldots$; and $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \ldots$, respectively, are sequences of independent copies of $\mathbf{R}_N$, $\mathbf{V}_N$, and $\mathbf{L}_N$. Since the random variables $\mathbf{R}_N$, $\mathbf{V}_N$, and $\mathbf{L}_N$ already have built into their definitions the condition $\{\mathbf{H} \leq N-1\}$, the probability generating function $K_{N-1} = E\{r^{\mathbf{R}_N} y^{\mathbf{V}_N} z^{\mathbf{L}_N}\}$

is calculated by Theorem 1.3.16. Thus by (1.95), and by calculating a geometric sum there holds:

$$E\{r^{\mathcal{R}_N}y^{\mathcal{V}_N}z^{\mathcal{L}_N}u^{\mathcal{M}_N}\} = \sum_{\nu=0}^{\infty} P(\mathcal{M}_N=\nu)\,(uK_{N-1})^{\nu} = \frac{P(\mathbf{H}\geq N)}{1-uP(\mathbf{H}<N)K_{N-1}[r,y,z]}. \tag{1.97}$$

We define (r_N, y_N, z_N) by (1.73), and also set $u_N := e^{-ita(b-a)/[(1-a)(1-b)N]}$. By (1.9), it suffices to show that $\lim_{N\to\infty} E\{e^{it(1+1/N)\mathcal{X}_{N+1}}\} = \hat{\psi}(t)/\hat{\varphi}(t)$; see (1.102). We define θ_1 and θ_2 by (1.74), so that also (1.75)–(1.76) hold. By the statement of Theorem 1.3.16 we must replace the calculation of $\overline{w}_{0,N}$, starting with (1.77), with instead $\overline{w}_{1,N+1}$. However, by (1.53), (1.62), and (1.77), the difference in the two calculations is simply accounted for by replacing f by $f-1$ in the calculation of $\overline{w}_{0,N}$, because j in (1.62) for $\overline{w}_{1,N+1}$ is determined by $j = N+1-f = N-(f-1)$, so $\frac{1}{\Lambda_2}\overline{w}_{1,N+1} = d_1(f')q^* + d_2(f')w^*$ with $f' := f-1$ in place of f in both (1.53) and (1.83). This is reflected by the fact that, by Lemma 1.3.8, $\overline{w}_{1,N+1} = \overline{w}_{N,0}$. We must now also calculate $\overline{q}_N = d_{q,1}(f)q^*_{N-f+1}(b) + d_{q,2}(f)w^*_{N-f+1}(b)$ given by Lemma 1.3.15, with $d_{q,j}(f) = \mu_{j,1}q^*_{f-1}(a) + \mu_{j,2}q^*_f(a)$, $j=1,2$, defined by (1.61) in the proof of Lemma 1.3.15. In summary, $f' = f-1$ yields (†) $\overline{q}_N = d_{q,1}(f'+1)q^*_{N-f'}(b) + d_{q,2}(f'+1)w^*_{N-f'}(b)$. Thus, because we simply replace f by $f-1$ in the required substitutions, and since $f \sim \eta N$, we will not change the name of f. With this understanding, we may use the calculation of $\overline{w}_{0,N}$ in (1.77)–(1.89) verbatim in place of the calculation of $\overline{w}_{1,N+1}$, and we will do this without changing the names of e_j, q^*, w^* and Λ_j; see (1.79) and (1.83). Further with this understanding, by (†), with f now recouping the role of f', and with q^* and w^* defined by (1.83), we have $\frac{1}{\Lambda_2}\overline{q}_N = d_{q,1}q^* + d_{q,2}w^*$ for

$$d_{q,j} := \mu_{j,1}q^*_f(a) + \mu_{j,2}q^*_{f+1}(a). \tag{1.98}$$

Here by (1.13), (1.41) and (1.74), in analogy with (1.78), we have

$$\begin{aligned} q^*_f(a) &= 2i\alpha_a^{-1}[az\tau_a]^f\left\{y^2\sin f\theta_1 - \sqrt{x_a}q^*_0(a)\sin(f-1)\theta_1\right\} \\ q^*_{f+1}(a) &= 2i\alpha_a^{-1}[az\tau_a]^f\sqrt{x_a}\left\{y^2\sin(f+1)\theta_1 - \sqrt{x_a}q^*_0(a)\sin f\theta_1\right\}. \end{aligned} \tag{1.99}$$

