GAMBLER'S RUIN WITH RANDOM STOPPING

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ABSTRACT. Let $\{\mathbf{X}_j, \quad j \geq 0\}$ denote a Markov process on $[-N-1, N+1] \cup \{\mathbf{c}\}$. Suppose $\mathbb{P}(\mathbf{X}_{j+1} = m+1 | \mathbf{X}_j = m) = ph$, $\mathbb{P}(\mathbf{X}_{j+1} = m-1 | \mathbf{X}_j = m) = (1-p)h$, all $j \geq 1$ and $|m| \leq N$, where $p = \frac{1}{2} + \frac{b}{N}$ and $h = 1 - c_N$ for $c_N = \frac{1}{2} a^2/N^2$. Define $\mathbb{P}(\mathbf{X}_{j+1} = \mathbf{c} | \mathbf{X}_j = m) = c_N$, $j \geq 0$, $|m| \leq N$. $\{\mathbf{X}_j\}$ terminates at the first j such that $\mathbf{X}_j \in \{-N-1, N+1, \mathbf{c}\}$. Let $\mathcal{L} = \max\{j \geq 0 : \mathbf{X}_j = 0\}$. On $\Omega^\circ = \{\mathbf{X}_j$ terminates at $\mathbf{c}\}$, denote by $\mathcal{R}^\circ$ and $\mathcal{L}^\circ$ respectively, as the numbers of runs and steps from $\mathcal{L}$ until termination. Denote $\Delta^\circ = \mathcal{L}^\circ - 2\mathcal{R}^\circ$. Then $\lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\Delta^\circ} | \ \Omega^\circ\} = C_{a,b} \frac{\sqrt{c^2+t^2}\left(\cosh\sqrt{c^2+t^2} - \cosh(2b)\right)}{(a^2+t^2)\sinh\sqrt{c^2+t^2}}$, where $c^2 = a^2 + 4b^2$.

1. Introduction

We introduce a model of gambler's ruin as follows. Let $\{\mathbf{X}_j, j \geq 0\}$ denote a Markov process on fortunes $\mathbb{Z} \cap [-N-1, N+1]$ together with an abstract *stopping state* $\mathbf{c}$, started from $\mathbf{X}_0 = 0$. Suppose that $\mathbb{P}(\mathbf{X}_{j+1} = m+1 | \mathbf{X}_j = m) = ph$ and $\mathbb{P}(\mathbf{X}_{j+1} = m-1 | \mathbf{X}_j = m) = (1-p)h$, all $j \geq 1$ and all $|m| \leq N$, where $0 < p < 1$. We define $\mathbb{P}(\mathbf{X}_{j+1} = \mathbf{c} | \mathbf{X}_j = m) = c_N$, independent of $j \geq 0$ and $|m| \leq N$, where $h = h_N = 1 - c_N$ for $c_N = \frac{1}{2} a^2/N^2$, with a *stopping parameter* $a > 0$. Thus at each *epoch* j the Markov chain either steps up one unit with probability ph, steps down one unit with probability $(1-p)h$, or transitions to $\mathbf{c}$ with small probability c_N. The process $\{\mathbf{X}_j\}$ terminates at the first epoch j such that $\mathbf{X}_j \in \{-N-1, N+1, \mathbf{c}\}$.

The value $\mathbf{X}_j$ is a model of a bettor's fortune at epoch j in gambling with one unit bets per play such that at any play for which the fortune is in $[-N, N]$, there is a random stoppage of play at discrete rate $\frac{1}{2} a^2/N^2$. In case $a = 0$ so that $h = 1$, we have a classical gambler's ruin model with no random stopping and *boundaries* $\pm(N+1)$. We define the parameter $\xi = p(1-p)h^2$ which in the classical case $h = 1$ is simply the variance of the step variable taking values ± 1.

In the symmetric case $p = \frac{1}{2}$ it is natural to identify $\mathbf{c}$ with fortune $m = 0$ in case the process comes to $\mathbf{c}$ before one of the boundaries $\pm(N+1)$, and thereby continue the process until one of these boundaries is reached. With this modification the extended process $\{(j, \widetilde{\mathbf{X}}_j)\}$ produces a *lattice path* up to a random termination epoch, though not of nearest neighbor type. Indeed we are inspired by the work [1] that treats nonnegative lattice paths, with no height restriction and on a fixed time scale, for which transition to the stopping state $\mathbf{c}$ corresponds there to a *catastrophe*, namely a return of the lattice path to the j–axis in one unit of time. The catastrophe is itself motivated by several contexts in [1], including a discrete time queuing model, which allows for the queue to reset to zero length on events with small constant probability. The study of catastrophes in Markov chains has a long history stemming from mathematical population biology; see [4] and the references therein, and also [2] and their references. These references study in particular a classical birth and death process for a population in the nonnegative integers with catastrophic deaths modeled by a transition at population i according to a binomial distribution with values in $[0, i]$, conditional on death; [2, section 1.1]. Obviously our model considers the catastrophe only as a complete collapse with subsequent transition to a cemetery state. In the symmetric case, by considering the absolute value process $\{|\widetilde{\mathbf{X}}_j|\}$ as a model for queuing, with $|\widetilde{\mathbf{X}}_j|$ being the length

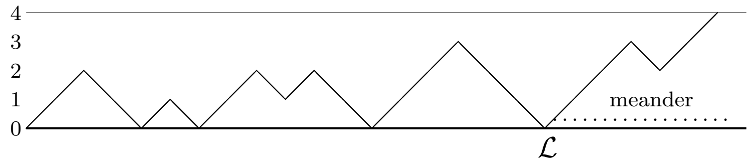

FIGURE 1. Last visit $\mathcal{L}$ and meander for $\{|\mathbf{X}_j|\}$; $N = 3$.

of the queue at epoch j, we have a nonnegative Markov chain with reflection at the j–axis. This model is easily handled by the Markov property and our Theorem 1.1. Note that our model in general allows a transition to the state $\mathbf{c}$ at the very first step, but this occurs with a negligible probability (order $O(1/N^2)$), and it turns out we can safely ignore this possibility.

To further state our motivations we need some definitions. A *nearest neighbor lattice path* is a sequence of vertices $\{(j, x_j), j = 0, \ldots, k\}$ in $\mathbb{Z}^2$ such that $|x_{j+1} - x_j| = 1$, $j = 0, \ldots, k-1$, for some $k \geq 1$. We call x_j the *level* of the path at epoch j. An excursion path satisfies the additional requirements that $x_0 = x_k = 0$ (so k is even), and $x_j \neq 0$, for all $0 < j < k$. A nonnegative excursion is an excursion that lies above the j-axis save for its endpoints. A *run* of a nearest neighbor lattice path is either an ascending incline or a descending incline of maximal length along the linearly interpolated path. Thus the number of runs of a nearest neighbor lattice path is one more than the number of *turns* of the path, where a turn simply corresponds to a change in direction: ascent to descent or vice versa. The number of *steps* of a lattice path is simply the length of the discrete time interval, or k, in the above nearest neighbor description. In Figure 1, the lattice path shown has 26 steps and 13 runs.

An earlier work for gambler's ruin with boundaries, [11], calculates the probability of ruin on a infinite time scale with both catastrophes and windfalls, and with constant probabilities of each of these events besides constant win and loss probabilities on each play, by utilizing a difference equation. Our main aim is to handle random stopping as well as the runs statistic of the gambler's ruin path that so far has not been handled by the difference equation method. Even without random stopping, our results involving the runs statistic have so far relied on generalized Fibonacci recurrences to compute probability generating functions for them, [13, 14]. We shall see that the random stopping gives rise to Fibonacci recurrences with an extra (driving) term that is not observed for the nearest neighbor only models; see Lemmas 3.4 and 3.8. A nice feature of our approach is that joint probability generating functions of runs and steps take on explicit closed forms, and limit distributions follow as the parameter N tends to infinity. While we will develop explicit generating function formulae for general p, in our application to Theorem 1.1 we will consider the *asymptotically symmetric* case $p = \frac{1}{2} + \frac{b}{N}$, for a constant b. Relative to the symmetric case, for certain problems such as Theorem 1.1(b), the presence of b produces a nonlinear effect on a limit law, while in other problems the parameter b simply increases the effective size of a. The nonlinear effect comes about for statistics along paths that are interrupted by random stopping.

The analogue of our model in continuous time is a Brownian motion with both drift and exponential killing, c.f. [8]. For our Markov chain there is a geometric stopping time with mean $\frac{2N^2}{a^2}$, and an exit boundary with order N scaling. So with time scaled by N^2 in the asymptotically symmetric case we have in the limit a Brownian process in $\mathbb{R}$ started from $x = 0$ with drift parameter $\mu = 2b$, exponential killing mean $\frac{2}{a^2}$, and termination after the killing time or exit from $[-1, 1]$, whichever comes first. We comment further on the Brownian motion limit with drift in Remark 1.5.

We introduce some definitions. Denote by $\mathcal{L} = \mathcal{L}_N$ the epoch of *last visit* to the fortune $m = 0$, namely

$$\mathcal{L} = \max\{j \geq 0 : \mathbf{X}_j = 0\}. \tag{1.1}$$

In Figure 1 the last visit is depicted relative to the absolute value process $\{|\mathbf{X}_j|\}$ and for the case when this process terminates at the boundary $N + 1 = 4$. We define the *meander* as the part of the absolute value path from the epoch of last visit until the process terminates, where we may call $|\mathbf{c}| = \mathbf{c}$. Define also

$$\mathbf{L} = \inf\{j \geq 1 : \mathbf{X}_j = 0 \text{ or } \mathbf{X}_j \in \{-(N+1), N+1, \mathbf{c}\}\ \}. \tag{1.2}$$

Thus $\mathbf{L}$ is either the first time until a return to fortune $m = 0$, namely an excursion time, or the termination time, whichever is smallest. Define $E_0 = \{\mathbf{X}_\mathbf{L} = 0\}$; on E_0 the process $\{\mathbf{X}_j\}$ makes an excursion from $m = 0$ back to $m = 0$ for a first time without terminating. Thus on E_0 we have that $\mathbf{L}$ is the number of steps in the excursion. Also on E_0, define $\mathbf{R}$ as the number of runs in the excursion of the absolute value process path $\{(j, |\mathbf{X}_j|), j = 0, 1, \ldots, \mathbf{L})\}$.

