BERNOULLI NUMBER IDENTITIES FOR ASSOCIATED STIRLING NUMBERS AND DERANGEMENTS

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Abstract. Denote by $b(n,k)$ the associated Stirling number of the second kind, that is the number of partitions of $[n]$ into k blocks, where each block contains at least 2 elements. Denote by $d(n,k)$ the number of derangements of $[n]$ into k cycles. Let B_n denote the sequence of Bernoulli numbers. We establish that $\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1} = -\,B_{n-1}$, for all $n \geq 3$, and $\sum_{k=1}^{n}(-1)^k d(n+k,k)/((n+k-1)(n+k-2)) = -\,B_{n-1}/(n-1)$, for all $n \geq 3$. These results are extended to the numbers $b_r(n,k)$ and $d_r(n,k)$ that, besides the defining properties of $b(n,k)$ and $d(n,k)$, satisfy also the condition that $1,2,\ldots,r$ fall in distinct blocks or cycles.

1. Introduction

Denote by $b(n,k)$ the number of partitions of $[n]$ into k blocks, where each block contains at least 2 elements, [13, A008299]. Denote by $d(n,k)$ the number of permutations of $[n]$ into k cycles such that there is no fixed point of the permutation, [13, A008306]. The number $d(n,k)$ is called the number of derangements of $[n]$ into k cycles, [4, p. 256], [3, p. 132]. The ordinary Stirling number of the second kind, $\left\{ {n \atop k} \right\}$, is defined as the number of partitions of $[n]$ into k blocks. The ordinary Stirling number of the first kind, $\left[{n \atop k} \right]$, is defined as the number of permutations of $[n]$ into k cycles. The numbers $b(n,k)$ are called the 2–associated Stirling numbers of the second kind as introduced by [4, p. 221], and are denoted variously in [12, p. 77], [4, p. 221], [8, p. 303], [3, p. 136], [5, p. 3], and [15]. The derangement numbers $d(n,k)$ are correspondingly referred to as associated Stirling numbers of the first kind, [15]. A generating function of $b(n,k)$ is by [4, p. 222]

$$\sum_{n,k\geq 0} b(n,k)u^k x^n/n! = \exp(u(e^x - 1 - x)), \tag{1.1}$$

where we define $b(0,0) = 1$ by convention. The corresponding generating function for derangements is, by [4, p. 256],

$$\sum_{n,k\geq 0} d(n,k)u^k x^n/n! = e^{-ux}(1-x)^{-u}, \tag{1.2}$$

where again we define $d(0,0) = 1$ by convention. One way to find (1.2) is to follow [3, p. 134]. By [4, p. 221, p. 256] we have the triangular recurrences

$$\begin{aligned} &\text{(i)} \quad b(n,k) = k\cdot b(n-1,k) + (n-1)b(n-2,k-1),\ n\geq 2, k\geq 1,\\ &\text{(ii)} \quad d(n,k) = (n-1)d(n-1,k) + (n-1)d(n-2,k-1),\ n\geq 2, k\geq 1; \end{aligned} \tag{1.3}$$

see also [3]. Here we define $b(n,0) = 0$ if $n \geq 1$, and $b(n,k) = 0$ if $n < 2k$ or $n < 0$, and likewise for $d(n,k)$.

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We investigate certain alternating sums involving the partition numbers $b(n+k,k)$, and derangement numbers $d(n+k,k)$. For example, by the recurrences (1.3) it may be seen that $\sum_{k=1}^{n}(-1)^k b(n+k,k) = (-1)^n n!$ and $\sum_{k=1}^{n}(-1)^k d(n+k,k) = (-1)^n$, [4, p. 221, p. 256], [3]. Bijective arguments for these identities are given by [3]. We are especially interested in proving results of this type but with Bernoulli numbers arising as the outcome of the summations. That an identity of this type exists for the partition numbers $b(n,k)$, as shown by Theorem 1.1, is perhaps not too surprising due to known formulae for the ordinary Stirling numbers of the second kind, such as the following identity of C. Jordan:

$$\sum_{q=0}^{m}(-1)^q \binom{m+1}{q+1} \frac{\left\{ {m+q \atop q} \right\}}{\binom{m+q}{m}} = B_m, \tag{1.4}$$

where B_m is the sequence of Bernoulli numbers, [9], [11, (15.10)]. Other evidence for partition numbers is given by the formula $B_m = \sum_{k=0}^{m}(-1)^k k! \left\{ {m \atop k} \right\}/(k+1)$; [4, p. 220], [11, (15.2)]. We are thus led after Theorem 1.1 to find a similar identity for a sum involving derangement numbers on one side and Bernoulli numbers on the other side of an equality. We note that the numbers B_n/n do arise in connection with the Stirling numbers of the first kind in [1].

In this paper we are motivated to prove the following results, and to generalize them to certain r–distinguished cases as we shall explain.

Theorem 1.1. *Let* $n \geq 3$. *Then*

$$\sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1} = -B_{n-1}.$$

Theorem 1.2. *Let* $n \geq 3$. *Then*

$$\sum_{k=1}^{n}(-1)^k d(n+k,k)/((n+k-1)(n+k-2)) = -\frac{B_{n-1}}{n-1}.$$

Since it is noted by [4, p. 256] that $\sum_{k=1}^{n}(-1)^k d(n+k,k)/(n+k-1) = 0$, in Theorem 1.2 we may drop the factor $1/(n+k-1)$ due to $1/((n+k-1)(n+k-2)) = 1/(n+k-2) - 1/(n+k-1)$. Yet we argue in section 4 that it is natural to leave the factor $1/(n+k-1)$ as stated for an extension of Theorem 1.2 to Theorem 4.2. This extension treats r–distinguished derangement numbers $d_r(n,k)$ that are defined as the number of permutations of $[n]$ into k cycles and no fixed points with the condition that the elements $1, 2, \ldots, r$ fall in distinct cycles; here $d_1(n,k) = d(n,k)$. These r–distinguished derangement numbers are not to be confused with Comtet's r–associated Stirling numbers of the first kind, [4, p. 257].

We prove Theorem 1.1 in section 2 by applying the generating function method. Besides the generating function approach, our proof of Theorem 1.2 in section 3 relies on both (1.4) and the following connection between Stirling numbers of the first and second kinds going back to Schläfli that is derived by [11, (13.32)]; see also [10, (1)].

$$\left[{k+m \atop k} \right] = (-1)^m \sum_{q=0}^{m}(-1)^q \binom{k+m+q-1}{m+q} \binom{k+2m}{m-q} \left\{ {m+q \atop q} \right\}. \tag{1.5}$$

In section 2.1 we extend Theorem 1.1 to the case of r–distinguished associated Stirling numbers of the second kind, $b_r(n,k)$; see Theorem 2.4. Here $b_r(n,k)$ is the number of partitions of $[n]$ into k blocks without singleton blocks such that $1, 2, \ldots, r$ fall in distinct blocks; in particular $b_1(n,k) = b(n,k)$. The pattern of proof of the extension Theorem 2.4 for $b_r(n,k)$ is simpler but in broad outline parallel to the pattern we use to prove the extension Theorem 4.2 for derangement numbers $d_r(n,k)$. First, for the case $r=1$, generating function arguments are used to represent the sums for both Theorems 1.1 and 1.2. This representation gives a direct link to the Bernoulli numbers in the case of $b(n,k)$ but, as shown by Proposition 3.2, not for the case of $d(n,k)$. For $r \geq 2$ we mention one other key step, that is to evaluate a companion alternating sum; already we see this in the context of $r=1$ for the associated partition numbers wherein the companion sum is given by (2.1). This companion sum is generalized for $b_r(n,k)$ in Lemma 2.3 via the generating function method, wherein we continue to find a zero sum. The companion sum for derangement numbers $d_r(n,k)$ is denoted $T_r(n)$ in (4.4), but only in the case $r=1$ does this companion sum evaluate to zero.

Overall the proofs for derangement numbers are more involved than those for associated partition numbers. This is borne out primarily in section 3 where we eventually apply (1.4)–(1.5) to convert the problem of a Stirling number representation for the sum of Theorem 1.2 to an interesting polynomial identity in Proposition 3.4. In section 4, the proof of a suitable recurrence relation for $d_r(n,k)$ with $r \geq 2$ in Lemma 4.3 requires some work beyond the case $r=1$. To obtain the integer part $P_r(n)$ of Definition 4.1 for the statement of Theorem 4.2, we extend the generating function argument of Proposition 3.2 to represent the companion sum $T_r(n)$ as an r–Stirling number sum; see Definition 4.6 and Lemma 4.8. We further generalize $T_r(n)$ to an r–Stirling number sum $U_r(n,N)$ defined by (4.18). We evaluate $U_r(n,N)$ in Lemma 4.9, which generalizes Corollary 3.3(ii), and thus find $T_r(n) = (-1)^r(r-1)^{n-r}$ for $n \geq r+1$. Since power sums are represented by Bernoulli's formula (4.27), we obtain the Corollary 4.10 to Theorem 4.2.

Acknowledgements

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2. Proof of Theorem 1.1 and its extension to $b_r(n,k)$

The motivation for the form that Theorem 1.1 takes is its similarity to the simple identity

$$\sum_{k=1}^{n} (-1)^k b(n+k,k) / \binom{n+k-1}{k-1} = 0, \text{ for all } n \geq 2, \tag{2.1}$$

which is the $r=1$ case of Lemma 2.3. One may easily prove (2.1) by applying the triangular recurrence (1.3)(i) to write $b(n+k,k) = k \cdot b(n+k-1,k) + (n+k-1)b(n+k-2,k-1)$. Therefore, starting from the case $k=n$ and working backwards, we have

$$\begin{aligned} &b(2n,n) = n \cdot b(2n-1,n) + (2n-1)b(2n-2,n-1) \\ &b(2n-1,n-1) = (n-1) \cdot b(2n-2,n-1) + (2n-2)b(2n-3,n-2) \\ &b(2n-2,n-2) = (n-2) \cdot b(2n-3,n-2) + (2n-3)b(2n-4,n-3), \ \ldots. \end{aligned} \tag{2.2}$$

Notice that, matching terms involving $b(2n-2,n-1)$ in (2.2) for the sum (2.1) we find

$$(-1)^n \left((2n-1)/\binom{2n-1}{n-1} - (n-1)/\binom{2n-2}{n-2} \right) b(2n-2,n-1) = 0.$$

lt is easy to see that the binomial coefficients in the denominators have been chosen to make the sum (2.1) telescope after the continuation of (2.2). In working backwards from $k = n$ to $k = 1$, after substituting (2.2) into the sum of (2.1), we have that the endpoint terms $(-1)^n\ n \cdot b(2n-1, n) = 0$ and $n \cdot b(n-1, 0) = 0$, as long as $n \geq 2$. So the proof of (2.1) follows. Notice that the same type of telescoping argument may be made for the identity $\sum_{k=1}^{n}(-1)^k d(n+k,k)/(n+k-1) = 0$, [4, p. 256], by applying instead (1.3)(ii).

In section 2.1 we prove an extension of Theorem 1.1 to the numbers $b_r(n,k)$ in Theorem 2.4. A key to that proof is the basis $r = 1$ of the extension, namely Theorem 1.1 itself.