Denote $e_{q,j} := d_{q,j}/\Lambda_1$, and $f_+ = f+1$ and f_- as before. Therefore, by (1.98)–(1.99),

$$e_{q,j} = \left(y^2\mu_{j,1} - \mu_{j,2}xq^*_0\right)\sin f\theta_1 + \sqrt{x}\left\{y^2\mu_{j,2}\sin f_+\theta_1 - \mu_{j,1}q^*_0\sin f_-\theta_1\right\}, \tag{1.100}$$

where $x = x_a$ and $q^*_0 = q^*_0(a)$. Rewrite (1.100) by applying the notations (1.81). Thus $e_{q,j}$ is written, with dependence on a suppressed, by

$$(y^2\mu_{j,1} - \mu_{j,2}xq^*_0)\mathbf{s}_1 + \sqrt{x}\{y^2\mu_{j,2}(\mathbf{s}_1\cos\theta_1 + \mathbf{c}_1\sin\theta_1) - \mu_{j,1}q^*_0(\mathbf{s}_1\cos\theta_1 - \mathbf{c}_1\sin\theta_1)\} \tag{1.101}$$

In summary, by (1.98), we have $\overline{q}_N/(\Lambda_1\Lambda_2) = e_{q,1}q^* + e_{q,2}w^*$, for $e_{q,j}$ in (1.101), and Λ_j defined by (1.79) and (1.83).

To guide the asymptotic expansions of (1.101) we rewrite (1.97) by substituting the last line of the proof of Theorem 1.3.16:

$$E\{e^{it(1+1/N)\mathcal{X}_{N+1}}\} = \frac{P(\mathbf{H} \geq N+1)\overline{w}_{1,N+1}}{\overline{w}_{1,N+1} - (1-a)u_N r_N^2 z_N^2 \overline{q}_N}. \tag{1.102}$$

It turns out that there is a cancellation in the order of the denominator of (1.102). That is, the leading order of each of $\overline{w}_{1,N+1}/(\Lambda_1\Lambda_2)$ and $\overline{q}_N/(\Lambda_1\Lambda_2)$ will be some order 1 trigonometric factor times it/N; in fact there holds $(1-a)\overline{q}_N/\overline{w}_{1,N+1} \sim 1$, as $N \to \infty$. Define

$$\Delta_N := \overline{w}_{1,N+1} - (1-a)u_N r_N^2 z_N^2 \overline{q}_N. \tag{1.103}$$

By direct calculation we will establish that $\Delta_N/(\Lambda_1\Lambda_2) = O(N^{-2})$, and we find the exact coefficient of the order N^{-2} term.

For the asymptotics of (1.101) we may still treat $\sin\theta_1 = \theta_1 + O(N^{-3})$ by (1.76), but must render precisely the $O(N^{-2})$ term in $\cos\theta_1 = 1 + O(N^{-2})$ of (1.75). In an appendix to [15], we display the many terms of the book–keeping method for this problem. For the the present, we simply exhibit the asymptotics of (1.103) obtained by machine computation with $\sin\theta_j$ substituted by the corresponding order $1/N$ term of (1.76):

$$\frac{\Delta_N}{\Lambda_1\Lambda_2} = \frac{1}{(1-a)(1-b)}\frac{t^2}{N^2}\{-ab\sigma_1\sigma_2\mathbf{c}_1\mathbf{c}_2 - a\sigma_1(a-b)^2\mathbf{c}_1\mathbf{s}_2 + a^2\sigma_1^2\mathbf{s}_1\mathbf{s}_2\} + \frac{O(1)}{N^3}. \tag{1.104}$$