Define the event $E_\mathbf{c} = \{\mathbf{X}_\mathbf{L} = \mathbf{c}\}$; on $E_\mathbf{c}$ the stopping state $\mathbf{c}$ is reached on the first attempt at excursion. On $E_\mathbf{c}$ we say we have a *stopped excursion*, that reaches $\mathbf{c}$ before ever returning to $m = 0$ or to a boundary. Finally, denote E_N as the event that on the first attempt at excursion the chain terminates at one of the boundaries $\pm(N + 1)$. Thus E_0, $E_\mathbf{c}$, and E_N are mutually disjoint and exhaustive.

On E_0, define $\mathbf{H} = \max\{|\mathbf{X}_j|, j = 0, 1, \ldots, \mathbf{L}\}$, while on $E_\mathbf{c}$, define $\mathbf{H} = \max\{|\mathbf{X}_j|, j = 0, 1, \ldots, \mathbf{L}-1\}$. So $\mathbf{H}$ denotes the *height* of an excursion on E_0, while on $E_\mathbf{c}$ it represents the maximum level of that portion of the absolute value process up until one step before reaching $\mathbf{c}$. On E_N, define $\mathbf{H} = N + 1$.

Note that E_N and $E_\mathbf{c}$ are both unusual events, each with probability of order $O(1/N)$, as we shall see. That is because it is difficult to exit the large region $[-N, N] \cap \mathbb{Z}$ before coming back to $m = 0$, or else to make a long stopped excursion (of order $O(N^2)$ steps) to the stopping state $\mathbf{c}$. Define now Ω° as the event that the process $\{\mathbf{X}_j\}$ terminates at state $\mathbf{c}$. Define Ω' as the complement of Ω°, that is the event that the process $\{\mathbf{X}_j\}$ terminates at one of the boundaries.

Denote by $\mathcal{M} = \mathcal{M}_N$ the number of consecutive bona fide excursions of the absolute value process $\{|\mathbf{X}_j|\}$, each of height at most N, until the last visit $\mathcal{L}$. Let $\mathbf{R}_\nu$ be the number of runs in the ν^{th} excursion of the absolute value process path $\{(j, |\mathbf{X}_j|)\}$. Define $\mathcal{R}_N = \sum_{\nu=1}^{\mathcal{M}} \mathbf{R}_\nu$. Here, if $\mathcal{M}_N = 0$, then the empty sum is taken as zero. Equivalently $\mathcal{R}_N$ is the total number of runs of the last visit portion of the absolute value path.

On the event Ω° that the process terminates at state $\mathbf{c}$, denote by $\mathcal{R}_N^\circ$ and $\mathcal{L}_N^\circ$ as the numbers of runs and steps of the absolute value process $\{(j, |\mathbf{X}_j|)\}$ starting from epoch $\mathcal{L}$ to the stopping state. For convenience, we shall count a new run on a final step to $\mathbf{c}$ if the step just before this final transition is away from the j–axis. Thus we may think of the final transition to $\mathbf{c}$ as in the direction of $m = 0$, but with a single long step, which we count as a single step. So, if the final nearest neighbor lattice path step, just before the actual final step to state $\mathbf{c}$, is toward the j-axis, then the actual final step may be regarded as a run continuation towards the j–axis, and therefore we count exactly one run on this continued incline toward the j–axis. On the complementary event Ω', denote by $\mathcal{R}'_N$ and $\mathcal{L}'_N$ as the numbers of runs and steps of the absolute value process $\{(j, |\mathbf{X}_j|)\}$ from epoch $\mathcal{L}$ to the terminal epoch.

We define an integer valued statistic for the last visit portion of gambler's ruin by $\Delta_N = \mathcal{L}_N - 2\mathcal{R}_N + \mathcal{M}_N$. We also define $\Delta_N^\circ = \mathcal{L}_N^\circ - 2\mathcal{R}_N^\circ$, and $\Delta'_N = \mathcal{L}'_N - 2\mathcal{R}'_N$. Define the overall process statistic $\boldsymbol{\Delta}_N = \Delta_N + \Delta_N^\circ \cdot 1_{\Omega^\circ} + \Delta'_N \cdot 1_{\Omega'}$. If $a = 0$, we take $\Delta_N^\circ \cdot 1_{\Omega^\circ} = 0$ and there is no content to

Theorem 1.1(b). In the asymptotically symmetric case we obtain limit laws after normalizing the various Δ–statistics by N, as follows.

Theorem 1.1. *Let $p = \frac{1}{2} + \frac{b}{N}$ for some constant b. Denote $c^2 = a^2 + 4b^2$. Let $t \in \mathbb{R}$. Then we have the following limiting characteristic functions:*

$$
\begin{aligned}
&\text{(a)}\ \lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\Delta_N}\} = \frac{c}{\tanh(c)} \frac{\tanh\sqrt{c^2+t^2}}{\sqrt{c^2+t^2}} \\
&\text{(b)}\ \lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\Delta_N^\circ} \mid \Omega^\circ\} = C_{a,b} \frac{\sqrt{c^2+t^2}\left(\cosh\sqrt{c^2+t^2} - \cosh(2b)\right)}{(a^2+t^2)\sinh\sqrt{c^2+t^2}} \\
&\text{(c)}\ \lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\Delta'_N} \mid \Omega'\} = \frac{\sinh(c)}{c} \frac{\sqrt{c^2+t^2}}{\sinh\sqrt{c^2+t^2}} \\
&\text{(d)}\ \lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\boldsymbol{\Delta}_N}\} = \frac{a^2\cosh\sqrt{c^2+t^2} + t^2\cosh(2b)}{(a^2+t^2)\cosh\sqrt{c^2+t^2}}
\end{aligned}
\tag{1.3}
$$

where $C_{a,b} = \frac{a^2\sinh(c)}{c(\cosh(c)-\cosh(2b))}$.

Remark 1.2. *In the symmetric case $b = 0$ and $a \neq 0$, the limiting joint characteristic function in Theorem 1.1(b) reduces to $\frac{1}{2}C_{a,0}\frac{\tanh\frac{1}{2}\sqrt{a^2+t^2}}{\sqrt{a^2+t^2}}$ due to the half angle identity $\frac{\cosh(\varphi)-1}{\sinh\varphi} = \tanh\frac{\varphi}{2}$. By* (4.36) *the events Ω° and Ω' are macrosopic, with $\lim_{N\to\infty}\mathbb{P}(\Omega') = \frac{\cosh(2b)}{\cosh(c)}$.*

Remark 1.3. *The probability density of the measure determined by the limiting conditional characteristic function given Ω° for Theorem 1.1(b) may be found via the Mittag–Leffler partial fraction expansions of $\frac{\tanh(u)}{u}$ and $\frac{u}{\sinh(u)}$. This is shown in Example 4.2. The inversion of the limiting characteristic function in the same context for part (a) is included in this example. The densities of Example 4.2 for parts (a)–(b) are not bounded, as illustrated in Figure 4. This method for inverting a limiting characteristic function via partial fraction expansion and term–wise inversion isn't tractable for the corresponding cases of Theorem 1.1(c)–(d). Instead, we may apply the residue theorem directly to implement Fourier inversion and obtain a bounded density for each of (c) and (d) since in these cases the limiting characteristic function is integrable.*

By the proof of Theorem 1.1 we obtain explicit limiting univariate Laplace transforms for the statistics involving runs alone or steps alone after scaling by N^2 as follows. Corollary 1.4 extends [13, Cor. 1], which handles the classical fair gambler's ruin model $c = 0$.

Corollary 1.4. *Let $p = \frac{1}{2} + \frac{b}{N}$ and denote $c^2 = a^2 + 4b^2$. For all $\lambda \geq 0$ there hold:*

$$
\begin{aligned}
&\text{(a)}\ \lim_{N\to\infty} \mathbb{E}\{e^{-\lambda\mathcal{R}_N/N^2}\} = \frac{c}{\tanh(c)} \frac{\tanh(\sqrt{c^2+\lambda})}{\sqrt{c^2+\lambda}}, \\
&\text{(b)}\ \lim_{N\to\infty} \mathbb{E}\{e^{-\lambda\mathcal{R}_N^\circ/N^2} \mid \Omega^\circ\} = C_{a,b} \frac{\sqrt{c^2+\lambda}\left(\cosh\sqrt{c^2+\lambda} - \cosh(2b)\right)}{(a^2+\lambda)\sinh\sqrt{c^2+\lambda}} \\
&\text{(c)}\ \lim_{N\to\infty} \mathbb{E}\{e^{-\lambda\mathcal{R}'_N/N^2} \mid \Omega'\} = \frac{\sinh(c)}{c} \frac{\sqrt{c^2+\lambda}}{\sinh(\sqrt{c^2+\lambda})}, \\
&\text{(d)}\ \lim_{N\to\infty} \mathbb{E}\{e^{-\lambda\left(\mathcal{R}_N + \mathcal{R}_N^\circ\cdot 1_{\Omega^\circ} + \mathcal{R}'_N\cdot 1_{\Omega'}\right)/N^2}\} = \frac{a^2\cosh\sqrt{c^2+\lambda} + \lambda\cosh(2b)}{(a^2+\lambda)\cosh\sqrt{c^2+\lambda}}.
\end{aligned}
\tag{1.4}
$$

Moreover the same transforms hold in parts (a)–(d) with $\frac{1}{2}\mathcal{L}_N$, $\frac{1}{2}\mathcal{L}_N^\circ$, and $\frac{1}{2}\mathcal{L}'_N$, respectively, in place of $\mathcal{R}_N$, $\mathcal{R}_N^\circ$, and $\mathcal{R}'_N$.