Proof of Theorem 1.1. We apply a generating function defined as follows:

$$f(u,z) = \sum_{n\geq 1, k\geq 1} \frac{b(n+k,k)}{\binom{n+k-2}{k-1}} \frac{u^k}{k!} \frac{z^{n-1}}{(n-1)!}. \tag{2.3}$$

Now reorganize the sum in (2.3) with the substitution $m = n + k$, and also substitute the simplification $k!(n-1)!\binom{n+k-2}{k-1} = k(m-2)!$. Hence we have

$$f(u,z) = \sum_{m\geq 2, k\geq 1} b(m,k) \frac{u^k}{k} \frac{z^{m-k-1}}{(m-2)!}.$$

Substitute the recurrence (1.3)(i) to find in turn that

$$f(u,z) = \sum_{m\geq 2, k\geq 1} (k \cdot b(m-1,k) + (m-1)b(m-2,k-1)) \frac{(u/z)^k}{k} \frac{z^{m-1}}{(m-2)!}. \tag{2.4}$$

Write the sum involving $k \cdot b(m-1,k)$ in (2.4) via the substitution $M = m-1$ and call it I:

$$I = \sum_{M\geq 1, k\geq 1} b(M,k)(u/z)^k \frac{z^M}{(M-1)!}. \tag{2.5}$$

Next, for the other term in (2.4), write the factor $(m-1) = (m-2)+1$ and thus break up the sum involving $(m-1)b(m-2,k-1)$ into the sum of two infinite sum expressions, II and III, after the substitutions $N = m-2$, $j = k-1$ as follows:

$$II = \sum_{N\geq 0, j\geq 0} b(N,j) \frac{(u/z)^{j+1}}{j+1} \frac{z^{N+1}}{(N-1)!}, \quad III = \sum_{N\geq 0, j\geq 0} b(N,j) \frac{(u/z)^{j+1}}{j+1} \frac{z^{N+1}}{N!}. \tag{2.6}$$

To calculate I, we note by (1.1) that $\sum_{M\geq 1, k\geq 1} b(M,k) v^k \frac{z^M}{M!} = e^{v(e^z-1-z)} - 1$, so by (2.5) we have $I = z\frac{\partial}{\partial z}(e^{v(e^z-1-z)} - 1)$ evaluated at $v = u/z$. Thus, with $v = u/z$, we have

$$I = zv(e^z-1)e^{v(e^z-1-z)} = (e^z-1) \cdot ue^{uB},$$

for $B = B(z) = \frac{e^z-1-z}{z}$.

Next, for the purpose of calculation write $h = h(z) = e^z - 1 - z$ and thus find II of (2.6) as follows: $II = z^2 \int_0^{u/z} \frac{\partial}{\partial z} e^{vh(z)} dv = z^2 h' \int_0^{u/z} v e^{vh(z)} dv = z^2 h' \left[\frac{v}{h} e^{vh} - \frac{1}{h^2} e^{vh}\right]_0^{u/z}$. Writing $B = h/z$ as before and putting in $h' = e^z - 1$, we therefore have

$$II = (e^z-1)\left(\frac{ue^{uB}}{B} - \frac{e^{uB}-1}{B^2}\right).$$

Lastly, write III of (2.6) as:

$$III = z\int_0^{u/z} e^{vh(z)}dv = z\left[\frac{e^{uh}-1}{h}\right]_0^{u/z} = \frac{e^{uB}-1}{B}.$$

Fix now $n \geq 3$. To calculate the sum of the statement of the theorem, by the definition (2.3) of f it remains to determine the left hand side of the theorem as

$$S(n) = \sum_{k=1}^{n}(-1)^k b(n+k,k)/\binom{n+k-2}{k-1} = (n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k]f(u,z).$$

Since $f(u,z) = I + II + III$, we expand $S(n) = (n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k](I+II+III)$ as follows:

$$\begin{aligned} S(n) &= (n-1)![z^{n-1}]\sum_{k=1}^{n}(-1)^k k![u^k]\left((e^z-1)\left(ue^{uB} + \frac{ue^{uB}}{B} - \frac{e^{uB}-1}{B^2}\right) + \frac{e^{uB}-1}{B}\right) \\ &= (n-1)![z^{n-1}]\left((e^z-1)\sum_{k=1}^{n}(-1)^k k!\left(\frac{B^{k-1}}{(k-1)!} + \frac{B^{k-2}}{(k-1)!} - \frac{B^{k-2}}{k!}\right) + \sum_{k=1}^{n}(-1)^k k!\frac{B^{k-1}}{k!}\right) \end{aligned} \tag{2.7}$$

Now for the first sum on the right side of (2.7) we have

$$\sum_{k=1}^{n}\left((-1)^k k\cdot B^{k-1} - (-1)^{k-1}(k-1)B^{k-2}\right) = (-1)^n n\cdot B^{n-1}$$

as a telescoping sum. For the second sum on the right side of (2.7) we have simply a finite geometric sum $\sum_{k=1}^{n}(-1)^k B^{k-1} = \frac{-1+(-1)^n B^n}{1+B}$. Hence by these calculations following from (2.7) we have shown that

$$S(n) = (n-1)![z^{n-1}]\left((e^z-1)(-1)^n n\cdot B^{n-1} + \frac{-1+(-1)^n B^n}{1+B}\right). \tag{2.8}$$

As power series, we have $B(z) = z/2 + \cdots$, and $e^z - 1 = z + \cdots$. Therefore by (2.8) we have

$$S(n) = (n-1)![z^{n-1}]\left(-\frac{1}{1+B}\right).$$

Finally, $-\frac{1}{1+B} = -\frac{z}{e^z-1} = -\sum_{n=0}^{\infty} B_n\frac{z^n}{n!}$, where B_n is the n–th Bernoulli number. Hence the proof is complete by $(n-1)![z^{n-1}]\left(-\sum_{n=1}^{\infty} B_n\frac{z^n}{n!}\right) = -B_{n-1}$. □

2.1. **Extension to $b_r(n,k)$.** Recall the definition, just after (1.5), of the r–distinguished associated Stirling numbers of the second kind $b_r(n,k)$. Define $b_0(n,k)$ by $b_0(n,k) = b_1(n,k) = b(n,k)$. Note that $b_r(n,k) = 0$ for $k < r$ or $n < 2k$. We take $b_r(0,0) = 0$ for $r \geq 2$.

Lemma 2.1. *Let $r \geq 1$. Then for all $n \geq 1$ and $k \geq 1$ we have*

$$b_r(n,k) = (k-r+1)b_{r-1}(n-1,k) + (n-r)b_{r-1}(n-2,k-1).$$

Proof. The second term accounts for the number $(n-r)$ of ways of forming doubleton sets with minimal element r that we can form with one of the elements x of $[n]\setminus[r]$ to comprise the k–th block, and thus complete the required partition from the $(k-1)$ blocks of $[n]\setminus\{r,x\}$ already counted by $b_{r-1}(n-2,k-1)$. The accounting provided by the first term on the right side of the lemma arises simply by adding the minimal element r to any one of the $(k-r+1)$ blocks of $[n]\setminus\{r\}$ that are not already distinguished by having a specified minimal element i for some $i \in [r-1]$. □

Lemma 2.2. *Let $r \geq 0$. Then*

$$\sum_{n\geq 0, k\geq 0} b_r(n+r, k+r)u^k \ \frac{z^n}{n!} = (e^z-1)^r e^{u(e^z-1-z)}. \tag{2.9}$$

Proof. We proceed by induction in r. By (1.1) the basis $r=0$ is verified. For the induction step, assume that the statement of the lemma is true for some $r \geq 0$. Denote by $f_r(u,z)$ the left side of (2.9). Then, by Lemma 2.1 write

$$f_{r+1}(u,z) = \sum_{n\geq 0,k\geq 0} (k+1)b_r(n+r, k+r+1)u^k \ \frac{z^n}{n!} + \sum_{n\geq 0,k\geq 0} n \cdot b_r(n+r-1, k+r)u^k \ \frac{z^n}{n!}$$

In the first sum change indices by $\ell = k+1$, and in the second sum write $N = n-1$. Thus

$$f_{r+1}(u,z) = \sum_{n\geq 0,\ell\geq 1} \ell \cdot b_r(n+r, \ell+r)u^{\ell-1} \ \frac{z^n}{n!} + \sum_{N\geq 0,k\geq 0} (N+1)b_r(N+r, k+r)u^k \ \frac{z^{N+1}}{(N+1)!}$$

Thus obtain $f_{r+1}(u,z) = \frac{\partial}{\partial u} f_r(u,z) + z f_r(u,z)$. By the induction hypothesis we easily compute this last expression to find $f_{r+1}(u,z) = \left((e^z-1-z)(e^z-1)^r + z(e^z-1)^r\right) e^{u(e^z-1-z)}$, or $f_{r+1}(u,z) = (e^z-1)^{r+1}e^{u(e^z-1-z)}$, as desired. □

Lemma 2.3. *Let $r \geq 1$ and $n \geq r+1$. Then*

$$\sum_{k=1}^{n} b_r(n+k,k)(-1)^k / \binom{n+k-r}{k-1} = 0.$$

Proof. Define

$$g_r(u,z) = \sum_{n\geq 1,k\geq 1} \frac{b_r(n+k,k)}{\binom{n+k-r}{k-1}} \frac{u^k}{(k-1)!} \frac{z^n}{(n-r+1)!}.$$

Substitute $m = n+k$ and use $\binom{n+k-r}{k-1}(k-1)!(n-r+1)! = (n+k-r)! = (m-r)!$ to write $g_r(u,z) = \sum_{m\geq 2,k\geq 1} b_r(m,k)u^k \frac{z^{m-k}}{(m-r)!} = z^r(u/z)^r \sum_{m\geq 2,k\geq 1} b_r(m,k)(u/z)^{k-r} \frac{z^{m-r}}{(m-r)!}$. Thus by Lemma 2.2 we have that $g_r(u,z) = u^r(e^z-1)^r e^{(u/z)(e^z-1-z)}$. Now write $B = B(z) = \frac{e^z-1-z}{z}$. Thus we have $g_r(u,z) = u^r(e^z-1)^r e^{uB}$. Therefore, fixing $r \geq 1$ and $n \geq r+1$, we compute

$$\sum_{k=1}^{n} b_r(n+k,k)(-1)^k / \binom{n+k-r}{k-1} = (n-r+1)![z^n](e^z-1)^r \sum_{k=r}^{n} [u^{k-r}](-1)^k(k-1)!e^{uB}, \tag{2.10}$$

where in the last sum we write the summation index k starting from $k=r$ because for $r \geq 1$ we have $b_r(m,k) = 0$ unless $k \geq r$. Now expand $e^{uB} = \sum_{j\geq 0} u^j B^j/j!$ and so rewrite the right side of (2.10) as

$$(n-r+1)![z^n](e^z-1)^r \sum_{k=r}^{n} \frac{(-1)^k(k-1)!B^{k-r}}{(k-r)!}. \tag{2.11}$$

Pull out a factor $(-1)^r(r-1)!$ and thus rewrite with a binomial coefficient to obtain the sum in (2.11) as

$$(-1)^r(r-1)! \sum_{k=r}^{n} (-1)^{k-r} B^{k-r} \binom{k-1}{r-1}.$$

Finally, because as a power series $B(z) = z/2 + \cdots$, and since likewise $e^z - 1 = z + \cdots$, we may replace the finite sum by the corresponding infinite sum in this last display because the terms beyond $k = n$ in the full series do not contribute to a calculation of (2.11). Thus we have that our desired expression (2.11) is written as

$$(n-r+1)![z^n](e^z-1)^r(-1)^r(r-1)!\sum_{k=r}^{\infty}(-1)^{k-r}B^{k-r}\binom{k-1}{r-1}.$$

But the infinite series collapses as the binomial series for $(1+B)^{-r}$. Hence, because an easy simplification yields $(e^z-1)^r(1+B)^{-r} = z^r$, the proof is complete by $[z^n](z^r) = 0$ for all $n \geq r+1$. □

The following result extends Theorem 1.1 to all $r \geq 1$.