Finally we compute the limit of the ratio (1.102) by the asymptotic relations (1.74), and by (1.89) and (1.104). Thus, because by (1.64) and Proposition 1.3.2 we have that $P(\mathbf{H} \geq N+1) \sim C_{a,b}N^{-1}$ for $C_{a,b} = ab/[(1-\eta)a + \eta b - ab]$, we find $E\{e^{it(1+1/N)\mathcal{X}_{N+1}}\}$ is asymptotic to

$$\begin{aligned} &C_{a,b}N^{-1}\left\{[a\sigma_1\mathbf{c}_1\mathbf{s}_2 + b\sigma_2\mathbf{s}_1\mathbf{c}_2 + (b-a)^2\mathbf{s}_1\mathbf{s}_2]\tfrac{it}{N} + O(N^{-2})\right\} \\ &\div\left\{\tfrac{1}{(1-a)(1-b)}[-ab\sigma_1\sigma_2\mathbf{c}_1\mathbf{c}_2 - a\sigma_1(a-b)^2\mathbf{c}_1\mathbf{s}_2 + a^2\sigma_1^2\mathbf{s}_1\mathbf{s}_2]\tfrac{t^2}{N^2} + \tfrac{O(1)}{N^3}\right\}. \end{aligned} \tag{1.105}$$

But, as in the proof of Theorem 1.1.1 we have $\mathbf{c}_j \sim \cosh(\kappa_j t)$, and $\mathbf{s}_j \sim i\sinh(\kappa_j t)$, $j = 1, 2$. Therefore, with $\tilde{C}_{a,b} := (1-a)(1-b)C_{a,b}$, we obtain that (1.105) has the following limit as $N \to \infty$, where we refer to (1.3) and statement of Theorem 1.1.6 for the definitions of $\hat{\varphi}(t)$ and $\hat{\psi}(t)$:

$$\lim_{N\to\infty} E\{e^{it(1+1/N)\mathcal{X}_N}\} = \frac{\tilde{C}_{a,b}}{t} \times \frac{(b\kappa_1\sigma_2 + a\kappa_2\sigma_1)t}{\hat{\varphi}(t)} \times \frac{\hat{\psi}(t)}{ab\sigma_1\sigma_2}.$$

We have $\tilde{C}_{a,b} = ab\sigma_1\sigma_2/(a\sigma_1\kappa_2 + b\sigma_2\kappa_1)$, so the proof is complete. □

******* DEFINE e_{q,1}, e_{q,2}, with auxiliary variables s1 and c1sin as in the proof of Theorem 1.1.1.

Also define the centering term u_N.

```
eq1[a_, b_, r_, y_, z_, s1_, c1sin_] :=
  (y^2 * m11[a, b, r, y, z] - m12[a, b, r, y, z] × xa[a, r, y, z] × q0stara[a, r, y, z]) s1 +
   a * z * tau[a, a, r, y, z] * (y^2 * m12[a, b, r, y, z] (s1 * costheta1[a, r, y, z] + c1sin) -
      m11[a, b, r, y, z] × q0stara[a, r, y, z] (s1 * costheta1[a, r, y, z] - c1sin));
eq2[a_, b_, r_, y_, z_, s1_, c1sin_] :=
  (y^2 * m21[a, b, r, y, z] - m22[a, b, r, y, z] × xa[a, r, y, z] × q0stara[a, r, y, z]) s1 +
   a * z * tau[a, a, r, y, z] * (y^2 * m22[a, b, r, y, z] (s1 * costheta1[a, r, y, z] + c1sin) -
      m21[a, b, r, y, z] × q0stara[a, r, y, z] (s1 * costheta1[a, r, y, z] - c1sin));
uN[a_, b_, t_, N_] := E^((-I * t * a (b - a)) / ((1 - a) (1 - b) N));
```

************** AUTOMATIC CALCULATION of the asymptotics of $ \Delta_{N}/(\Lambda_1\Lambda_2) [rN, yN, zN, uN]$ for (1.104)

```
Collect[
 Factor[Normal[Series[e1[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1, c1 *
        sintheta1asymp[a, b, t, sig1, N]] * (s2 * s2qstar[a, b, rN[a, b, t, N],
          yN[a, b, t, N], zN[a, b, t, N]] + c2 * sintheta2asymp[a, b, t, sig2, N] *
         c2sinqstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]]) +
     e2[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1,
       c1 * sintheta1asymp[a, b, t, sig1, N]] *
      (s2 * s2wstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]] +
        c2 * sintheta2asymp[a, b, t, sig2, N] *
         c2sinwstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]])
     - (1 - a) uN[a, b, t, N] × rN[a, b, t, N]^2 × zN[a, b, t, N]^2 *
      (eq1[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1,
          c1 * sintheta1asymp[a, b, t, sig1, N]] * (s2 * s2qstar[a, b, rN[a, b, t, N],
             yN[a, b, t, N], zN[a, b, t, N]] + c2 * sintheta2asymp[a, b, t, sig2, N] *
            c2sinqstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]]) +
        eq2[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N], s1,
          c1 * sintheta1asymp[a, b, t, sig1, N]] * (s2 * s2wstar[a, b, rN[a, b, t, N],
             yN[a, b, t, N], zN[a, b, t, N]] + c2 * sintheta2asymp[a, b, t, sig2, N] *
            c2sinwstar[a, b, rN[a, b, t, N], yN[a, b, t, N], zN[a, b, t, N]]))

    , {t, 0, 2}]]], N]
```