Remark 1.5. *In the context of Corollary 1.4, if there is no random stopping* ($a = 0$)*, we have* $\mathcal{R}^\circ 1_{\Omega^\circ} = \mathcal{L}^\circ 1_{\Omega^\circ} = 0$*, and part (b) has no content. In this case, if* $\mu = 2b$ *is the drift parameter, then the limiting Laplace transform of part (d), that is* $\frac{\cosh \mu}{\cosh \sqrt{\mu^2+\lambda}}$*, is established for the time until Brownian motion with drift started from* $x = 0$ *exits the interval* $[-1, 1]$ *in* [6, Example (c), pp. 631–633]. *The Laplace transform of Corollary 1.4(d) with geometric stopping included* ($a \neq 0$) *should correspond to the Laplace transform of the distribution of the termination time of an exponentially killed Brownian motion with drift, where termination occurs when either the process exits the interval* $[-1, 1]$ *or is killed, whichever comes first.*

Some of the key results of the paper have not been stated in this introduction due to several definitions that must be given. So, we now outline the structure of the proofs. In section 2, we introduce the Fibonacci recurrences (2.7) which are employed from subsection 2.2 onward under the parameters (2.17) that carry the generating function variables r and z in various definitions, including the upward first passage joint generating function g_n of definition (2.3). Our development of the Fibonacci method for the last visit portion of Theorem 1.1(a) culminates in Proposition 2.6, which generalizes [13, Theorem 1], after the generating function definitions (2.1). The subsection 2.1 recovers the first step of the method used to prove Proposition 2.6 by employing a lattice path reformulation of the original approach of [13] to establish the recurrence for g_n of Lemma 2.2. Starting from subsection 2.2, the outline of the remaining steps leading to Proposition 2.6 is used as a guide to construct a trail of proof for section 3. To see that already in section 2 the stopping parameter comes into play, consider the probabilities (2.4) for upward and downward first passages that are derived by induction based on the recurrence (2.20) in Lemma 2.5. Even in the symmetric case $b = 0$ these would be classical gambler's ruin probabilities, conditioned by the event $\mathbf{L}_n < \infty$ as defined by (2.2), are written in terms of the reciprocals of the Fibonacci polynomials $w_n(\xi, 1) = w_n[\mathbf{1}]$ after (2.7), with parameter $\xi = \frac{h^2}{4} \neq \frac{1}{4}$, as long as $a \neq 0$; the classical gambler's ruin probabilities are recovered in the case $a = b = 0$.

In section 3 we show that the fundamental approach of section 2 can be extended for the definitions (3.1) of G_n° and K_N° that are analogues for G_n and K_N of (2.1), while still maintaining explicit generating function formulae. Specifically, we develop extended Fibonacci recurrences each with a driving term to compute the joint probability generating function of runs and steps over stopped excursions first for the analogue g_n° of g_n defined by (3.6), in a parallel construction to (2.24), as follows. To establish the building block $g_{n,k}^\circ$ of g_n° defined by (3.2)–(3.3), we prove Lemma 3.1 wherein we make use of the Markov property. Then via Lemma 3.4 we prove the formula for g_n° of Proposition 3.5 in terms of the Fibonacci sequence u_n of (3.17). Finally, the Lemma 3.8 shows a companion formula (3.27) involving u_n and the Fibonacci sequence v_n of Definition 3.7. This companion relation plays the same role as the original companion formula (2.30) motivated by the theory of continued fractions, and (3.27) is used to establish the key Proposition 3.10. Lemma 3.9 allows us to make Proposition 3.10 explicit as a closed formula via the closed solutions (2.9).

In section 4 we prove Theorem 1.1 and Corollary 1.4 by using the explicit formula provided by Propositions 2.6 and 3.10 after (2.9) and Lemma 3.9. The method is to use the substitution (4.4) which leads via trigonometric manipulations including Lemma 4.1, together with asymptotic analysis, to trigonometric formulae for the characteristic functions of Theorem 1.1 and the Laplace transforms of Corollary 1.4.

2. Method of Proof for Proposition 2.6

Our goal in this section is to prove Proposition 2.6, that is a formula for the generating function K_N of 2.1. Our basic strategy for the proofs is to decompose generating functions according to the events $\{\mathbf{H} = n\}$, $1 \leq n \leq N$. Our method is based on calculation of the following two joint probability generating functions in turn. Define

$$\begin{aligned} G_n &= G_n(r,z) = \mathbb{E}\left(r^{\mathbf{R}} z^{\mathbf{L}} \mid E_0 \cap \{\mathbf{H}=n\}\right), \ 1 \le n \le N; \\ K_N &= K_N(r,z) = \mathbb{E}\left(r^{\mathbf{R}} z^{\mathbf{L}} \mid E_0 \cap \{1 \le \mathbf{H} \le N\}\right). \end{aligned} \tag{2.1}$$

At first we focus on the calculation of G_n by means of a nonnegative first passage joint generating function g_n defined by (2.3). The method we follow was first established in [13]. A key feature of the method even without the presence of random stopping is to find an explicit formula for the generating function g_n. The formula we seek for g_n is given by Proposition 2.3. The condition that the process exits via the boundaries $\pm(N+1)$ during an attempted excursion, and thus the proof of Theorem 1.1(c), is straightforward to handle via Proposition 2.3.

For $1 \le n \le N$, on $\{\mathbf{H} \ge n\}$, define the *first passage* number of steps $\mathbf{L}_n$ along the absolute value process $\{|\mathbf{X}_j|, j \ge 0\}$ to reach level n by:

$$\mathbf{L}_n = \inf\{j \ge 1 : |\mathbf{X}_j| = n\}. \tag{2.2}$$

We define this first passage number $\mathbf{L}_n = +\infty$ on the event $\{\mathbf{H} < n\}$; thus $\{\mathbf{L}_n < \infty\} = \{\mathbf{H} \ge n\}$. On E_0, denote by $\mathbf{R}_n$ the number of runs along the absolute value path $\{(j, |\mathbf{X}_j|), j = 0, 1, \ldots, \mathbf{L}_n\}$. Define g_n as the following *upward* conditional joint probability generating function for $\mathbf{R}_n$ and $\mathbf{L}_n$ given the following condition: the path is a first passage path that starts at 0 and stays at or above level $m = 0$ until it first reaches level n. Thus for all $n \ge 1$ we define

$$g_n = \mathbb{E}\{r^{\mathbf{R}_n} z^{\mathbf{L}_n} | \mathbf{L}_n < \infty;\ \mathbf{X}_0 = 0;\ \mathbf{X}_j \ge 0,\ j = 0, \ldots, \mathbf{L}_n\}. \tag{2.3}$$

Since any nonnegative nearest neighbor lattice path that starts at level $m = 0$ and ends when it first reaches level n may be reflected to give a nonpositive path starting from $m = 0$ and reaching level $-n$ for a first time, and since there is a constant conversion factor $((1-p)/p)^n$ to find the probability of the reflected path from the probability of the original nonnegative path, if we condition instead on a nonpositive path in (2.3) then we obtain the same generating function g_n, that we may refer to as a *downward* first passage generating function.

We introduce a terminology. Let $m \le n-2$. Call a path that starts at level m that stays at levels in $[m, n]$ until level n is reached for a first time as an *upward first passage path from level m to level n*. For brevity, we also refer to such paths as making an *upward transition*. Similarly, call a path that starts at level n that stays at levels in $[m, n]$ until level m is reached for a first time as a *downward first passage path from level n to level m*, or *downward transition*. See Figure 2 for one such downward first passage path from $n = 5$ to $m = 0$.

For all $n \ge 1$ we define

$$\begin{aligned} \rho_n &= \mathbb{P}(\mathbf{L}_n < \infty;\ \mathbf{X}_0 = 0;\ \mathbf{X}_j \ge 0,\ j = 1, \ldots, \mathbf{L}_n); \\ \rho_{-n} &= \mathbb{P}(\mathbf{L}_n < \infty;\ \mathbf{X}_0 = 0;\ \mathbf{X}_j \le 0,\ j = 1, \ldots, \mathbf{L}_n). \end{aligned} \tag{2.4}$$

Define also $\rho_0 = 1$. In case $a = 0$, so that $h = 1$ and transition to $\mathbf{c}$ cannot occur, and if further we have the symmetric case $p = \frac{1}{2}$, then ρ_n is determined by the classical solution for a fair gambler's ruin started from $m = 0$ to come to the boundary of the interval $[-1, n]$ at state n, namely $\rho_n = 1/(n+1)$. For $h < 1$, even in the symmetric case we no longer have the reciprocal of a linear term in n as we shall find in Lemma 2.5. That lemma gives for example $\rho_2 = (ph)^2/(1-\xi)$, $\rho_3 = (ph)^3/(1-2\xi)$, and $\rho_4 = (ph)^4/(1-3\xi+\xi^2)$. Here we derive

$$\rho_2 = (ph)^2 \sum_{k=0}^{\infty} \left(p(1-p)h^2\right)^k = (ph)^2/(1-\xi). \tag{2.5}$$

Since the probability of a single nonnegative path for the event defining ρ_n is simply multiplied by the conversion factor $((1-p)/p)^n$ to obtain the probability of the reflected nonpositive path for the event under ρ_{-n}, we have that

$$\rho_{-n} = ((1-p)/p)^n \rho_n. \tag{2.6}$$

The calculation of g_n is fundamental to our method. Our proofs of this step feature bivariate generalized Fibonacci polynomials $\{q_n(x,\beta)\}$ and $\{w_n(x,\beta)\}$, defined as follows.

Definition 2.1. *Let $\beta, x \in \mathbb{C}$. Define sequences $q_n(x,\beta)$ and $w_n(x,\beta)$ generated by the following recurrence relations, valid for all $n \geq 1$.*

$$q_{n+1} = \beta q_n - x q_{n-1},\ q_0 = 0, q_1 = 1; \quad w_{n+1} = \beta w_n - x w_{n-1},\ w_0 = 1, w_1 = 1. \tag{2.7}$$

The polynomials $q_n(x,\beta)$ generalize the univariate Fibonacci polynomials $q_n(x,1)$, [10, p. 327]; also for the special case $\beta = 1$ we have $w_n(x,1) = q_{n+1}(x,1)$. We have $\{w_n(x,1)\} = \{1, 1, 1-x, 1-2x, 1-3x+x^2, \dots\}$; the numerical Fibonacci sequence arises with $x = -1$. We write an *interlacing* property of any two term recurrence $v_{n+1} = \beta v_n - x v_{n-1}$, $n \geq 1$, with coefficients β and x independent of n:

$$v_{n+1}v_{n-1} - v_n^2 = \beta^{-1}x^{n-1}(v_3 v_0 - v_2 v_1),\ \beta \neq 0; \tag{2.8}$$

see [13, (2.7)–(2.8)]. By standard generating function techniques the fundamental sequences (2.7) have closed formulae given by (2.9); see [15, (2.1) and (2.3)], or [13, (2.11)–(2.12)]. Define $\alpha = \sqrt{\beta^2 - 4x}$. Then for all $n \geq 1$ we have

$$q_n(x,\beta) = \tfrac{2^{-n}}{\alpha}\left((\beta+\alpha)^n - (\beta-\alpha)^n\right); \quad w_n(x,\beta) = q_n(x,\beta) - x q_{n-1}(x,\beta). \tag{2.9}$$

The second identity holds by the fact that $q_1 - xq_0 = 1 = w_1$, $q_2 - xq_1 = \beta - x = w_2$, and q_n and w_n satisfy the same two term recurrence, (2.7). Moreover the following identities hold for all $n \geq 1$.

$$\begin{aligned} &\text{(i)}\ \ w_{n+1}w_{n-1} - w_n^2 = x^{n-1}(\beta - x - 1);\\ &\text{(ii)}\ \ q_{n+1}q_{n-1} - q_n^2 = -x^{n-1}; \quad \text{(iii)}\ \ q_n w_{n+1} - w_n q_{n+1} = -x^n. \end{aligned} \tag{2.10}$$

Indeed, (2.10)(i)–(ii) follow directly from (2.7) and (2.8). By $w_n = q_n - xq_{n-1}$, $n \geq 1$, we obtain $q_n w_{n+1} - w_n q_{n+1} = x(q_{n+1}q_{n-1} - q_n^2)$, so (2.10)(iii) follows from (ii); see [13, Lemma 4].