Theorem 2.4. *Let $r \geq 1$ and $n \geq r+2$. Then*

$$\sum_{k=r}^{n} b_r(n+k,k)(-1)^k/\binom{n+k-r-1}{k-1} = (-1)^r r! B_{n-r}.$$

Proof. Define $S_r(n) = \sum_{k=r}^{n}(-1)^k b[r,n+k,k)/\binom{n+k-r-1}{k-1}$. By Lemma 2.1 we make the reduction $b_r(n+k,k) = (k-r+1)b_{r-1}(n+k-1,k) + (n+k-r)b_{r-1}(n+k-2,k-1)$. Hence by substituting this reduction and making a change of index $j = k+1$ in the first sum of the resulting expression for $S_r(n)$ we have that $S_r(n)$ is given by

$$\sum_{j=r+1}^{n+1}(-1)^{j-1}\frac{(j-r)b_{r-1}(n+j-2,j-1)}{\binom{n+j-r-2}{j-2}} + \sum_{k=r}^{n}(-1)^k\frac{(n+k-r)b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-1}{k-1}}.$$

Then in turn, by combining the sums into one, using that the terms in the first sum at both $j = n+1$ (by $b_{r-1}(2n-1,n) = 0$) and $j = r$ are zero, we rewrite $S_r(n)$ under a single summation by

$$S_r(n) = \sum_{k=r}^{n}(-1)^k b_{r-1}(n+k-2,k-1)\left(\frac{-(k-r)}{\binom{n+k-r-2}{k-2}} + \frac{n+k-r}{\binom{n+k-r-1}{k-1}}\right).$$

Now calculate

$$\frac{-(k-r)}{\binom{n+k-r-2}{k-2}} + \frac{n+k-r}{\binom{n+k-r-1}{k-1}} = (k-2)!\frac{-(k-r)(n+k-r-1)+(k-1)(n+k-r)}{(n+k-r-1)\cdots(n-r+1)}.$$

The numerator of the last fraction may be rewritten $r(n+k-r-1)-(n-r)$. Correspondingly we obtain that $S_r(n) = I + II$ where

$$I = r\sum_{k=r}^{n}(-1)^k\ \frac{b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-2}{k-2}}, \quad II = \frac{-(n-r)}{n-r+1}\sum_{k=r}^{n}(-1)^k\ \frac{b_{r-1}(n+k-2,k-1)}{\binom{n+k-r-1}{k-2}}.$$

Now change both indices $n' = n-1$ and $k' = k-1$ and also put $r' = r-1$ to obtain

$$II = \frac{n-r}{n-r+1}\sum_{k'=r'}^{n'}(-1)^{k'}\ \frac{b_{r'}(n'+k',k')}{\binom{n'+k'-r'}{k'-1}}.$$

Hence we have $II = 0$ by Lemma 2.3. Further we have

$$I = (-r)\sum_{k'=r'}^{n'}(-1)^{k'}\ \frac{b_{r'}(n'+k',k')}{\binom{n'+k'-r'-1}{k'-1}} = (-r)S_{r'}(n').$$

Therefore we have shown $S_r(n) = (-r)S_{r'}(n')$. Hence because the case $r = 1$ in the statement of the theorem is the result of Theorem 1.1, by backward induction the proof is complete. $\square$

3. Proof of Theorem 1.2

In this section we reduce the problem of Theorem 1.2 by a series of steps. The first step in Proposition 3.2 is to apply a generating function argument to write the sum in the statement of this theorem in terms of a sum involving ordinary Stirling numbers of the first kind. To prepare for this step we want an arithmetical identity for the ordinary Stirling numbers of the first kind as follows.

Lemma 3.1. [2, Thm. 7] *Let $m \geq 2$ and $m > k \geq 1$. Then we have*

$$\begin{bmatrix} m \\ m-k \end{bmatrix} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k < m} i_1 i_2 \cdots i_k. \tag{3.1}$$

Further $\begin{bmatrix} m \\ m \end{bmatrix} = 1$, *for all* $m \geq 0$.

For example, $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot 2 = 2$, while $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1 + 2 = 3$. Likewise $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11$, since we consider products with $k = 4 - 2 = 2$ factors.

Proposition 3.2. *Let $n \geq 2$. Then we have*

$$\sum_{k=1}^{n} (-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} = \sum_{k=1}^{n} \frac{(-1)^k}{n+k-2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \binom{2n-2}{n+k-2}.$$

Proof. Define

$$f(u,z) = \sum_{n \geq 1, k \geq 1} \frac{d(n+k,k)}{n+k-1} \, u^k \frac{z^n}{(n+k-2)!}. \tag{3.2}$$

Make the substitution $m = n + k$ and write $z^n = z^{n+k}/z^k = z^m/z^k$ to rewrite (3.2) as

$$f(u,z) = \sum_{m \geq 2, k \geq 1} d(m,k)(u/z)^k \frac{z^m}{(m-1)!} \tag{3.3}$$

Therefore, by denoting $g(u,z)$ the generating function of $d(n,k)$ defined by (1.2), we have by (3.3) that

$$f(u,z) = z \frac{\partial}{\partial z} g(v,z) \text{ evaluated at } v = u/z.$$

Here we have

$$z \frac{\partial}{\partial z} g(v,z) = z \frac{\partial}{\partial z} \left(e^{-tv}(1-z)^{-v} \right) = zve^{-tv} \left((1-z)^{-v-1} - (1-z)^{-v} \right) = z^2 v e^{-tv} (1-z)^{-v-1}.$$

Therefore, evaluation at $v = u/z$ yields

$$f(u,z) = zue^{-u}(1-z)^{-u/z-1} \tag{3.4}$$

Now expand $(1-z)^{-v-1}$ as a binomial expansion about $z = 0$ to obtain

$$(1-z)^{-v-1} = 1 + \frac{(v+1)}{1!} z + \frac{(v+1)(v+2)}{2!} z^2 + \frac{(v+1)(v+2)(v+3)}{3!} z^3 + \cdots.$$

Thus, after plugging in $v = u/z$ we find by (3.4) that

$$f(u,z) = ze^{-u} \cdot u \left(1 + \frac{(u+z)}{1!} + \frac{(u+z)(u+2z)}{2!} + \frac{(u+z)(u+2z)(u+3z)}{3!} + \cdots \right). \tag{3.5}$$

By Lemma 3.1 we have the basic representation:

$$u(u+z)(u+2z)(u+3z)\cdots(u+pz) = \sum_{n=0}^{p} \begin{bmatrix} p+1 \\ p+1-n \end{bmatrix} z^n u^{p+1-n}. \tag{3.6}$$

Therefore by (3.4)–(3.6), putting the factor of z now under the sum we must expand $f(u,z) = e^{-u}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n=0}^{p}\begin{bmatrix} p+1 \\ p+1-n \end{bmatrix} z^{n+1}u^{p+1-n}$. We write $e^{-u} = \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{u^{\ell}}{\ell!}$ and thus find:

$$\begin{aligned} f(u,z) &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\ \frac{u^{\ell}}{\ell!}\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n=0}^{p}\begin{bmatrix} p+1 \\ p+1-n \end{bmatrix} z^{n+1}u^{p+1-n} = u\left(\begin{bmatrix}1\\1\end{bmatrix}\frac{z}{0!} + \begin{bmatrix}2\\1\end{bmatrix}\frac{z^2}{1!} + \begin{bmatrix}3\\1\end{bmatrix}\frac{z^3}{2!} + \cdots\right) \\ &+ u^2\left(\left(-\begin{bmatrix}1\\1\end{bmatrix}\frac{z}{1!0!} + \begin{bmatrix}2\\2\end{bmatrix}\frac{z}{0!1!}\right) + \left(-\begin{bmatrix}2\\1\end{bmatrix}\frac{z^2}{1!1!} + \begin{bmatrix}3\\2\end{bmatrix}\frac{z^2}{0!2!}\right)\cdots\right) + \\ &+ u^3\left(\left(\begin{bmatrix}1\\1\end{bmatrix}\frac{z}{2!0!} - \begin{bmatrix}2\\2\end{bmatrix}\frac{z}{1!1!} + \begin{bmatrix}3\\3\end{bmatrix}\frac{z}{0!2!}\right) + \left(\begin{bmatrix}2\\1\end{bmatrix}\frac{z^2}{2!1!} - \begin{bmatrix}3\\2\end{bmatrix}\frac{z^2}{1!2!} + \begin{bmatrix}4\\3\end{bmatrix}\frac{z^2}{0!3!}\right) + \cdots\right) \end{aligned} \tag{3.7}$$

Here the first expanded sum for the triple sum expression is the case $p-n=0$, $\ell=0$, for all $n=0,1,2,\cdots$; the second expanded sum is the case $p-n=0, \ell=1$ or $p-n=1, \ell=0$ for each grouped parenthetical term involving powers of z^{n+1}, where $n=0,1,\cdots$ for successive grouped terms; and the third expanded sum is the case $p-n=0, \ell=2$, or $p-n=1, \ell=1$ or $p-n=2, \ell=0$, again with $n=0,1,\cdots$ picking up powers of z^{n+1} in successive grouped terms. The Stirling number coefficients for the term $u^k z^{n+1}/(p!\ell!)$, where $p+\ell=n+k-1$, have top index $n+j$ and bottom index j, for $j=k-\ell$, $0\le\ell\le k-1$. That is, we take $j=p+1-n$ for the bottom index and $n+j$ for the top index of the Stirling number coefficient. So we have $k=(p+1-n)+\ell=j+\ell$ for the power on u, and $n+1$ for the power on z. Thus we convert to summation indices k, n, and j, with $k\ge1$, $n\ge0$ and $1\le j\le k$. The sign of the term $u^k z^{n+1}/(p!\ell!) = u^k z^{n+1}/((n+j-1)!(k-j)!)$ is $(-1)^{\ell}=(-1)^j(-1)^k$. Therefore we have

$$f(u,z) = \sum_{k=1}^{\infty}(-1)^k\ u^k\sum_{n=0}^{\infty}z^{n+1}\sum_{j=1}^{k}(-1)^j\begin{bmatrix} n+j \\ j \end{bmatrix}\frac{1}{(n+j-1)!(k-j)!}. \tag{3.8}$$

We note that the $u^k z^{n+1}$ terms in the expansion of $f(u,z)$ where $n<k$ must have coefficient zero by $d(n+k,k)=0$ in this case. For example, the $u^3 z$ term has coefficient $\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{2!0!} - \begin{bmatrix}2\\2\end{bmatrix}\frac{1}{1!1!} + \begin{bmatrix}3\\3\end{bmatrix}\frac{1}{0!2!} = \frac{1}{2}-1+\frac{1}{2}=0$, and the u^3z^2 term has coefficient $\begin{bmatrix}2\\1\end{bmatrix}\frac{1}{2!1!} - \begin{bmatrix}3\\2\end{bmatrix}\frac{1}{1!2!} + \begin{bmatrix}4\\3\end{bmatrix}\frac{1}{0!3!} = \frac{1}{2}-\frac{3}{2}+1=0$.