$$-\frac{1}{(-1+a)\ (-1+b)\ N^2}\ a\ \left(-a^2\ s1\ s2+2\ a^2\ b\ s1\ s2-a\ b^2\ s1\ s2+ a^2\ c1\ s2\ sig1-2\ a\ b\ c1\ s2\ sig1+b^2\ c1\ s2\ sig1+b\ c1\ c2\ sig1\ sig2\right)\ t^2$$

```
Factor[-a (-a^2 s1 s2 + 2 a^2 b s1 s2 - a b^2 s1 s2) / (sigma1[a, b]^2)]
```

$$a^2\ s1\ s2$$

```
Factor[-a (a^2 c1 s2 sig1 - 2 a b c1 s2 sig1 + b^2 c1 s2 sig1)]
```

```
-a (a - b)^2 c1 s2 sig1
```

```
Factor[-a (b c1 c2 sig1 sig2)]
```

```
-a b c1 c2 sig1 sig2
```

*** CONCLUSION: we already distributed the factor (- a) from $-\frac{1}{(-1+a)\ (-1+b)\ N^2}$ a . Hence \Delta_{N}/(\Lambda_1\Lambda_2) [rN, yN, zN, uN]

= [a^2 sig1^2 s1 s2 - a (a - b)^2 c1 s2 sig1 - a b c1 c2 sig1 sig2] $\frac{t^2}{(-1+a)\ (-1+b)\ N^2}$ + O(N^{-3}).

Corollary 1.4.1. *Assume $a = b$. Define*
$Z_1 = \frac{1}{N}\left(\mathcal{R}_N - \frac{1}{(1-a)}\mathcal{V}_N + a\mathcal{M}_N\right); Z_2 = \frac{1}{N}\left(\mathcal{L}_N - \frac{1}{(1-a)}\mathcal{R}_N + \frac{a}{1-a}\mathcal{M}_N\right) - Z_1.$
Then, $\lim_{N\to\infty} E\{e^{i(sZ_1+tZ_2)}\} = \frac{\tanh(\sqrt{1-a)s^2+at^2})}{\sqrt{1-a)s^2+at^2}}.$

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Proof (Corollary 1.4.1). One simplifies the lines of proof Theorem 1.1.6. We leave details in an appendix to [15]. □

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1.5 Appendix

We display the explicit asymptotic expansions through order N^{-2} following from (1.82).

$$[\mathbf{s}_1](e_1) = -(1-a)(1-b) + 2(1-ab)\frac{it}{N} + \frac{2-a^2+a^3-3ab-a^2b+2a^2b^2}{(1-a)(1-b)}\frac{t^2}{N^2} + \frac{O(1)}{N^3}. \tag{1.106}$$

$$[\mathbf{s}_1](e_2) = 1 - a - \frac{a(b-a)}{1-b}\frac{it}{N} - \frac{a(-a-5a^2+2a^3+8ab+4a^2b+2b^2-10ab^2)}{2(1-a)(1-b)^2}\frac{t^2}{N^2} + \frac{O(1)}{N^3}. \tag{1.107}$$

For the proof of Theorem 1.1.1 we do not require the explicit order N^{-2} terms in (1.106)–(1.107), but we display them because they are implicitly required for the proof of Theorem 1.1.6. Again by (1.82) and direct calculation,

$$[\mathbf{c}_1 \sin\theta_1](e_1) = \frac{O(1)}{N^2}; \quad [\mathbf{c}_1 \sin\theta_1](e_2) = a + \frac{a(b-a)}{1-b}\frac{it}{N} + \frac{O(1)}{N^2}. \tag{1.108}$$