Let γ be a nearest neighbor lattice path. Denote by $\mu(\gamma)$ the product of the probabilities ph for a step up and $(1-p)h$ for a step down all along the lattice path. Let $R(\gamma)$ and $L(\gamma)$ denote respectively the number of runs and steps along γ. For any unnormalized generating function of path statistics $f = \sum \mu(\gamma) r^{R(\gamma)} z^{L(\gamma)}$ over some collection of paths γ, we may refer to a (conditional) probability generating function by normalizing the sum f by $\sum \mu(\gamma)$. For this purpose we denote $\mathbf{1} = (1,1)$ and evaluation of any expression $f(r,z)$ at $(r,z) = \mathbf{1}$ by $f[\mathbf{1}]$. Then we form a probability generating function by normalization to the form $f/f[\mathbf{1}]$.

2.1. **Recurrence for g_n.** In this subsection we prove Lemma 2.2, that is a recurrence relation for g_n defined by (2.3). Here and in the sequel, *const.* denotes a generic constant that may depend on n but is independent of the generating function variables.

Lemma 2.2. *Define for any $j \geq 2$ the factor*

$$\lambda_j = \frac{1}{1 - \rho_j \rho_{-j} g_j^2}. \tag{2.11}$$

Then for all $n \geq 2$ we have

$$g_{n+1} = \textit{const.}\ g_n^2 \lambda_n / g_{n-1}. \tag{2.12}$$

Proof. It is convenient to focus on a downward path decomposition for g_n with some $n \geq 3$. We introduce the following notation. Let U or D stand for one step up or down, respectively, in a lattice path, and let $(UD)^k$ be shorthand for $UDUD\cdots$ with k repetitions of the pattern UD for

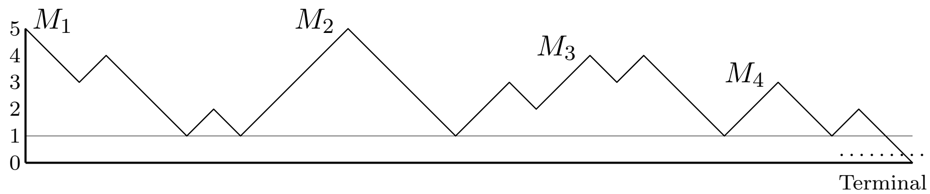

FIGURE 2. Downward First Passage from Level 5 to Level 0: Future Maxima $M_1 = 5$, $M_2 = 5$, $M_3 = 4$, $M_4 = 3$.

some $k \geq 0$. Since any downward lattice path from n to 0 must first reach the level $m = 1$ we have an initial factor g_{n-1} in a product formula for g_n.

Still assuming $n \geq 3$, any section of a downward lattice path from level n to $m = 1$ for g_{n-1} must end in DD. Following this, we have either a sequence of steps of the form $(UD)^k UU$, for some $k \geq 0$, or a *terminal* sequence $(UD)^k D$, for some $k \geq 0$. See Figure 2 where we have a transition $(UD)^1 UU$ at the first point where level $m = 1$ is reached. Define a *turning point* of a lattice path as step up to step down or vice versa. When two lattice paths are concatenated at a common point that is the terminal point of the first path and the initial point of the second path, such that the joining point is a turning point for the whole path, then the runs statistics add along the two paths. To calculate g_2 we introduce:

$$\omega = 1 - \xi r^2 z^2; \quad \kappa = (1 - \xi)/\omega. \tag{2.13}$$

By (2.13), $\kappa[\mathbf{1}] = 1$. We may easily see that κ is the probability generating function for all paths of the form $(UD)^k$, for some $k \geq 0$, and also for all paths of the form $(DU)^k$, for some $k \geq 0$. Thus, because any upward path for g_2 can be decomposed into a preamble $(UD)^k$ for some $k \geq 0$ followed by the path UU, we have

$$g_2 = \kappa r z^2 \tag{2.14}$$

where κ is defined by (2.13). Similarly, the probability generating function of all terminal sequences $(UD)^k D$, $k \geq 0$, for downward first passage paths is as follows:

$$\textit{const.}\ z(1 + \xi r^2 z^2 + \xi^2 r^4 z^4 + \cdots) = \textit{const.}\ z/\omega = \kappa z. \tag{2.15}$$

By symmetry in ξ after interchanging the roles of ph and $(1 - p)h$, we have that κz is also the probability generating function of all terminal sequences $(DU)^k U$, $k \geq 0$, for upward first passage paths from $-n$ to 0.

Suppose still $n \geq 3$, and that the continuation of a path in a downward representation of g_n after the first downward passage to level $m = 1$ is not yet passing into a terminal sequence. Then the path makes an upward first passage from level $m = 1$ to level n again (or not), and the pattern "upward transition from level $m = 1$ to level n followed by downward transition to level $m = 1$" repeats for an indefinite number of times, $\ell \geq 0$. After the path would no longer reach level n again from the starting level $m = 1$, and if $n - 1 \geq 2$, then we would similarly take account of up and down transitions between levels $m = 1$ and $n - 1$, and so forth. Since for any $j \geq 2$ we have that ρ_j is the total probability of paths making an upward first passage from level $m = 1$ to level $j + 1$, and ρ_{-j} is the total probability of paths making a downward first passage from level $j + 1$ to level $m = 1$, and since runs statistics add for the concatenation of paths that connect at a turning point, we have that $\rho_j \rho_{-j} g_j^2$ is the unnormalized generating factor for an upward transition from level $m = 1$ to level $j + 1$ followed by a downward transition to level $m = 1$. Accordingly, for each $j \geq 2$, it follows that $\lambda_j = \frac{1}{1 - \rho_j \rho_{-j} g_j^2} = \sum_{\ell=0}^{\infty} \left(\rho_j \rho_{-j} g_j^2\right)^\ell$ as defined by (2.11) has the property that the factor $\lambda_j/\lambda_j[\mathbf{1}]$ is a probability generating function for a class of paths, including the empty path, starting and ending at the same level 1, where each path besides the empty path makes a positive number of consecutive up and down first passage transitions between levels $m = 1$ and $j + 1$.

Define $M_1 = n$ as the starting level for a downward transition from n to $m = 0$. The path must reach the level $m = 1$ for a first time. Define M_2 as the maximum possible level in the remainder of the downward lattice path after the first passage down to level $m = 1$, as long as $M_2 \geq 3$. Here the successive maximum levels $n = M_1 \geq M_2 \geq \cdots \geq M_r$ over the whole future of the path, determined in turn from the points of each of its returns to level $m = 1$ from the previous such maximum, and determined at the first opportunity under this condition, are the *future maxima* (cf. [13]) of a downward path from level $n \geq 3$ to level $m = 0$. See Figure 2, in which we have $M_1 = 5$, $M_2 = 5$, $M_3 = 4$, $M_4 = 3$; there is no second future maximum of level 4, for example, because there is no return to level 1 between the two successive peaks at level 4. By definition here, we have $M_r \geq 3$, and the downward path goes into a terminal sequence after a return to level 1 from M_r. Eventually the path will never rise to level n again but only to lower future maxima at levels $3 \leq m \leq n-1$; thus the product $\lambda_{n-1}\lambda_{n-2}\ldots\lambda_2$. Therefore, since by (2.15) the factor κz corresponds to the terminal sequence, we have

$$g_n = const.\ \kappa z \cdot g_{n-1} \prod_{j=2}^{n-1} \lambda_j, \quad \text{for all } n \geq 3. \tag{2.16}$$

for a normalization constant such that $g_n[\mathbf{1}] = 1$. By (2.16) we simply have $g_{n+1}/g_n = const.\ g_n\lambda_n/g_{n-1}$, so (2.12) follows for $n \geq 3$. By (2.16), $g_3 = const.\ \kappa z \cdot g_2\lambda_2 = const.\ \kappa r z^2 \cdot g_2\lambda_2/g_1$ since $g_1 = rz$. Therefore by (2.14) the recurrence (2.12) holds also for $n = 2$. □

2.2. **Closed form for g_n.** In this section we first solve the recurrence (2.12) for g_n in Proposition 2.3 by employing the Fibonacci recurrences (2.7) for certain parameters x and β of (2.17) that carry the generating function variables. Once we have the formula for g_n in hand we go on to prove Proposition 2.6.

We define

$$x = x(r,z) = \xi z^2; \quad \beta = \beta(r,z) = 1 + \xi z^2\left(1 - r^2\right) = \omega + x. \tag{2.17}$$

Using the parameters (2.17) we then define the recurrence

$$w_{n+1} = \beta w_n - x w_{n-1},\ n \geq 1,\ \text{with } w_0 = 1,\ w_1 = 1. \tag{2.18}$$

This recurrence is simply (2.7) with the specified values of x and β of (2.17). The companion recurrence for $\{q_n\}$ is likewise defined by (2.7) under the definition (2.17). It follows from (2.17)–(2.18) that $w_2 = \omega$ for ω defined by (2.13). The recurrence (2.18) is the same as in [13, (1.10)] except here for the definition (2.17) of the coefficients x and β we have $\xi = p(1-p)h^2$ with $h < 1$ for the stopping parameter $a \neq 0$, whereas [13] only treats $a = 0$, that is $h = 1$.

Proposition 2.3. *We have that the following formula is valid for all $n \geq 1$.*

$$g_n = C_n\ r z^n / w_n, \quad \textit{with} \quad C_n = w_n[\mathbf{1}].$$

In this statement $C_n = w_n[\mathbf{1}]$ is the Fibonacci polynomial in the variable ξ defined by (2.7) with $\beta = 1$ and $x = \xi$. In concert with this statement we define $g_0 = r$ by convention. To prove the proposition we need two lemmas to follow.