Now we rewrite $f(u,z)$ in (3.8) by replacing n by $n-1$ so as to make a power of z^n, consistent with (3.2). Correspondingly we introduce $\frac{1}{(n+j-2)!(k-j)!} = \binom{n+k-2}{n+j-2}\frac{1}{(n+k-2)!}$. We rewrite (3.8) accordingly as follows.

$$f(u,z) = \sum_{k=1}^{\infty}(-1)^k\ u^k\sum_{n=1}^{\infty}z^n\sum_{j=1}^{k}(-1)^j\begin{bmatrix} n-1+j \\ j \end{bmatrix}\binom{n+k-2}{n+j-2}\frac{1}{(n+k-2)!}. \tag{3.9}$$

We simply write, by the definition (3.2) of $f(u,z)$ and by (3.9), that for any $n\ge1$ and $k\ge1$,

$$(-1)^k\frac{d(n+k,k)}{(n+k-1)} = (-1)^k[u^kz^n](n+k-2)!f(u,z) = \sum_{j=1}^{k}(-1)^j\begin{bmatrix} n-1+j \\ j \end{bmatrix}\binom{n+k-2}{n+j-2}. \tag{3.10}$$

Finally, fix $n \geq 2$. Divide through (3.10) by $(n+k-2)$ and sum over k from 1 to n to find

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} &= \sum_{k=1}^{n}\sum_{j=1}^{k}(-1)^j \left[{n-1+j \atop j}\right]\binom{n+k-2}{n+j-2}\frac{1}{n+k-2} \\ &= \sum_{j=1}^{n}(-1)^j \left[{n-1+j \atop j}\right]\sum_{k=j}^{n}\binom{n+k-2}{n+j-2}\frac{1}{n+k-2},\end{aligned} \tag{3.11}$$

where we changed the order of summation of the double sum in k and j at the last step. However, we have the simple binomial identity $\sum_{m=a}^{b}\frac{1}{m}\binom{m}{a} = \sum_{m=a}^{b}\frac{1}{a}\binom{m-1}{a-1} = \frac{1}{a}\binom{b}{a}$, because $\binom{m-1}{a-1} = \binom{m}{a} - \binom{m-1}{a}$, so we have a telescoping sum in m. Therefore, with $m = n+k-2$, $a = n+j-2$, and $b = n+n-2 = 2n-2$, we have $\sum_{k=j}^{n}\binom{n+k-2}{n+j-2}\frac{1}{n+k-2} = \binom{2n-2}{n+j-2}\frac{1}{n+j-2}$. Thus by (3.11) we have

$$\sum_{k=1}^{n}(-1)^k \frac{d(n+k,k)}{(n+k-1)(n+k-2)} = \sum_{j=1}^{n}\frac{(-1)^j}{n+j-2}\left[{n-1+j \atop j}\right]\binom{2n-2}{n+j-2}. \tag{3.12}$$

This completes the proof of the proposition. □

Corollary 3.3. *The following hold under the stated conditions.*

$$\begin{aligned}&(i) \quad \sum_{j=1}^{N-n+2}(-1)^j \left[{n+j-1 \atop j}\right]\binom{N}{n+j-2} = 0, \quad \text{for all } n \geq 1,\ N \geq 2n-1; \\ &(ii) \quad \sum_{j=1}^{N-n+1}(-1)^j \left[{n+j-1 \atop j}\right]\binom{N}{n+j-1} = 0, \quad \text{for all } n \geq 2,\ N \geq 2n-1.\end{aligned} \tag{3.13}$$

Proof. First, let $k \geq n+1$ in (3.10). Then because $d(n+k,k) = 0$, the sum on the right side of (3.10) is zero. Now put $N = n+k-2$. So the sum (3.13)(i) is zero for all $N \geq n+(n+1)-2 = 2n-1$. To prove (3.13)(ii), if $n \geq 2$ we have $\sum_{k=1}^{n}(-1)^k d(n+k,k)/(n+k-1) = 0$, [4, p. 256]. Therefore because $d(n+k,k) = 0$ for $k \geq n+1$, we have $\sum_{k=1}^{M}(-1)^k d(n+k,k)/(n+k-1) = 0$ for any $M \geq n$. Therefore by (3.10), if $M \geq n$, we have $\sum_{k=1}^{M}\sum_{j=1}^{k}(-1)^j \left[{n-1+j \atop j}\right]\binom{n+k-2}{n+j-2} = \sum_{j=1}^{M}(-1)^j\left[{n-1+j \atop j}\right]\sum_{k=j}^{M}\binom{n+k-2}{n+j-2} = 0$. But the inner sum of this last double sum is simply $\binom{n+M-1}{n+j-1}$. Now put $M = N-n+1$ and substitute this binomial expression in place of the inner sum of the last double sum to obtain (3.13)(ii). It only remains to check the condition on the size of N. By the definition of M in terms of N for all $N \geq 2n-1$ we have $M \geq n$. Therefore the proof is complete. □

The remaining strategy to prove Theorem 1.2 is to obtain the statement (3.16) that, when shown to be true, is sufficient to imply the theorem as follows. To obtain (3.16) we will first plug in Schläfli's formula (1.5) to the right side of Proposition 3.2. The resulting double sum will then involve ordinary Stirling numbers of the second kind. Due to experimental evidence we then simply match a sum of binomial expressions with the coefficient of the C. Jordan formula (1.4) to find (3.16). Note that statement (3.16) in not an algebraic identity; it only holds for the specified range in its statement. The last part of the proof is to delve in to establishing this statement by passing first to Lagrange interpolation.

We proceed to plug in (1.5) with $m = n-1$ to the sum on the right side of Proposition 3.2 and thus obtain

$$\begin{aligned}&\sum_{k=1}^{n} \frac{(-1)^k}{n+k-2} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \binom{2n-2}{n+k-2} \\ &= \sum_{k=1}^{m+1} \frac{(-1)^k}{m+k-1} (-1)^m \sum_{q=0}^{m} (-1)^q \binom{k+m+q-1}{m+q} \binom{k+2m}{m-q} \begin{Bmatrix} m+q \\ q \end{Bmatrix} \binom{2m}{m+k-1},\end{aligned} \tag{3.14}$$

where we have substituted $n = m+1$ throughout the terms from the first line of (3.14) into the second line.

Now trivially reverse the order of summation in the second line of (3.14) and rearrange to rewrite write this second line as

$$(-1)^m \sum_{q=0}^{m} (-1)^q \sum_{k=1}^{m+1} \frac{(-1)^k}{m+k-1} \binom{k+m+q-1}{m+q} \binom{k+2m}{m-q} \binom{2m}{m+k-1} \begin{Bmatrix} m+q \\ q \end{Bmatrix} \tag{3.15}$$

Consider now the formula (1.4). In view of Proposition 3.2 and (3.14)–(3.15), for Theorem 1.2 to hold it suffices that that the following statement holds:

$$\begin{aligned}&\sum_{k=1}^{m+1} \frac{(-1)^k}{m+k-1} \binom{k+m+q-1}{m+q} \binom{k+2m}{m-q} \binom{2m}{m+k-1} = -\frac{1}{m} \binom{m+1}{q+1} / \binom{m+q}{m}, \\ &\text{for all } q = 0, 1, \cdots, m, \text{ and all } m \geq 2.\end{aligned} \tag{3.16}$$

Indeed, if the statement (3.16) holds, then because already $m = n-1$, then the factor $-\frac{1}{m}$ on the right side of (3.16) will result via (1.4) into an evaluation of (3.14)–(3.15) as the value $-B_{n-1}/(n-1)$ whenever m is even. In (3.16) we have dropped the factor $(-1)^m$ from (3.15) because if $m = n-1 \geq 3$ is odd, we have that the Bernoulli number $B_m = 0$. Therefore by (3.15) and (1.4) it suffices to show that (3.16) holds as stated in the range $q = 0, 1, 2, \cdots, m$ (with $m \geq 2$). It is easily verified by experiment, say at $q = m+1$, that the statement (3.16) fails outside this range.

3.1. **Verification of statement** (3.16)**.** We first undertake to simplify statement (3.16) by isolating the variable q. First, by simply expanding binomial coefficients and rearranging factors we have

$$\binom{k+m+q-1}{m+q} \binom{k+2m}{m-q} \binom{2m}{m+k-1} = \frac{m+1}{m+k+q} \binom{2m}{m-q} \binom{m}{k-1} \binom{k+2m}{m+1}. \tag{3.17}$$

Indeed, the left side of (3.17) is

$$\begin{aligned}&\frac{(k+m+q-1)!}{(m+q)!(k-1)!} \frac{(k+2m)!}{(k+m+q)!(m-q)!} \frac{(2m)!}{(m+k-1)!(m-k+1)!} \\ &= \frac{1}{k+m+q} \frac{(k+2m)!}{(k-1)!} \frac{(2m)!}{(m+q)!(m-q)!} \frac{m+1}{(m+k-1)!(m+1)!} \frac{m!}{(m-k+1)!} \\ &= \frac{m+1}{k+m+q} \frac{(2m)!}{(m+q)!(m-q)!} \frac{(k+2m)!}{(m+k-1)!(m+1)!} \frac{m!}{(k-1)!(m-k+1)!}\end{aligned}$$

where we canceled $(k+m+q)!$ in numerator and denominator and multiplied and divided by $m!$ in the form $(m+1)\frac{m!}{(m+1)!}$ in the second line, and moved the factors $\frac{(k+2m)!}{(k-1)!}$ in the third line. This third line is evidently the right side of (3.17). Next we rewrite the right side of (3.16) modulo the minus sign as follows:

$$\frac{1}{m}\binom{m+1}{q+1}\Big/\binom{m+q}{q} = \frac{1}{q+1}\binom{2m}{m-q}\Big/\binom{2m}{m+1}. \tag{3.18}$$

Indeed, $\frac{1}{m}\frac{(m+1)!}{(q+1)!(m-q)!}\frac{q!m!}{(m+q)!} = \frac{1}{q+1}\frac{m+1}{m}\frac{m!m!}{(2m)!}\frac{(2m)!}{(m-q)!(m+q)!} = \frac{(m-1)!(m+1)!}{(2m)!}\frac{(2m)!}{(m-q)!(m+q)!}$, where we multiplied and divided by $(2m)!$. So (3.18) is verified. Finally, by plugging in (3.17) and (3.18) respectively into the left and right sides of (3.16), and canceling the common factor $\binom{2m}{m-q}$ on the two sides, and finally changing the index of summation with k in place of $k-1$ running from 0 to m, and thus introducing a minus sign on the left side, we see that statement (3.16) is equivalent to the following statement:

$$\sum_{k=0}^{m}(-1)^k\frac{m+1}{(m+1+k+q)(m+k)}\binom{m}{k}\binom{k+2m+1}{m+1} = \frac{1}{q+1}\frac{1}{\binom{2m}{m+1}}, \tag{3.19}$$
$$\text{for all } q=0,1,\cdots,m, \text{ and all } m\geq 2.$$

We now treat q in (3.19) as a real variable. Define a polynomial of degree $m+1$ by

$$P(x) = \prod_{k=0}^{m}(m+1+k+x), \quad \text{for all real } x. \tag{3.20}$$

Since we want to prove that the two sides of (3.19) indeed interpolate each other at the points $q=0,1,\cdots,m$ and since multiplication of both sides by $(1+q)P(q)$ leaves a polynomial of degree $(m+1)$ in q on the left and a constant multiple of $P(q)$ on the right, there would be a polynomial of degree $(m+1)$ that makes up the difference. Define the polynomial

$$\phi(x) = (-1)^{m+1}x(x-1)(x-2)\cdots(x-m) = \prod_{j=0}^{m}(j-x). \tag{3.21}$$

Then we see that the problem of the verification of (3.19) will be solved by the following polynomial identity.

Proposition 3.4. *Define $P(x)$ and $\phi(x)$ by (3.20)–(3.21). Then we have the following polynomial identity in the real variable x.*

$$\phi(x)+(1+x)P(x)\sum_{k=0}^{m}(-1)^k\frac{m+1}{(m+1+k+x)(m+k)}\binom{m}{k}\binom{k+2m+1}{m+1} = P(x)\Big/\binom{2m}{m+1}. \tag{3.22}$$

Before proving Proposition 3.4 we state some useful formulae for polynomials from [11, Chp. 7].

Theorem 3.5. *Lagrange Interpolation Formula* [11, Thm. 7.1] *Let $\phi(x) = \sum_{i=0}^{n} a_i x^i$ with $n\geq 1$. Then*

$$\phi(x) = \sum_{k=0}^{n}\phi(x_k)\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{x-x_i}{x_k-x_i},$$

whenever $\{x_i\}_{i=0}^{n}$ is a set of cardinallity $n+1$.

Theorem 3.6. *Melzak's Formula* [11, (7.1)] *Let* $f(x) = \sum_{i=0}^{n} a_i x^i$ *with* $n \geq 0$*. Let* y *be an arbitrary complex number with* $y \neq 0, -1, -2, \ldots, -n$*. Then*

$$f(x+y) = y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-y)}{y+k}.$$

Proof of Proposition 3.4. The idea of the proof is to apply the Lagrange interpolation formula to $\phi(x)$ with $m+1$ in place of n and with the interpolation points $\{x_i\}_{i=0}^{m+1}$ chosen in such a way that the polynomial term $(1+x)P(x)/(m+1+k+x) = (1+x)\prod_{i=0,\ i\neq k}^{m}(m+1+i+x)$ is matched in the formula for ϕ instead by $\prod_{i=0,\ i\neq k}^{m+1}(m+1+i+x)$, where the upper index $m+1$ in this last product agrees with the Lagrange formula. By Theorem 3.5 and the definition (3.21) of $\phi(x)$ this means that we simply take as interpolation points $x_i = -(m+1+i)$, for all $i = 0, 1, \ldots, , m+1$. Now we develop a combinatorial formula for $\phi(x)$ based on Lagrange's formula. Define

$$Q(x) = \prod_{i=0}^{m+1}(m+1+i+x), \tag{3.23}$$

so $Q(x)$ has degree $n+1 = m+2$. Then by (3.21) and our choice of the interpolation points x_i in the Lagrange formula, the k–th summand of the interpolation formula for $\phi(x)$ may be written as

$$\phi(x_k)\prod_{i=0,\ i\neq k}^{m+1}\frac{x-x_i}{x_k-x_i} = \frac{\prod_{j=0}^{m}(j-x_k)}{\prod_{i=0,\ i\neq k}^{m+1}(x_k-x_i)}\frac{Q(x)}{m+1+k+x},\ \ k = 0, 1, 2, \ldots, m+1, \tag{3.24}$$

since by (3.23) we have $\prod_{i=0,\ i\neq k}^{m+1}(x-x_i) = Q(x)/(m+1+k+x)$. Since $(j-x_k) = (m+1+k+j)$, and $(x_k - x_i) = (i-k)$, for our representation of $\phi(x)$ it remains to evaluate the coefficient

$$\frac{\prod_{j=0}^{m}(j-x_k)}{\prod_{i=0,\ i\neq k}^{m+1}(x_k-x_i)} = \frac{\prod_{j=0}^{m}(m+1+k+j)}{\prod_{i=0,\ i\neq k}^{m+1}(i-k)}. \tag{3.25}$$

It is easy to see that the denominator of (3.25) is $(-1)^k k!(m+1-k)!$. Moreover the numerator of this fraction may be regarded as a descending product starting from the top factor $(k+2m+1)$ and running for $m+1$ factors. Therefore

$$\frac{\prod_{j=0}^{m}(m+1+k+j)}{\prod_{i=0,\ i\neq k}^{m+1}(i-k)} = \frac{(m+1)!}{\prod_{i=0,\ i\neq k}^{m+1}(i-k)}\frac{\prod_{j=0}^{m}(m+1+k+j)}{(m+1)!} = (-1)^k\binom{m+1}{k}\binom{k+2m+1}{m+1}. \tag{3.26}$$

Therefore by Theorem 3.5 and (3.24)–(3.26) the assertion (3.22) of the proposition may be rewritten as follows.

$$\begin{aligned}\sum_{k=0}^{m+1}(-1)^k\frac{Q(x)}{m+1+k+x}\binom{m+1}{k}\binom{k+2m+1}{m+1}\\+\sum_{k=0}^{m}(-1)^k\frac{(m+1)(1+x)P(x)}{(m+1+k+x)(m+k)}\binom{m}{k}\binom{k+2m+1}{m+1}=P(x)/\binom{2m}{m+1}.\end{aligned}\tag{3.27}$$

Notice that in the second sum of (3.27) the binomial coefficient $\binom{m}{k}$ is automatically zero when $k=m+1$, so we may regard the sum of the two sums on the left side as a single sum as k runs from 0 to $m+1$. We claim that we can reduce the sum of the two corresponding k–th summands modulo the common factor $(-1)^k\binom{k+2m+1}{m+1}$ as follows.

$$\frac{Q(x)}{m+1+k+x}\binom{m+1}{k}+\frac{(m+1)(1+x)P(x)}{(m+1+k+x)(m+k)}\binom{m}{k}=P(x)\left(\frac{m+1}{m+k}\binom{m}{k}+\binom{m+1}{k}\right),\tag{3.28}$$

so that the two summands add simply to a combinatorial multiple of $P(x)$. Now by the definitions (3.20) and (3.23) of $P(x)$ and $Q(x)$, we have $Q(x)=(2m+2+x)P(x)$. Thus to verify (3.28) we must simply check, after transposing the binomial terms on the right side of (3.28) to their matching terms on the left, that

$$\left(\frac{2m+2+x}{m+1+k+x}-1\right)\binom{m+1}{k}+\left(\frac{(m+1)(1+x)}{(m+1+k+x)(m+k)}-\frac{m+1}{m+k}\right)\binom{m}{k}=0.\tag{3.29}$$

After obtaining common denominators in the two differences in (3.29) we must simply verify that $\frac{m+1-k}{m+1+k+x}\binom{m+1}{k}-\frac{m+1}{(m+1+k+x)}\binom{m}{k}=0$. Obviously we can dispense with the denominators in this last difference so the verification boils down to $(m+1)\binom{m+1}{k}-(m+1)\binom{m}{k}-k\binom{m+1}{k}=(m+1)\binom{m+1}{k-1}-k\binom{m+1}{k}=0$. Thus (3.28) has been verified. Hence, by (3.28) the assertion (3.27), which we recall is equivalent to the statement of the proposition, takes the form

$$P(x)\sum_{k=0}^{m+1}(-1)^k\left(\frac{m+1}{m+k}\binom{m}{k}+\binom{m+1}{k}\right)\binom{k+2m+1}{m+1}=P(x)/\binom{2m}{m+1}.\tag{3.30}$$

We now dispense with the common factor $P(x)$, and so to complete the proof of the proposition we are left in (3.30) with a final combinatorial identity to prove. To handle the proposed combinatorial identity (3.30) we will first rewrite $\frac{m+1}{m+k}\binom{m}{k}=\frac{k+1}{m+k}\binom{m+1}{k+1}$, and then change index by $j=k+1$. Thus (3.30) becomes

$$-\sum_{j=0}^{m+1}(-1)^j\frac{j}{m-1+j}\binom{m+1}{j}\binom{j+2m}{m+1}+\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\binom{k+2m+1}{m+1}=1/\binom{2m}{m+1}\tag{3.31}$$

Here, for the first sum on the left, the term $j=0$ has contribution zero and the term $j=m+2$ has contribution zero, so we've included the case $j=0$ and excluded the case $j=m+2$ to match the same initial and final indices of the sum over k. Thus convert both sums to sums over k and rewrite $\frac{k}{m-1+k}=1-\frac{m-1}{m-1+k}$ in the first sum and thus obtain that the left side of

(3.31) is written

$$\begin{aligned}&(m-1)\sum_{k=0}^{m+1}(-1)^k\frac{1}{m-1+k}\binom{m+1}{k}\binom{k+2m}{m+1}\\&+\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\left(\binom{k+2m+1}{m+1}-\binom{k+2m}{m+1}\right)=I+II.\end{aligned}\tag{3.32}$$

Now by the binomial recurrence we have $II=\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\binom{k+2m}{m}$. But by [11, (6.35)] we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+x}{m}=(-1)^n\binom{x}{m-n}$. Thus with $n=m+1$ and $x=2m$ we find that $II=0$.