Lemma 2.4. *Let w_n be defined as the solution to* (2.18)*. Then for all $n \geq 1$ we have:*

$$(w_n)^2 - w_{n+1}w_{n-1} = r^2 x^n.$$

Proof. Since the definition (2.18) conforms to (2.7), by (2.10)(i) and (2.17) we have:

$$(w_n)^2 - w_{n+1}w_{n-1} = -x^{n-1}(\beta - x - 1) = r^2x^n \tag{2.19}$$

since $\beta - x - 1 = -r^2x$ □

Lemma 2.5. *Let $n \geq 0$. Then we have the following formula:*

$$\rho_n = \frac{(ph)^n}{w_n[\mathbf{1}]}.$$

Proof. By the same argument as used for (2.16),

$$\rho_{n+1} = \frac{(1-p)h}{1-\xi} \cdot \rho_n \cdot \prod_{j=2}^{n} \frac{1}{1-\rho_j\rho_{-j}}; \quad \text{for all } n \geq 2, \tag{2.20}$$

where the factor $(1-p)h/(1-\xi)$ corresponds to the terminal sequence; compare (2.15) and Figure 2. Since by (2.6) we have $\rho_j\rho_{-j} = ((1-p)/p)^n\rho_j^2$, then

$$\rho_{n+1} = \frac{\rho_n^2}{\rho_{n-1}} \frac{1}{1-((1-p)/p)^n\rho_n^2}; \ \rho_1 = ph, \quad \text{for all } n \geq 2. \tag{2.21}$$

Obviously $\rho_0 = 1$ and ρ_1 satisfy the statement of the lemma by $w_0 = w_1 = 1$. By (2.5) and $w_2[\mathbf{1}] = \omega[\mathbf{1}] = 1-\xi$, we have that also $\rho_2 = (ph)^2/w_2[\mathbf{1}]$. Assume by induction that the statement of the proposition holds for all $1 \leq m \leq n$ for some $n \geq 2$. Then by (2.21) we have

$$\rho_{n+1} = \left(\frac{(ph)^n}{w_n[\mathbf{1}]}\right)^2 \left(\frac{w_{n-1}[\mathbf{1}]}{(ph)^{n-1}}\right) \frac{w_n[\mathbf{1}]^2}{w_n[\mathbf{1}]^2 - \xi^n}. \tag{2.22}$$

But by Lemma 2.4 we have $w_n[\mathbf{1}]^2 - \xi^n = w_{n+1}[\mathbf{1}]w_{n-1}[\mathbf{1}]$. Therefore after this substitution and subsequent cancellations in (2.22) the induction step is complete. □

Proof of Proposition 2.3. The proof is similar to that of [13, Proposition 2], except here we apply Lemma 2.5 in place of [13, Lemma 1]. The case $n = 1$ is trivial with $g_1 = rz$ by the definition (2.3). Next, since we have $w_2 = \omega$, the given formula for g_2 evaluates to $C_2\ rz^2/\omega = \kappa rz^2$, as long as we take $C_2 = 1-\xi$. But this C_2 is simply $\omega[\mathbf{1}]$ as desired. Thus by (2.14) the case $n = 2$ is verified.

We assume by induction that the statement of the proposition holds with m in place of n for all $1 \leq m \leq n$, where $n \geq 2$. We first compute λ_n. By Lemma 2.5 and (2.6) we have $\rho_n\rho_{-n} = \xi^n/w_n[\mathbf{1}]^2$. Therefore by the induction hypothesis we have from the definition (2.11) that $1 - 1/\lambda_n$ is given by

$$1 - \frac{1}{\lambda_n} = \rho_n\rho_{-n}g_n^2 = \frac{\xi^n C_n^2 r^2 z^{2n}}{w_n[\mathbf{1}]^2(w_n)^2}.$$

By the induction hypothesis we have $C_n^2/w_n^*[\mathbf{1}]^2 = 1$. We also write that $\xi^n r^2 z^{2n} = r^2x^n$. Hence we have

$$\lambda_n = \frac{(w_n)^2}{(w_n)^2 - r^2x^n}.$$

By Lemma 2.4 the denominator of this last expression for λ_n is simply $w_{n+1}w_{n-1}$. Therefore $\lambda_n = \frac{(w_n)^2}{w_{n+1}w_{n-1}}$. By (2.12) we then have

$$g_{n+1} = const.\ \frac{r^2z^{2n}w_{n-1}}{rz^{n-1}(w_n)^2} \frac{(w_n)^2}{w_{n+1}w_{n-1}} = const.\ \frac{rz^{n+1}}{w_{n+1}}. \tag{2.23}$$

Since the normalizing constant must make $g_{n+1}[\mathbf{1}] = 1$, the proof is complete. □

Proposition 2.6. *Let w_N and q_N be defined by (2.7) with x and β defined by (2.17). Then the generating function K_N of (2.1) has the following formula:*

$$K_N = k_N r^2 z^2 \frac{q_N}{w_N}, \quad \text{for all } N \geq 1; \quad \text{with } k_N = 2\xi / \mathbb{P}\left(E_0 \cap \{1 \leq \mathbf{H} \leq N\}\right).$$

Proof. The idea of the proof parallels the construction of convergents to a continued fraction, [5, Chp. III]. The proof follows the same lines as the proof of [13, Theorem 1]. First, the probability generating function G_n of (2.1) may be written for all $n \geq 2$ as follows.

$$G_n = z g_{n-1} g_n, \quad \text{if } n \geq 2, \tag{2.24}$$

and trivially $G_1 = r^2 z^2$.

Now the generating function K_N of (2.1) is written

$$K_N = \sum_{n=1}^{N} G_n \, \mathbb{P}(E_0 \cap \{\mathbf{H} = n\}) / \, \mathbb{P}(E_0 \cap \{1 \leq \mathbf{H} \leq N\}). \tag{2.25}$$

Furthermore, by Lemma 2.5,

$$\mathbb{P}(E_0 \cap \{\mathbf{H} = n\}) = ph\rho_{n-1}\rho_{-n} + (1-p)h\rho_{-n+1}\rho_n = \frac{2\xi^n}{w_{n-1}[\mathbf{1}]w_n[\mathbf{1}]}, \tag{2.26}$$

where we wrote $(ph)(ph)^{n-1}((1-p)h)^n + ((1-p)h)((1-p)h)^{n-1}(ph)^n = 2\xi^n$. By Proposition 2.3 and (2.24)–(2.26), we have that $\mathbb{P}(E_0 \cap \{1 \leq \mathbf{H} \leq N\})K_N$ is written for all $N \geq 2$ as

$$2\xi r^2 z^2 + \sum_{n=2}^{N} C_{n-1} C_n \frac{2\xi^n}{w_{n-1}[\mathbf{1}]w_n[\mathbf{1}]} \frac{r^2 z^{2n}}{w_{n-1} w_n}, \tag{2.27}$$

Since by Proposition 2.3 we have $\frac{C_{n-1}C_n}{w_{n-1}[\mathbf{1}]w_n[\mathbf{1}]} = 1$, we are left with the problem of evaluating

$$\sum_{n=2}^{N} \frac{2\xi^n r^2 z^{2n}}{w_{n-1} w_n} = 2\xi r^2 z^2 \sum_{n=2}^{N} \frac{x^{n-1}}{w_{n-1} w_n} \tag{2.28}$$

as the main term of (2.27). Notice that the $n = 1$ term of (2.27) also conforms to satisfy $2\xi^2 r^2 z^2 = 2\xi r^2 z^2 \frac{x^0}{w_0 w_1}$. Thus by (2.27)–(2.28) we have

$$\mathbb{P}(E_0 \cap \{1 \leq \mathbf{H} \leq N\})K_N = 2\xi r^2 z^2 \sum_{n=1}^{N} \frac{x^{n-1}}{w_{n-1} w_n}. \tag{2.29}$$

Next we collapse the summation in (2.29). Indeed, we claim that for all $N \geq 1$ we have

$$\sum_{n=1}^{N} \frac{x^{n-1}}{w_{n-1} w_n} = \frac{q_N}{w_N}. \tag{2.30}$$

The claim (2.30) follows by induction as in [13, (2.19)] by applying (2.10)(iii). Therefore by (2.29)–(2.30) the proof is complete. ☐

Remark 2.7. *If $a = 0$, so $h = 1$, and if also $p = \frac{1}{2}$, then the statement of Proposition 2.6 becomes $K_N = \frac{1}{2} \mathbb{P}\left(1 \leq \mathbf{H} \leq N\right)^{-1} r^2 z^2 \frac{q_N}{w_N} = \frac{N+1}{2N} r^2 z^2 \frac{q_N}{w_N}$, as shown by* [13, Theorem 1].

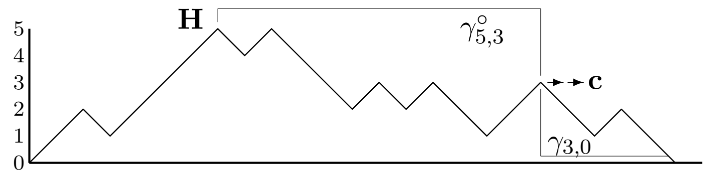

FIGURE 3. Excursion Height $n = 5$; Transition to $\mathbf{c}$ at $k = 3$.

3. JOINT GENERATING FUNCTION OVER A STOPPED EXCURSION.

The method of section 2 applies to bona fide excursion statistics. In this section we extend the method by working instead over an excursion attempt that is interrupted by stopping. On the event $E_\mathbf{c} = E_\mathbf{c} \cap \{\mathbf{H} \le N\}$ that the first excursion attempt is stopped, define $\mathbf{R}^\circ = \mathbf{R}^\circ_N$ and $\mathbf{L}^\circ = \mathbf{L}^\circ_N$ as the number of runs and steps, respectively, on this stopped excursion. We apply the same convention as described in the second paragraph before the statement of Theorem 1.1; that is, for the definition of $\mathbf{R}^\circ$ we shall count a new run on a final step to $\mathbf{c}$ in the transition from epoch $\mathbf{L} - 1$ to $\mathbf{L}$ if the step just before this final transition is away from the j–axis. We now define

$$\begin{aligned} G^\circ_n &= \mathbb{E}\left\{ r^{\mathbf{R}^\circ} z^{\mathbf{L}^\circ} \,|E_\mathbf{c} \cap \{\mathbf{H} = n\};\ \mathbf{X}_0 = 0;\ \mathbf{X}_j > 0,\ j = 1, 2, \ldots \mathbf{L} - 1 \right\}, \\ K^\circ_N &= \mathbb{E}\left\{ r^{\mathbf{R}^\circ} z^{\mathbf{L}^\circ} \mid E_\mathbf{c} \cap \{1 \le \mathbf{H} \le N\} \right\}; \end{aligned} \tag{3.1}$$

where G°_n is defined for all $1 \le n \le N$. In words we have that G°_n is the joint probability generating function of runs and steps along all nonnegative stopped excursions of height n.

Our main result of this section is the culminating formula for K°_N given by Proposition 3.10 in subsection 3.2. Our first step is to compute G°_n that we find by computing g°_n of definition (3.6) together with the formula for g°_n of Proposition 3.5 derived in subsection 3.1.

We now focus on G°_n for $n \ge 2$. On $E_\mathbf{c}$ the stopped excursion path comes to a *jump–off* point $(\mathbf{L} - 1, \mathbf{X}_{\mathbf{L}-1})$ before exiting to state $\mathbf{c}$. For any finite nonnegative stopped nearest neighbor lattice path γ started from level $m = 0$, denote $\mathbf{R}^\circ(\gamma)$ and $\mathbf{L}^\circ(\gamma)$ respectively as the number of runs and steps along γ. Since the generating function G°_n is a normalized infinite sum $\sum \mu(\gamma) r^{\mathbf{R}^\circ(\gamma)} z^{\mathbf{L}^\circ(\gamma)}$ such that $G^\circ_n[\mathbf{1}] = 1$, where the sum runs over all stopped excursions γ of maximum level n, we may break up the sum in this expression by decomposing the collection of paths γ into those paths that have a jump–off point at level k. This can be formalized in terms of generating functions as follows. For any $1 \le k \le n$ and $1 \le n \le N$ define

$$G^\circ_{n,k} = \mathbb{E}\left\{ r^{\mathbf{R}^\circ} z^{\mathbf{L}^\circ} \mid E_\mathbf{c} \cap \{\mathbf{H} = n\};\ \mathbf{X}_0 = 0;\ \mathbf{X}_j > 0, j = 1, 2, \ldots, \mathbf{L} - 1;\ \mathbf{X}_{\mathbf{L}-1} = k \right\}. \tag{3.2}$$

Then define the probability generating function $g^\circ_{n,k}$ implicitly by

$$G^\circ_{n,k} = z g_{n-1} g^\circ_{n,k} \quad \text{for all } \ 1 \le k \le n \text{ and } n \ge 2. \tag{3.3}$$

If $n = k = 1$, we define $g^\circ_{1,1} = rz$. Because the paths that contribute to $G^\circ_{n,k}$ must stay above the j–axis save for the starting point $m = 0$ and reach the level n for a first time, then for all $n \ge 2$ we have a factor $z g_{n-1}$ corresponding to this upward first passage. Thus $g^\circ_{n,k}$ is the probability generating factor corresponding to the remaining portion of the paths after the maximum level n has been first achieved, with the condition that this remaining portion achieves levels only in $[1, n]$ until a jump-off point at level k.

Lemma 3.1. *Let $n \geq 1$. Then for all $1 \leq k \leq n$, we have*

$$g^\circ_{n,k} = r\frac{g_n}{g_{k-1}},$$

where by convention $g_0 = r$.

Proof. First fix $2 \leq k \leq n$. Consider a stopped nonnegative excursion with maximum level n and jump–off point at level k. Denote by $\gamma^\circ_{n,k}$ the part of the stopped excursion path starting when the maximum level n is first achieved until the jump–off epoch $\mathbf{L} - 1$. We may extend the stopped excursion to a full excursion by appending a path $\gamma_{k,0}$, starting at the jump–off point, such that $\gamma_{k,0}$ goes from level k to $k-1$ at the first step, its level thereafter stays in $[0, k-1]$, and $\gamma_{k,0}$ ends by reaching level 0 for a first time. We can denote then the downward first passage from level n to $m = 0$ of the full excursion, starting when the maximum level n is first achieved until the excursion ends, as the concatenation $\gamma_n = \gamma^\circ_{n,k}\gamma_{k,0}$. In Figure 3 we have $\gamma_5 = \gamma^\circ_{5,3}\gamma_{3,0}$.

To compute $g^\circ_{n,k}$ of (3.3), we have that there is a one–to–one correspondence between all downward first passage paths γ_n from level n to 0 (that stay at levels of $[1, n]$ until a final step) and all possible concatenations $\gamma^\circ_{n,k}\gamma_{k,0}$, where $\gamma^\circ_{n,k}$ stays at levels in $[1, n]$ until level k is achieved. That is because any such γ_n must attain level k for a last time. Assume that $2 \leq k \leq n$. Define $g_{k,0} = const.\ \sum \mu(\gamma_{k,0}) r^{R(\gamma_{k,0})} z^{L(\gamma_{k,0})}$, where $R(\gamma_{k,0})$ and $L(\gamma_{k,0})$ denote the number of runs and steps along $\gamma_{k,0}$ and the constant is determined to make $g_{k,0}[\mathbf{1}] = 1$. It is obvious then that

$$g_{k,0} = zg_{k-1}, \quad \text{for all } k \geq 2. \tag{3.4}$$

Hence, up to reckoning the number of runs and steps accounted for in $\gamma^\circ_{n,k}$ concatenated with the one–step transition to $\mathbf{c}$ and $\gamma_{k,0}$ separately, versus the number of runs and steps in the concatenation $\gamma_n = \gamma^\circ_{n,k}\gamma_{k,0}$, by (3.4) and the Markov property we have

$$g^\circ_{n,k} \cdot zg_{k-1} = rzg_n, \text{ for all } 2 \leq k \leq n. \tag{3.5}$$

Our reckoning of the factor r on the right side in (3.5) is as follows. (1) Suppose the step of $\gamma^\circ_{n,k}$ that comes to the jump–off point at level k is in the direction of the j–axis. Then there is a run accounted for in the descent of $\gamma^\circ_{n,k}$ to this point, while the factor zg_{k-1} also accounts for a run along the same descent continued from the jump–off point into a path $\gamma_{k,0}$. Thus we have counted an extra run for this continued descent on the left side of (3.5), so we compensate by adding a factor of r on the right side. (2) Suppose instead that the step of $\gamma^\circ_{n,k}$ that comes to the jump–off point is in the opposite direction of the j–axis. Then the descent at the beginning of $\gamma_{k,0}$ from the jump–off point is a fresh descent, so no run is added by the product on the left side of (3.5), but in this case we agreed to add a run for $\gamma^\circ_{n,k}$ for the transition to the stopping state $\mathbf{c}$. Thus the extra factor of r again in this situation for the right side of (3.5). Our reckoning for the factor of z on the right side of (3.5) is that there is a step counted in all cases for the transition to $\mathbf{c}$, so we add a factor of z on the right side for this step; the factor of z on the left side is already from (3.4).

For $k = 1$, the approach to the jump–off point in $\gamma^\circ_{n,k}$ must be toward the j–axis. Therefore $\gamma_{1,0}$ is simply one step down, and there in no added run for the exit to $\mathbf{c}$, so $g^\circ_{n,1} = g_n = rg_n/g_0$. □

Once more, in concert with (3.3), define g°_n implicitly by

$$G^\circ_n = zg_{n-1}g^\circ_n \quad \text{for all } n \geq 1. \tag{3.6}$$

In parallel with (3.3), define $\rho^\circ_{n,k}$ for all $1 \leq k \leq n$ and $1 \leq n \leq N$ by

$$phc_N \cdot \rho_{n-1}\rho^\circ_{n,k} = \mathbb{P}\left(E_\mathbf{c} \cap \{\mathbf{H} = n\};\ \mathbf{X}_0 = 0;\ \mathbf{X}_j > 0, j = 1, 2, \ldots, \mathbf{L} - 1;\ \mathbf{X}_{\mathbf{L}-1} = k\right). \tag{3.7}$$

In words, $\rho^\circ_{n,k}$ is the total probability of all nearest neighbor lattice paths among paths starting from level n that stay in levels of $[1, n]$ and and stop at level k. The factor c_N on the left accounts for the fact that there is a factor c_N implicit for the transition to $\mathbf{c}$ on the right side of (3.7). We define $\rho^\circ_{-n,-k}$ analogously by reflection of the paths for $\rho_{n,k}$: $\rho^\circ_{-n,-k}$ is the total probability of all nearest neighbor lattice paths among paths starting from level $-n$ that stay in levels of $[-n,-1]$ and and stop at level $-k$. By the same argument given for (2.6), where now paths representing $\rho_{n,k}$ have $n-k$ more down steps than up steps, we have

$$\rho^\circ_{-n,-k} = (p/(1-p))^{n-k}\,\rho^\circ_{n,k}, \quad \text{for all } 1 \le k \le n. \tag{3.8}$$

By the proof of Lemma 3.1 and definition (2.4), we simply have

$$\rho^\circ_{n,k}(1-p)h\rho_{-k+1} = \rho_{-n}, \quad \text{for all } 1 \le k \le n. \tag{3.9}$$

3.1. **Formula for g°_n.** Our main novelty in this subsection is to obtain Proposition 3.5, that is a formula for g°_n as defined by (3.6) and rewritten by (3.10). This formula involves an extension u_n of the Fibonacci recurrence (2.7) by a certain forcing term as shown in (3.17). Let $n \ge 2$. By (3.1)–(3.3), (3.6), and (3.7), we have, for a normalizing constant C°_n, that

$$g^\circ_n = C^\circ_n \sum_{k=1}^{n} \rho^\circ_{n,k} g^\circ_{n,k}. \tag{3.10}$$

The identity (3.10) arises simply because $g^\circ_{n,k}$ is a normalized sum along lattice paths discussed just after (3.3) such that $g^\circ_{n,k}[\mathbf{1}] = 1$, so we must multiply $g^\circ_{n,k}$ by its normalizing factor, which is $\rho^\circ_{n,k}$, and then sum on $1 \le k \le n$ to cover all possible paths for g°_n. Thus it follows that C°_n equals $1/\sum_{k=1}^{n} \rho^\circ_{n,k}$ by (3.10) and $g^\circ_n[\mathbf{1}] = 1$. By (2.6), (3.9), and Lemma 2.5 we have for all $1 \le k \le n$ that

$$\rho^\circ_{n,k} = \frac{((1-p)/p)^n\,\rho_n}{(1-p)h\,((1-p)/p)^{k-1}\,\rho_{k-1}} = ((1-p)h)^{n-k}\,\frac{w_{k-1}[\mathbf{1}]}{w_n[\mathbf{1}]}. \tag{3.11}$$