Finally to handle I of (3.32) we apply Melzak's formula. The procedure is similar to that shown by [11, (7.29)–(7.30)]. To match the form of Melzak's formula in Theorem 3.6, put back $n=m+1$ and define $y=m-1=n-2$. Also take the polynomial function of degree n as $f(x)=\binom{x+2n-2}{n}$. For our application of Theorem 3.6 we choose the real variable x by $-x=3n-3$. To motivate this choice of x, notice that to apply Melzak's formula as stated we must apply the -1 transformation $\binom{x}{n}=(-1)^n\binom{-x+n-1}{n}$ to rewrite $f(x-k)$ as follows. We have $f(x-k)=(-1)^n\binom{-(x-k+2n-2)+n-1}{n}=(-1)^n\binom{k+2n-2}{n}$, where we applied our choice of x. On the other hand, using the original formula for $f(x)$, we have $f(x+y)=\binom{-(3n-3)+2n-2+n-2}{n}=\binom{-1}{n}=(-1)^n$. By our choices of n, x, and y the term I from (3.32) is indeed written in the form

$$I=y(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x-k)/(y+k).$$

Therefore we have by Melzak's formula that

$$I=y(-1)^nf(x+y)/\left(y\binom{y+n}{n}\right)=1/\binom{2n-2}{n},\tag{3.33}$$

since $f(x+y)=(-1)^n$ and $y+n=2n-2$. But $2n-2=2m$ and $n=m+1$, so $I=1/\binom{2m}{m+1}$, as desired. That is, by (3.31)–(3.33), and using $II=0$, we have verified (3.30). Thus the proposition is proved. □

Proof of Theorem 1.2. Let $q\in\{0,1,2,\ldots,m\}$. Plug in $x=q$ in the polynomial identity of Proposition 3.4 with $\phi(x)$ defined by (3.21). Since $\phi(q)$ vanishes, and since $P(q)\neq 0$ for $P(x)$ defined by (3.20), and since $P(q)$ is a common factor of the remaining identity in m with $x=q$ fixed in the given range resulting from the evaluation of (3.22), we obtain that (3.19) holds. Therefore, working backwards through (3.17)–(3.18), we have that the statement (3.16) holds. We already argued just after (3.16) that, via Proposition 3.2 and (1.4)–(1.5), the theorem follows from the statement (3.16). □

4. Extension of Theorem 1.2 to $d_r(n,k)$

Let $d_r(n,k)$ be the number of permutations of $[n]$ into k cycles with no fixed points such that $1,2,\ldots,r$ fall in distinct cycles; this is the r–distinguished case for derangements. Define $d_0(n,k)=d_1(n,k)=d(n,k)$. In this section we obtain an extension of Theorem 1.2 to $d_r(n,k)$ for all $r\geq 1$. The pattern of proof follows roughly the proof of Theorem 2.4 for the case of the partition numbers $b_r(n,k)$. For the derangement numbers we work backwards to the $r=1$ case proved in Theorem 1.2 by applying Lemma 4.5. However, in the case of $d_r(n,k)$, besides the appearance of the Bernoulli numbers in the form $B_{n-r}/(n-r)$, we get a nonzero integer part $P_r(n)$ for the evaluation of the appropriately extended alternating sum. It is not hard to find this integer part experimentally, but in contrast to the case of partitions, the proof of its form takes some work that doesn't appear to be readily available in the literature. For the statement of the extension we first define $P_r(n)$ as follows.

Definition 4.1. *Let $r \geq 1$ and let $n \geq r+2$. Define*

$$P_r(n) = (-1)^r \sum_{i=0}^{r-1} i^{n-r-1}.$$

Theorem 4.2. *Let $r \geq 1$. Then for all $n \geq r+2$ we have*

$$\sum_{k=1}^{n} (-1)^k d_r(n+k,k)/((n+k-r)(n+k-r-1)\cdots(n+k-2r)) = P_r(n) + (-1)^r \frac{B_{n-r}}{n-r},$$

where $P_r(n)$ is given by Definition 4.1.

Here we see that the product $(n+k-1)(n+k-2)$ that appears in the denominator of the alternating sum of Theorem 1.2 becomes in general a falling factorial power (that is a descending product), denoted $(n+k-r)^{\underline{r+1}} = \prod_{j=0}^{r}(n+k-r-j)$, [7, p. 47].

The first step in the pattern of section 2.1 is to find a suitable recurrence for $d_r(n,k)$. A recurrence obtained by emulating a proof of the triangular recurrence (1.3)(ii) is as follows:

$$d_r(n,k) = r \cdot d_{r-1}(n-2,k-1) + (n-r-1)d_r(n-2,k-1) + (n-1)d_r(n-1,k) \qquad (4.1)$$

Indeed we have one of the following three disjoint and exhaustive possibilities. (1) Element n makes a 2–cycle with one of the elements of $x \in [r]$, where the other $(r-1)$ elements $[r] \setminus \{x\}$ are already distinguished for $(k-1)$–cycle derangements of $[n-1] \setminus \{x\}$. (2) Element n makes a 2–cycle with one of the elements $y \in [n-1] \setminus [r]$, where there are already r distinguished cycles from $(k-1)$–cycle derangements of $[n-1] \setminus \{y\}$. (3) We already have derangements of $[n-1]$ into k cycles with r distinguished cycles and we place element n in any one of the $(n-1)$ places available, one place after each element of $[n-1]$, in any such derangement. The value of (4.1) lies in the simplicity of its proof. Since $d_0(n,k)$ is the same as $d_1(n,k) = d(n,k)$ we recover (1.3)(ii) by the case $r=1$. Yet, despite its straightforward derivation, the recurrence (4.1) does not lend itself easily to developing a generating function for $d_r(n,k)$. Moreover, besides wanting a nicer recurrence that we can use to establish a generating function, we want a nicer recurrence for our proofs to follow. Surprisingly, even though the following Lemma 4.3 does yield such a recurrence, our proof of it requires some construction.

Lemma 4.3. *Let $r \geq 1$ and let $n \geq 2k \geq 2r$. Then*

$$d_r(n,k) = (n-r)d_r(n-1,k) + (n-r)d_{r-1}(n-2,k-1). \qquad (4.2)$$

Proof. Denote $\Pi(r,n,k)$ as the derangements of $[n]$ into k cycles with r distinguished cycles. We break up $\Pi(r,n,k)$ into two disjoint sets A and B as follows. Extend each derangement $\alpha \in \Pi[r,n-1,k]$ for each $y \in [r,n-1]$ to obtain a derangement $\alpha^+ \in \Pi[r,n,k]$ by inserting the element n after the element y in α. The set A consists of all these extended derangements α^+ as y ranges between r and $n-1$. In particular we never insert the element n immediately after one of the elements i with $1 \leq i \leq r-1$. We get accordingly the cardinality of the subset A as $|A| = (n-r)|\Pi(r,n-1,k)| = (n-r)d_r(n-1,k)$.

The definition of the set B proceeds in several cases. Case (B)(1): First consider $y = n$. Consider the derangements of $[n] \setminus \{r, y\} = [n] \setminus \{r, n\}$ into $(k-1)$ cycles with $(r-1)$ distinguished cycles. For each such derangement β simply extend it to a derangement $\beta^+ \in \Pi(r, n, k)$ by adding the 2–cycle (r, n) as a k–th cycle. Case (B)(2): Consider next $r+1 \le y \le n-1$. Let β be a derangement of $[n]\setminus\{y, n\}$ into $(k-1)$ cycles with $(r-1)$ distinguished cycles. Form the doubleton cycle (y, n) as a k–th cycle that we add to β to obtain a derangement β^* that is a k–cycle decomposition of $[n]$ with $(r-1)$ distinguished cycles: $\beta^* \in \Pi(r-1, n, k)$.

Now consider the main subcases (I) and (II) under case (B)(2). In case (I) the element r (where of course $r \le k \le n/2 < n$) does not belong to any of the $(r-1)$ distinguished cycles of β^*. In this case we define $\beta^+ = \beta^*$ and take this derangement β^+ to belong to B. There are no subcases under B(2)(I).

In subcase (II) of case (B)(2), the element r does belong to one of the $(r-1)$ distinguished cycles of β^*. Say r belongs to the cycle with leading entry i where $1 \le i \le r-1$. There are two further subcases under subcase (II). Subcase (II)(i): If the cycle containing both i and r has at least 3 elements, then starting from β^*, we move element n to just follow element i and simultaneously move element r to take the former place of element n, so that we end with the doubleton (r, y) in place of (y, n). The cycle containing element i as a minimal element that remains after the swap now has the element i directly preceding element n and this revised cycle still has at least three elements. This revision of β^* is thus a derangement $\beta^+ \in \Pi(r, n, k)$ that we define to be an element of B. Subcase (II)(ii). If the cycle containing both i and r in β^* is a doubleton cycle, then switch the element n of the doubleton (y, n) with the element i so as to form two doubletons (r, y) and (i, n) in the final revised derangement β^+, which we take to belong to B

We now argue that A and B are disjoint and that $A \cup B = \Pi(r, n, k)$, and that there is no overlap in the various cases for the definition of the set B.

To prove A and B are disjoint, first by n belongs to a doubleton cycle in cases (B)(1) and (B)(2)(I), we have that the derangements from these cases for the set B do not belong to the set A, simply because every derangement in A has element n in a cycle of length at least 3. In case (B)(2)(II)(i) we have that element i with $1 \le i \le r-1$ precedes element n in a cycle c of at least 3 elements, and this cannot occur for any derangement in A, again by definition of A, since a cycle of length at least 3 containing both i and n where n does not immediately succeed i is not equivalent to c. Obviously in case (B)(b)(II)(ii) the doubleton cycle (i, n) cannot occur for any derangement in A. Thus A and B are disjoint.

We next prove that $A \cup B$ exhausts all derangements in $\Pi(r, n, k)$. Let $\pi \in \Pi(r, n, k)$. Then element n belongs to one of the k cycles of π. If element n belongs to a cycle of at least 3 elements, then this is handled by set A except for the caveat that n must not directly succeed an element i with $1 \le i \le r-1$ in such a cycle; it does handle cycles of length at least 3 for minimal element r that also contain element n whether n follows r or not. If element n belongs to a 2–cycle then this is handled for all cases by the union of cases (B)(1), (B)(2)(I), and (B)(2)(II)(ii). If finally n belongs to a cycle of length at least 3 and directly succeeds an element i with $1 \le i \le r-1$, then this is covered by case (B)(2)(II)(i).

Finally we prove that there is no overlap among the cases for the construction of the set B. Cases (B)(2)(II)(i) and (B)(2)(II)(ii) are disjoint by construction. Also cases (B)(1) and (B)(2)(I) are disjoint by the fact that the doubleton cycle containing the element n is paired with element r in case B(1) while with element y with $r+1 \le y \le n-1$ in case (B)(2)(I). The cases (B)(2)(II)(ii) and (B)(2)(I) are disjoint because we have a doubleton cycle (i, n) with $1 \le i \le r-1$ in case (B)(2)(II)(ii) while we have a doubleton cycle (y, n) with $y \in [r+1, n-1]$ in case (B)(2)(I). This completes the discussion showing no overlap in the various cases for the set B. Since there is only one y–value for case (B)(1) while there are $(n-r-1)$ such y–values for case (B)(2), we obtain that the cardinality of B is $(n-r)|\Pi(r-1,n-2,k-1)| = (n-r)d_{r-1}(n-2,k-1)$. Thus the proof of the lemma is complete. □

With the Lemma 4.3 in hand we obtain a generating function formula for the r–distinguished derangement numbers in parallel to the corresponding case for partitions of Lemma 2.2.