Lemma 3.2. *Let $n \ge 2$. Then we have*

$$\rho^\circ_{n,k} g^\circ_{n,k} = ((1-p)h)^{n-k}\,rz^{n-k+1}\frac{w_{k-1}}{w_n} \text{ for all } 1 \le k \le n.$$

Proof. By Proposition 2.3, Lemma 3.1 and (3.11), for all $1 \le k \le n$ we have

$$\rho^\circ_{n,k} g^\circ_{n,k} = ((1-p)h)^{n-k}\,\frac{w_{k-1}[\mathbf{1}]}{w_n[\mathbf{1}]}\cdot r\frac{C_n r z^n}{C_{k-1} r z^{k-1}}\frac{w_{k-1}}{w_n} = ((1-p)h)^{n-k}\,rz^{n-k+1}\frac{w_{k-1}}{w_n}, \tag{3.12}$$

since by Proposition 2.3 we have $\frac{w_{k-1}[\mathbf{1}]}{w_n[\mathbf{1}]}\frac{C_n}{C_{k-1}} = 1$. □

We now wish to find a closed formula for g°_n as given by (3.10) and Lemma 3.2. For this purpose, to simplify the notation we introduce the square root of the ratio of down–step to up–step probabilities as follows

$$d = \sqrt{(1-p)/p}. \tag{3.13}$$

In the symmetric case we simply have $d = 1$. However, the parameter d comes into the formula for $\rho^\circ_{n,k} g^\circ_{n,k}$ of Lemma 3.2 as follows. We apply the reformulation that, for all $k \ge 1$,

$$((1-p)h)^{n-k}\,rz^{n-k+1} = rz\left((1-p)h\sqrt{x/\xi}\right)^{n-k} = rz\left(d\sqrt{x}\right)^{n-k}. \tag{3.14}$$

Here, we keep one factor of z by itself because that corresponds to the step to the state $\mathbf{c}$. By (3.10), Lemma 3.2 and (3.14), we have for all $n \ge 2$ that

$$g^\circ_n = C^\circ_n \sum_{k=1}^{n} \rho^\circ_{n,k} g^\circ_{n,k} = C^\circ_n\, rz\left(\sum_{k=1}^{n}\left(d\sqrt{x}\right)^{n-k} w_{k-1}\right)\frac{1}{w_n}. \tag{3.15}$$

We now study the summation under the parentheses on the right side of (3.15). It is convenient to make a change of index in this expression as follows.

Definition 3.3. *Define the sequence u_n, $n \geq 1$, by*

$$u_n = \sum_{\ell=0}^{n-1} \left(d\sqrt{x}\right)^{n-1-\ell} w_\ell, \quad \textit{for all } n \geq 1, \tag{3.16}$$

with $u_0 = 0$ and $u_1 = 1$.

By (3.15) and Definition 3.3 we indeed have that $g_n^\circ = C_n^\circ \, rzu_n/w_n$ for all $n \geq 2$. We now find a recurrence for u_n via the following ansatz:

$$u_{n+1} = \beta u_n - xu_{n-1} + A\left(d\sqrt{x}\right)^{n-1}, \quad \text{with } u_0 = 0 \text{ and } u_1 = 1. \tag{3.17}$$

Lemma 3.4. *We have that the sequence u_n, $n \geq 1$, defined by* (3.16) *satisfies the recurrence* (3.17) *with $A = 1 + d\sqrt{x} - \beta$.*

Proof. We proceed by induction. Consider the basis $n = 1$ for (3.17) with A defined by the statement of the lemma. By definition (3.16) we have $u_2 = 1 + (d\sqrt{x})\, w_1 = 1 + d\sqrt{x}$. On the other hand we have that the right side of (3.17) at $n = 1$ is given by $\beta u_1 - xu_0 + A = \beta + A = 1 + d\sqrt{x}$. Thus the two sides of (3.17) are equal at $n = 1$, so the basis of induction is verified.

Assume now that (3.17) holds for some $n \geq 1$ with A given by the statement of the lemma. Then by definition (3.16) we have

$$u_{n+2} = \sum_{\ell=0}^{n+1} \left(d\sqrt{x}\right)^{n+1-\ell} w_\ell = w_{n+1} + d\sqrt{x} \sum_{\ell=0}^{n} \left(d\sqrt{x}\right)^{n-\ell} w_\ell = w_{n+1} + \left(d\sqrt{x}\right) u_{n+1}. \tag{3.18}$$

Now by definition (2.18) write $w_{n+1} = \beta w_n - xw_{n-1}$ and by the induction hypothesis write $u_{n+1} = \beta u_n - xu_{n-1} + A\,(d\sqrt{x})^{n-1}$. Then by (3.18) we have

$$u_{n+2} = \beta\left(w_n + d\sqrt{x}u_n\right) - x\left(w_{n-1} + d\sqrt{x}u_{n-1}\right) + d\sqrt{x}A\left(d\sqrt{x}\right)^{n-1}. \tag{3.19}$$

Finally, apply definition (3.16) to rewrite u_n and u_{n-1} in the right side of (3.19) as sums, and apply the factor $d\sqrt{x}$ on these terms so as to raise the powers of $d\sqrt{x}$ by 1 in each of these sums. Hence, by (3.19),

$$u_{n+2} = \beta\left(w_n + \sum_{\ell=0}^{n-1} \left(d\sqrt{x}\right)^{n-\ell} w_\ell\right) - x\left(w_{n-1} + \sum_{\ell=0}^{n-2} \left(d\sqrt{x}\right)^{n-1-\ell} w_\ell\right) + A\left(d\sqrt{x}\right)^n. \tag{3.20}$$

Here, if $n = 1$ there is an empty sum evaluating to zero in the second parenthetical term. But obviously the two total sums in parentheses in (3.20) are by the definition (3.16) simply u_{n+1} and u_n, respectively, and the term $A\,(d\sqrt{x})^n$ is of the correct form for the induction step. □

We summarize the calculations of this section as follows.

Proposition 3.5. *Let g_n° be defined by* (3.1) *and* (3.6)*. Let u_n, $n \geq 1$, be given by* (3.17) *with A defined by the statement of Lemma 3.4. Then, for all $n \geq 2$ we have*

$$g_n^\circ = C_n^\circ \; rz\frac{u_n}{w_n}, \quad \textit{where } C_n^\circ = 1/\sum_{k=1}^{n} \rho_{n,k}^\circ.$$

Proof. The proposition follows by (3.15), Definition 3.3, and Lemma 3.4, where we noted the evaluation of C_n° after (3.10) with $\rho_{n,k}^\circ$ defined by (3.7). □

3.2. **Calculation of K_N°.** Our goal in this subsection is to obtain Proposition 3.10, that is a formula for K_N° of definition (3.1). In Proposition 3.5 we focussed on nonnegative stopped excursions to calculate g_n°. However, while the analogue for nonpositive stopped excursions of the probability generating function $g_{n,k}$ defined by (3.2)–(3.3) will still be the same as given by the formula of Lemma 3.1, the corresponding analogue g_{-n}° of g_n° itself will not be as given by Proposition 3.5. Indeed define the analogue G_{-n}° of G_n° for nonpositive stopped excursions as follows.

$$G_{-n}^\circ = \mathbb{E}\left(r^{\mathbf{R}^\circ} z^{\mathbf{L}^\circ} \; | E_{\mathbf{c}} \cap \{\mathbf{H} = n\};\; \mathbf{X}_0 = 0;\; \mathbf{X}_j < 0,\; j = 1, 2, \dots \mathbf{L} - 1\right). \tag{3.21}$$

In parallel with (3.6), define g_{-n}° implicitly for all $n \geq 1$ by

$$G_{-n}^\circ = z g_{n-1} g_{-n}^\circ. \tag{3.22}$$

Then just as in (3.10), we have

$$g_{-n}^\circ = C_{-n}^\circ \sum_{k=1}^{n} \rho_{-n,-k}^\circ \cdot g_{n,k}^\circ, \tag{3.23}$$

where $\rho_{-n,-k}$ is defined by (3.7)–(3.8). However since the relation (3.8) depends on k in the asymmetric case $p \neq \frac{1}{2}$, we have in this case that $g_{-n}^\circ \neq g_n^\circ$. Hence the formula for K_N° involves separate contributions from nonnegative and nonpositive stopped excursions as follows: $\mathbb{P}\left(E_{\mathbf{c}} \cap \{1 \leq \mathbf{H} \leq N\}\right) K_N^\circ$ is written as:

$$\begin{aligned} &\sum_{n=1}^{N} G_n^\circ \, \mathbb{P}\left(E_{\mathbf{c}} \cap \{\mathbf{H} = n\} \cap \{\mathbf{X}_0 = 0;\; \mathbf{X}_j > 0, j = 1, 2, \dots \mathbf{L} - 1\}\right) \\ &+ \sum_{n=1}^{N} G_{-n}^\circ \, \mathbb{P}\left(E_{\mathbf{c}} \cap \{\mathbf{H} = n\} \cap \{\mathbf{X}_0 = 0;\; \mathbf{X}_j < 0, j = 1, 2, \dots \mathbf{L} - 1\}\right) = \sum_{n=1}^{N} \sigma_n + \sum_{n=1}^{N} \widetilde{\sigma}_n \end{aligned} \tag{3.24}$$

We note that the case $\mathbf{H} = 0$ occurs if and only if there is transition to $\mathbf{c}$ on the first step, which occurs only with very small probability $c_N = O(1/N^2)$. Therefore we may safely ignore this case. For the calculation of the probabilities involved in the expression σ_n of (3.24) we have

$$\mathbb{P}\left(E_{\mathbf{c}} \cap \{\mathbf{H} = n\} \cap \{\mathbf{X}_0 = 0;\; \mathbf{X}_j > 0, j = 1, 2, \dots \mathbf{L} - 1\}\right) = phc_N \cdot \rho_{n-1} \sum_{k=1}^{n} \rho_{n,k}^\circ. \tag{3.25}$$

Due to the easy interchange of parameters p and $1-p$ to handle the contributions for nonpositive stopped excursions from the nonnegative ones, so that $\widetilde{\sigma}_n$ is determined by applying this interchange to a formula for σ_n, we focus on the sum $\sum_{n=1}^{N} \sigma_n$ in (3.24). We first summarize our calculations to this point to find a nice expression for σ_n.