Lemma 4.4. *Let $r \geq 0$. Then for all $r \geq 0$ we have*

$$\sum_{n\geq 0,k\geq 0} d_r(n+r,k+r)u^k\,\frac{z^n}{n!} = \left(\frac{z}{1-z}\right)^r e^{-uz}(1-z)^{-u} \tag{4.3}$$

Proof. We proceed by induction. By (1.2) the basis $r=0$ is verified. For the induction step, assume that the statement of the lemma is true for some $r \geq 0$. Denote by $f_r(u,z)$ the left side of (4.3). Then, since $f_{r+1}(u,z) = \sum_{n\geq 0,k\geq 0} d_{r+1}(n+r+1,k+r+1)u^k\,\frac{z^n}{n!}$, by Lemma 4.3 we obtain

$$f_{r+1}(u,z) = \sum_{n\geq 0,k\geq 0} n\cdot d_{r+1}(n+r,k+r+1)u^k\,\frac{z^n}{n!} + \sum_{n\geq 0,k\geq 0} n\cdot d_r(n+r-1,k+r)u^k\,\frac{z^n}{n!}$$

We change indices by $N=n-1$ in both sums. Thus

$$f_{r+1}(u,z) = \sum_{N\geq 0,k\geq 0} d_{r+1}(N+r+1,k+r+1)u^k\,\frac{z^{N+1}}{N!} + \sum_{N\geq 0,k\geq 0} d_r(N+r,k+r)u^k\,\frac{z^{N+1}}{N!}$$

Thus obtain $f_{r+1}(u,z) = zf_{r+1}(u,z) + zf_r(u,z)$. Hence $f_{r+1}(u,z) = \frac{z}{1-z}f_r(u,z)$. By the induction hypothesis we therefore have $f_{r+1}(u,z) = \left(\frac{z}{1-z}\right)^{r+1} e^{-uz}(1-z)^{-u}$, as desired. □

To motivate the remaining steps we take to prove Theorem 4.2, we first show the backwards recurrence step we will apply. To help make this step clear we introduce two notations as follows. Denote

$$\begin{aligned} S_r(n) &= \sum_{k=1}^{n}(-1)^k d_r(n+k,k)/((n+k-r)(n+k-r-1)\cdots(n+k-2r)),\\ T_r(n) &= \sum_{k=1}^{n}(-1)^k d_r(n+k,k)/((n+k-r)(n+k-r-1)\cdots(n+k-2r+1)). \end{aligned} \tag{4.4}$$

Here the only difference in the two notations is that the denominator in the sum defining $S_r(n)$ is $(n+k-r)^{\underline{r+1}}$, while the corresponding denominator of $T_r(n)$ has the last factor $(n+k-2r)$ removed, so its denominator is instead $(n+k-r)^{\underline{r}}$.

Lemma 4.5. *Let $r \geq 2$ and $n \geq r+2$. Then we have*

$$S_r(n) = T_r(n-1) - S_{r-1}(n-1). \tag{4.5}$$

Proof. Apply Lemma 4.3 to write $S_r(n)$ as

$$\sum_{k=1}^{n-1}(-1)^k\frac{n+k-r}{(n+k-r)^{\underline{r+1}}}d_r(n+k-1,k) + \sum_{k=2}^{n-1}(-1)^k\frac{n+k-r}{(n+k-r)^{\underline{r+1}}}d_{r-1}(n+k-2,k-1) \tag{4.6}$$

Call the first sum in (4.6) as I and the second sum as II. In I, by canceling the first factor $(n+k-r)$ of the denominator, and then by putting $m = n-1$, we have

$$I = \sum_{k=1}^{n-1} (-1)^k \frac{d_r(n+k-1,k)}{(n+k-r-1)^{\underline{r}}} = \sum_{k=1}^{m} (-1)^k \frac{d_r(m+k,k)}{(m+k-r)^{\underline{r}}}.$$

Also put $m = n-1$ in II and in addition change the index of summation by $j = k-1$ and denote $s = r-1$, so that $(n+k-r-1) = (m+j-s)$, $r = s+1$, and $(-1)^k = (-1)(-1)^j$. Thus

$$II = \sum_{k=2}^{n-1} (-1)^k \, \frac{d_{r-1}(n+k-2,k-1)}{(n+k-r-1)^{\underline{r}}} = (-1) \sum_{j=1}^{m} (-1)^j \frac{d_s(m+j,j)}{(m+j-s)^{\underline{s+1}}}.$$

Clearly by our notation (4.4) we have $I = T_r(m) = T_r(n-1)$ and $II = -S_s(m) = -S_{r-1}(n-1)$. Thus by $S_r(n) = I + II$ the proof is complete. □

We want to prove an evaluation of $T_r(n)$ defined by (4.4). To do this we first find a certain Stirling number representation of $T_r(n)$ by an extension of the generating function argument of Proposition 3.2. The Stirling numbers that arise naturally are the r-Stirling numbers of the first kind introduced by [2].

Definition 4.6. [2, p. 241]. *Let r, k, and $n \geq 0$. Define the r–Stirling number of the first kind, ${n \brack k}_r$, as the number of permutations of $[n]$ into k cycles such that each of $1, 2, \ldots, r$ appears in a different cycle. The ordinary Stirling number of the second kind, namely the case $r = 1$, is denoted without subscript. By convention, ${n \brack k}_0 = {n \brack k}$. Further ${n \brack k}_r = 0$, if $n < r$ or $k < r$, while ${r \brack k}_r = \delta_{k,r}$, for $r \geq 0$, and ${n \brack 0}_r = 0$, for $n > r$.*

We have the following recurrences.

Lemma 4.7. [2, Theorems 1 and 3]

$$\begin{aligned} &\textit{(triangular recurrence)} \quad {n \brack m}_r = (n-1) {n-1 \brack m}_r + {n-1 \brack m-1}_r, \; n > r. \\ &\textit{(cross recurrence)} \quad (r-1) {n \brack m}_r = {n \brack m-1}_{r-1} - {n \brack m-1}_r, \; n \geq r > 1. \end{aligned} \tag{4.7}$$

We also have the following extension of (3.6) via the r–Stirling extension of Lemma 3.1 given by [2, Thm. 7]; compare [2, Cor. 9].

$$u^r(u+rz)(u+(r+1)z)(u+(r+2)z)\cdots(u+(r+p-1)z) = \sum_{n=0}^{p} {p+r \brack p+r-n}_r z^n u^{p+r-n}. \tag{4.8}$$

Lemma 4.8. *Define $T_r(n)$ by (4.4). Let $n \geq r \geq 1$. Then we have*

$$T_r(n) = \sum_{j=r}^{n} (-1)^j {n+j-r \brack j}_r \binom{2n-2r+1}{n+j-2r+1}. \tag{4.9}$$

Proof. Define

$$f_r(u,z) = \sum_{n,k \geq r} \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r}}} \; u^k \frac{z^n}{(n+k-2r)!}. \tag{4.10}$$

Make the substitution $m = n + k$, and write $u^k z^n = (u/z)^k z^m = u^r (u/z)^{k-r} z^{m-r}$, to rewrite

$$f_r(u,z) = \sum_{m \ge k \ge r} d_r(m,k)(u/z)^k \frac{z^m}{(m-r)!} = u^r \sum_{m \ge k \ge r} d_r(m,k)(u/z)^{k-r} \frac{z^{m-r}}{(m-r)!} \tag{4.11}$$

where we have incorporated $(m-r)^{\underline{r}} \cdot (m-2r)! = (m-r)!$. Therefore by (4.11) and Lemma 4.4 we have (compare the case $r = 1$ in (3.4)):

$$f_r(u,z) = u^r z^r e^{-u} (1-z)^{-u/z-r}. \tag{4.12}$$

Develop $(1-z)^{-v-r}$ as a binomial expansion about $z = 0$ to obtain

$$(1-z)^{-v-r} = 1 + \frac{(v+r)}{1!} z + \frac{(v+r)(v+r+1)}{2!} z^2 + \frac{(v+r)(v+r+1)(v+r+2)}{3!} z^3 + \cdots.$$

Thus, after plugging in $v = u/z$ we find by (4.12) that

$$f_r(u,z) = z^r e^{-u} \cdot u^r \left(1 + \frac{(u+rz)}{1!} + \frac{(u+rz)(u+(r+1)z)}{2!} + \cdots\right). \tag{4.13}$$

Hence by (4.8) applied to (4.13), incorporating the factor z^r under the sum, we have

$$f_r(u,z) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{u^\ell}{\ell!} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{n=0}^{p} \begin{bmatrix} p+r \\ p+r-n \end{bmatrix}_r z^{n+r} u^{p+r-n}. \tag{4.14}$$

Following the development after (3.7) but now applied to (4.14) with here $k = \ell + p + r - n$ as the power of u and $j = k - \ell$ as the lower index of the r–Stirling coefficient, we have the formula

$$f_r(u,z) = \sum_{k=r}^{\infty} (-1)^k u^k \sum_{n=0}^{\infty} z^{n+r} \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n+j \\ j \end{bmatrix}_r \frac{1}{(n+j-r)!(k-j)!}; \tag{4.15}$$

compare the case $r = 1$ of (4.15) in (3.8). Now we rewrite $f_r(u,z)$ in (4.15) by replacing n by $n - r$ so as to make a power of z^n, consistent with (4.10). Correspondingly we introduce $\frac{1}{(n+j-2r)!(k-j)!} = \binom{n+k-2r}{n+j-2r} \frac{1}{(n+k-2r)!}$. We rewrite (4.15) accordingly as follows.

$$f_r(u,z) = \sum_{k=r}^{\infty} (-1)^k u^k \sum_{n=r}^{\infty} z^n \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n-r+j \\ j \end{bmatrix}_r \binom{n+k-2r}{n+j-2r} \frac{1}{(n+k-2r)!}. \tag{4.16}$$

Fix $n \ge r + 1$. We simply write, by the definition (4.10) of $f_r(u,z)$ and by (4.16), that for any $k \ge 1$,

$$(-1)^k \frac{d_r(n+k,k)}{(n+k-r)^{\underline{r}}} = (-1)^k [u^k z^n](n+k-2r)! f_r(u,z) = \sum_{j=r}^{k} (-1)^j \begin{bmatrix} n+j-r \\ j \end{bmatrix}_r \binom{n+k-2r}{n+j-2r}. \tag{4.17}$$

Finally by constructing $T_r(n)$ by its definition (4.4) as a sum over k of the left side of (4.17), where we may take k running from r to n, and interchanging the order of summation in the resulting double sum following from the right side of (4.17), the proof is complete by noting the binomial identity $\sum_{k=j}^{n} \binom{n+k-2r}{n+j-2r} = \binom{2n-2r+1}{n+j-2r+1}$. □

We will compute $T_r(n)$ by using the Stirling number representation of Lemma 4.8. We generalize slightly this representation by introducing $U_r(n,N)$ with $T_r(n) = U_r(n, 2n-(2r-1))$

by introducing a parameter N in the upper index of the binomial coefficient. We define a companion sum $V_r(n,N)$ too as follows. For all parameters r, n, and N as shown, we define

$$\begin{aligned} U_r(n,N) &= \sum_j (-1)^j \left[{n+j-r \atop j} \right]_r \binom{N}{n+j-2r+1};\ r\geq 1, n\geq r+1, N\geq 2n-2r+1 \\ V_r(n,N) &= \sum_j (-1)^j \left[{n+j-r \atop j} \right]_r \binom{N}{n+j-2r};\ r\geq 1, n\geq r, N\geq 2n-2r+1 \end{aligned} \tag{4.18}$$

Here and in the sequel, the sums defining $U_r(n,N)$ and $V_r(n,N)$ are finite and extend to all nonzero values of the summand. By (3.13)(i) we have evaluated $V_1(n,N)=0$ for all $n\geq 1$ and $N\geq 2n-1$. Also, by (3.13)(ii) we have evaluated $U_1(n,N)=0$ for all $n\geq 2$ and $N\geq 2n-1$.

Lemma 4.9. *Define $U_r(n,N)$ and $V_r(n,N)$ by (4.18). Then we have*

$$\begin{aligned} &(i)\quad V_r(n,N)=0, \quad \text{for all}\ \ r\geq 1,\ n\geq r,\ N\geq 2n-2r+1; \\ &(ii)\quad U_r(n,N)=(-1)^r(r-1)^{n-r}, \quad \text{for all}\ \ r\geq 1,\ n\geq r+1,\ N\geq 2n-2r+1. \end{aligned} \tag{4.19}$$

Additionally, for $r\geq 2$ we have $U_r(r,N)=(-1)^r$ when $N\geq 1$.

Proof. We prove statement (i) of the lemma by applying the proof of (3.13)(i) to (4.17). Indeed let $r\geq 1$ and $k\geq n+1$ in (4.17). Then $d_r(n+k,k)=0$. Hence the sum on the right side of (4.17) is zero. Now put N in place of $n+k-2r$ in the upper index of the binomial coefficient for the sum on the right side of (4.17). Then we have that this sum is zero for all $N\geq n+(n+1)-2r$. Hence by the definition of $V_r(n,N)$ in (4.18) we have verified that $V_r(n,N)=0$ under the constraints shown in (4.19)(i).

To prove statement (ii) of the lemma, we first apply the cross recurrence of (4.7) to $U_r(n,N)$ defined by (4.18). For the application of the recurrence we transpose the terms to write $\left[{n \atop m-1} \right]_r = -(r-1)\left[{n \atop m} \right]_r + \left[{n \atop m-1} \right]_{r-1}$. Hence by (4.18) and this last relation we have

$$\begin{aligned} U_r(n,N) &= -(r-1)\sum_j (-1)^j \left[{n+j-r \atop j+1} \right]_r \binom{N}{n+j-2r+1} \\ &+ \sum_j (-1)^j \left[{n+j-r \atop j} \right]_{r-1} \binom{N}{n+j-2r+1}. \end{aligned} \tag{4.20}$$

Now put $k=j+1$ and $m=n-1$ in the first sum on the right side of (4.20) and put $m=n-1$ in the second sum. Thus

$$\begin{aligned} U_r(n,N) &= +(r-1)\sum_k (-1)^k \left[{m+k-r \atop k} \right]_r \binom{N}{m+k-2r+1} \\ &+ \sum_j (-1)^j \left[{m+j-(r-1) \atop j} \right]_{r-1} \binom{N}{m+j-2r+2} = (r-1)U_r(m,N)+V_{r-1}(m,N). \end{aligned} \tag{4.21}$$

By $m=n-1$ in (4.21) we have thus shown

$$U_r(n,N) = (r-1)U_r(n-1,N)+V_{r-1}(n-1,N), \quad \text{for any}\ \ N. \tag{4.22}$$

To complete the proof of (ii), we first observe that the case $r=1$ for $U_r(n,N)$ is verified by (3.13)(ii) since the powers of $r-1=0$ are zero for $n-r\geq 1$. Thus let $r\geq 2$. Assume that $n\geq r+1$ and $N\geq 2n-2r+1$. By (4.22) we have

$$U_r(n,N) = (r-1)\cdot U_r(n-1,N)+V_{r-1}(n-1,N) = (r-1)\cdot U_r(n-1,N), \tag{4.23}$$

where we have applied $V_{r-1}(n-1,N)=0$. We can make this application by (4.19)(i) because $n-1\ge r\ge r-1$ and $N\ge 2n-2r+1=2(n-1)-2(r-1)+1$, as required. Now iterate (4.23), which is possible for the evaluation of $V_{r-1}(n',N)=0$ with $n'=n-k$ because the inequality constraint $N\ge 2(n'-1)-2(r-1)+1$ is satisfied for $k\ge 1$ and N fixed. Therefore, by iterating k times with $k=n-r$ we obtain

$$U_r(n,N)=(r-1)^{n-r}\cdot U_r(r,N). \tag{4.24}$$

Finally we compute $U_r(r,N)$ by using the definition (4.18) with $r+j-r=j$ in the top index of the r–Stirling number and $r+j-2r+1=j-r+1$ in the bottom index of the binomial coefficient. Thus we have $U_r(r,N)=\sum_{j\ge r}(-1)^j\begin{bmatrix} j\\ j\end{bmatrix}_r\binom{N}{j-r+1}=(-1)^r\sum_{k=0}^{N-1}(-1)^k\binom{N}{k+1}=(-1)^r(1-(1-1)^N)=(-1)^r$, where we made the change of index $k=j-r$. Hence, $U_r(n,N)=(-1)^r(r-1)^{n-r}$. Thus we have verified (4.19). The additional statement of the lemma has been proved by our evaluation of $U_r(r,N)=(-1)^r$ for all $r\ge 2$. Hence the lemma is proved. □

Proof of Theorem 4.2. We compute $S_r(n)$, which is defined by (4.4), and which stands as the left side of the statement of the theorem, as follows. Let $r\ge 2$ and let $n\ge r+2$. By Lemma 4.5 we have that the backward recurrence (4.5) holds, which we restate here:

$$S_r(n)=T_r(n-1)-S_{r-1}(n-1),$$

for $T_r(n)$ also defined by (4.4). By Lemma 4.8 and the definition (4.18) of $U_r(n,N)$ we have $T_r(n)=U_r(n,2n-2r+1)$. So by Lemma 4.9, putting $n-1$ in place of n in this last relation, we have $T_r(n-1)=U_r(n-1,2(n-1)-2r+1)=(-1)^r(r-1)^{n-1-r}$. Therefore in turn, recalling by Definition 4.1 that $P_r(n)=(-1)^r\sum_{i=0}^{r-1}i^{n-r-1}$, we have $T_r(n-1)=P_r(n)-(-1)P_{r-1}(n)=P_r(n)+P_{r-1}(n)$. Hence, by the backward recurrence (4.5) for $S_r(n)$, we have

$$S_r(n)=P_r(n)+P_{r-1}(n)-S_{r-1}(n-1),\ \text{ for all }\ r\ge 2,\ n\ge r+2. \tag{4.25}$$

Put $n'=n-1$ and $r'=r-1$. Since $n'=n-1\ge r+1=r'+2$, we may iterate (4.25) and thus obtain

$$\begin{aligned}
S_r(n)&=P_r(n)+P_{r-1}(n)-S_{r-1}(n-1)\\
&=P_r(n)+P_{r-1}(n)-(P_{r-1}(n-1)+P_{r-2}(n-1))+S_{r-2}(n-2)=\cdots\\
&=(-1)^{r-1}S_1(n-r+1)+\sum_{k=0}^{r-2}(-1)^k(P_{r-k}(n-k)+P_{r-k-1}(n-k))\\
&=(-1)^{r-1}S_1(n-r+1)+P_r(n)-P_1(n-r+2),
\end{aligned} \tag{4.26}$$

where we used that the sum in the second to last line is telescoping. Finally, $P_1(n-r+2)=(-1)^1 0^{n-r+2}=0$, and the definition (4.4) of $S_1(n-(r-1))$ and Theorem 1.2, we have $(-1)^{r-1}S_1(n-r+1)=(-1)^{r-1}(-B_{n-r}/(n-r))$. Therefore by (4.26) we have $S_r(n)=P_r(n)+(-1)^rB_{n-r}/(n-r)$. By the definition of $S_r(n)$, this is what we wanted to prove. □

Power sums are evaluated in terms of Bernoulli numbers in [11, (15.25)] as the following Bernoulli's formula:

$$\sum_{k=0}^{N}k^p=\frac{1}{p+1}\sum_{k=0}^{p}(-1)^k\binom{p+1}{k}N^{p+1-k}B_k,\quad p\ge 1, \tag{4.27}$$

where we have the definition $\sum_{n=0}^{\infty}\frac{x^n}{n!}B_n=\frac{x}{e^x-1}$ of the Bernoulli numbers B_n. Therefore by the statement of Theorem 4.2, and by taking $p=n-r-1$ and $N=r-1$ in (4.27) to represent $P_r(n)$ of Definition 4.1, and because only when n and r have the same parity with $n-r\ge 2$

do we have $B_{n-r} \neq 0$, we obtain the following by adding a last term $k = p+1 = n-r$ to the Bernoulli number sum of (4.27).

Corollary 4.10. *Let $r \geq 1$. Then for all $n \geq r+2$ we have*

$$\sum_{k=1}^{n}(-1)^k d_r(n+k,k)/(n+k-r)^{\underline{r+1}} = \frac{(-1)^r}{n-r}\sum_{k=0}^{n-r}(-1)^k \binom{n-r}{k}(r-1)^{n-r-k}B_k,$$

where in case $r=1$ we interpret the sum on the right as $(-1)^{n-r}\ 0^0 B_{n-1} = (-1)^{n-r}B_{n-1}$.

References

[1] C. M. Bender, D. C. Brody, and B. K. Meister, *Bernoulli-like polynomials associated with Stirling Numbers*, arXiv:math-ph/0509008v1.
[2] A. Z. Broder, The r–Stirling numbers, *Discrete Mathematics* 49 (1984), 241–259.
[3] D. Callan, T. Mansour, and M. Shattuck, Some identities for derangement and Ward number sequences and related bijections, *Pure Math. Appl. (PU. M. A.)* 25 (2) (2015), 132–143.
[4] L. Comtet, *Advanced Combinatorics*, Reidel (1974).
[5] H. W. Gould, H. Kwong, and J. Quaintance, On certain sums of Stirling numbers with binomial coefficients, *J. Integer Sequences* 18 (2015), Article 15.9.6.
[6] H. W. Gould, Stirling Number representation problems, *Proc. Amer. Math. Soc.* 11 (1960), 447–451.
[7] R. L. Graham, D. R. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd Edition, New York: Addison–Wesley (1994).
[8] T. Howard, Associated Stirling numbers, *Fibonacci Quarterly* (1980), 303–315.
[9] C. Jordan, *Calculus of Finite Differences*, New York (1957).
[10] A q–analogue of Schlaäfli and Gould identities on Stirling numbers, *Ramanujan J.* 46 (2018), 483?507.
[11] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific: Singapore (2016).
[12] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley (1958).
[13] N. J. A. Sloane, *On–Line Encyclopedia of Integer Sequences*, at `http://oeis.org`.
[14] Wolfram *Mathematica*, http://www.wolfram.com/mathematica/
[15] F–Z. Zhao, Some properties of associated Stirling numbers, *J. Integer Sequences* 11 (2008), Article 08.1.7.

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