Lemma 3.6. *Denote $\sigma_n = G_n^\circ \, \mathbb{P}\left(E_{\mathbf{c}} \cap \{\mathbf{H} = n\} \cap \{\mathbf{X}_0 = 0;\; \mathbf{X}_j > 0, j = 1, 2, \dots \mathbf{L} - 1\}\right)$. Then for each $n \geq 1$ we have*

$$\sigma_n = phc_N r^2 z^2 \frac{\left(\sqrt{x}/d\right)^{n-1} u_n}{w_{n-1} w_n},$$

where u_n is defined by (3.16).

Proof. By definition (3.6), Propositions 2.3 and 3.5, and (3.25), for all $n \geq 1$ we have that $\sigma_n/(phc_N)$ is written

$$z g_{n-1} g_n^\circ \cdot \rho_{n-1} \sum_{k=1}^{n} \rho_{n,k}^\circ = z \rho_{n-1} C_{n-1} r z^{n-1} \cdot r z \frac{u_n}{w_{n-1} w_n} C_n^\circ \sum_{k=1}^{n} \rho_{n,k}^\circ, \tag{3.26}$$

where if $n = 1$ we have $C_0 = 1$, $g_0 = r$, and $\rho_0 = 1$, consistent with $\sigma_1/(phc_N) = r^2z^2$. Here, by the definition, $C_n^\circ \sum_{k=1}^{n} \rho_{n,k}^\circ = 1$. Also by Lemma 2.5 we have

$$\rho_{n-1}C_{n-1} = \frac{(ph)^{n-1}}{w_{n-1}[\mathbf{1}]} w_{n-1}[\mathbf{1}] = (ph)^{n-1}.$$

Therefore by these cancellations in (3.26) we have

$$\sigma_n = phc_N r^2 z^2 \cdot (ph)^{n-1} z^{n-1} \frac{u_n}{w_{n-1}w_n}.$$

Finally, as before, write $z = \sqrt{x/\xi}$ and thus find that $(ph)^{n-1}z^{n-1} = (\sqrt{x}/d)^{n-1}$, so the proof is complete. $\square$

Let w_n and u_n be defined respectively by (2.18) and Definition 3.16, where u_n is determined as a recurrence by Lemma 3.4. We wish to find a recurrence for a sequence v_n, $n \geq 1$, with $v_0 = 0$ and $v_1 = 1$ such that for all $N \geq 1$ we have:

$$\sum_{n=1}^{N} \frac{(\sqrt{x}/d)^{n-1}\, u_n}{w_{n-1}w_n} = \frac{v_N}{w_N}. \tag{3.27}$$

The reason for this desired form is that by Lemma 3.6 the sum $\sum_{n=1}^{N} \sigma_n$ takes the form of the sum in (3.27) modulo some factors that are constant in n. The relation (3.27) is true for $N = 1$ by $u_1 = v_1 = w_1 = 1$. For $N = 2$, since $u_2 = 1 + d\sqrt{x}$, by (3.27) we therefore want $1 + \frac{\sqrt{x}}{d}\left(1 + d\sqrt{x}\right)/\omega = v_2/\omega$. So we take $v_2 = \omega + \frac{\sqrt{x}}{d} + x = \beta + \frac{\sqrt{x}}{d}$. This term is of the form $v_2 = \beta v_1 - xv_0 + (\sqrt{x}/d)^1$.

For our proofs we prefer the following definition of v_n, leaving the problem to show that (3.27) does indeed follow from it for all $N \geq 1$.

Definition 3.7. *Define a sequence v_n, $n \geq 1$, by*

$$v_{n+1} = \beta v_n - xv_{n-1} + \left(\sqrt{x}/d\right)^n, \ \textit{for } n \geq 1; \quad \textit{with } v_0 = 0,\ v_1 = 1. \tag{3.28}$$

Lemma 3.8. *Let w_n, u_n, and v_n be defined respectively by* (2.18)*, Definition 3.3, and Definition 3.7, where u_n is determined as a recurrence by Lemma 3.4. Then* (3.27) *holds for all $N \geq 1$.*

Before proving Lemma 3.8, we record solutions for u_n and v_n in terms of the fundamental sequence $\{q_n,\ n \geq 0\}$ of (2.7) via generating function manipulations.

Lemma 3.9. *Let u_n and v_n be defined respectively by Definition 3.3, and Definition 3.7, where u_n is determined by Lemma 3.4. Also denote $B = \beta - d\sqrt{x} - \sqrt{x}/d$. Then, with q_n defined by* (2.7)*, we have for all $n \geq 1$ that*

$$u_n = q_n + \frac{A}{B}\left(-(d\sqrt{x})^{n-1} + q_n - \frac{\sqrt{x}}{d}q_{n-1}\right); \quad v_n = q_n + \frac{1}{B}\left(-(\sqrt{x}/d)^n + \frac{\sqrt{x}}{d}q_n - xq_{n-1}\right).$$

Proof. By a standard generating function manipulation we have by (2.7) that

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n = \frac{s}{1 - \beta s + xs^2}, \tag{3.29}$$

[13, Lemma 2]. Denote $U = U(s) = \sum\limits_{n=0}^{\infty} u_n s^n$ and $V = V(s) = \sum\limits_{n=0}^{\infty} v_n s^n$. By standard manipulations under (3.17) and (3.28) we have, by $u_0 = v_0 = 0$ and $u_1 = v_1 = 1$ that

$$U = s + \beta s U - xs^2 U + \frac{As^2}{1 - d\sqrt{x}s}; \quad V = s + \beta s V - xs^2 V + \frac{(\sqrt{x}/d)s^2}{1 - (\sqrt{x}/d)s}. \tag{3.30}$$

Therefore by algebraic manipulation of (3.30), in view of (3.29) we obtain

$$U = Q + \frac{As^2}{(1 - d\sqrt{x}s)(1 - \beta s + xs^2)}; \quad V = Q + \frac{(\sqrt{x}/d)s^2}{(1 - (\sqrt{x}/d)s)(1 - \beta s + xs^2)}. \tag{3.31}$$

Next, apply partial fractions to the last terms in the expansions for U and V, as follows:

$$\begin{aligned} \frac{As^2}{(1 - d\sqrt{x}s)(1 - \beta s + xs^2)} &= \frac{A}{B}\left(-\frac{1}{d\sqrt{x}(1 - d\sqrt{x})} + \frac{1}{d\sqrt{x}}\frac{1 + (d\sqrt{x} - \beta)s}{1 - \beta s + xs^2}\right); \\ \frac{(\sqrt{x}/d)s^2}{(1 - (\sqrt{x}/d)s)(1 - \beta s + xs^2)} &= \frac{1}{B}\left(-\frac{1}{1 - (\sqrt{x}/d)s} + \frac{1 + (\sqrt{x}/d - \beta)s)}{1 - \beta s + xs^2}\right). \end{aligned} \tag{3.32}$$

Thus, by (3.31)–(3.32) we read off u_n and v_n for all $n \geq 1$ as follows.

$$\begin{aligned} u_n &= q_n + \frac{A}{B}\left(-(d\sqrt{x})^{n-1} + \frac{1}{d\sqrt{x}}\left((d\sqrt{x} - \beta)q_n + q_{n+1}\right)\right) \\ v_n &= q_n + \frac{1}{B}\left(-(\sqrt{x}/d)^n + \left(\sqrt{x}/d - \beta)q_n + q_{n+1}\right)\right). \end{aligned} \tag{3.33}$$

Finally, rewrite (3.33) by applying the recurrence $q_{n+1} = \beta q_n - xq_{n-1}$, $n \geq 1$. This completes the proof. □

Proof of Lemma 3.8. We proceed by induction. We have already shown that the lemma holds for $N = 1, 2$. As the induction step, it now suffices to show that for any $n \geq 2$ we have

$$\frac{v_{n+1}}{w_{n+1}} - \frac{v_n}{w_n} = \frac{(\sqrt{x}/d)^n u_{n+1}}{w_n w_{n+1}} \iff w_n v_{n+1} - v_n w_{n+1} = (\sqrt{x}/d)^n u_{n+1}. \tag{3.34}$$

We will verify the last relation of (3.34) by direct calculation using Lemma 3.9, and also $w_n = q_n - xq_{n-1}$ from (2.9). We start with $w_n v_{n+1} - v_n w_{n+1} = (q_n - xq_{n-1})v_{n+1} - v_n(q_{n+1} - xq_n) = (v_{n+1}q_n - v_n q_{n+1}) - x(v_{n+1}q_{n-1} - v_n q_n) = I + II$. We reduce I and II separately; there is cancellation in each. For the term I we substitute the expressions for v_{n+1} and v_n from Lemma 3.9. It follows that we have cancellation of the resulting $q_n q_{n+1}$ terms of I. There is another term $\frac{x}{B}\left(-q_n^2 + q_{n+1}q_{n-1}\right) = \frac{-x^n}{B}$ by (2.10)(ii). The remaining terms thus give

$$I = \frac{1}{B}\left(-x^n + \left(\sqrt{x}/d\right)^n q_{n+1} - \left(\sqrt{x}/d\right)^{n+1} q_n\right).$$

Next, in II, after again substituting for v_{n+1} and v_n from Lemma 3.9, there is cancellation of the $q_{n-1}q_n$ terms. We also obtain a term $-x\left(q_{n+1}q_{n-1} - q_n^2\right)\left(1 + \frac{\sqrt{x}}{Bd}\right) = \frac{\beta - d\sqrt{x}}{B}x^n$, where we again applied (2.10)(ii) and also applied $Bd + \sqrt{x} = \beta d - d^2\sqrt{x}$ for B defined in Lemma 3.9. Thus, after adding in the remaining terms of II, we have

$$II = \frac{1}{B}\left((\beta - d\sqrt{x})x^n + x\left(\left(\sqrt{x}/d\right)^{n+1} q_{n-1} - \left(\sqrt{x}/d\right)^n q_n\right)\right).$$

Then, to match terms in I and II, rewrite I by substituting the basic recurrence $q_{n+1} = \beta q_n - xq_{n-1}$. We then obtain

$$I + II = \frac{1}{B}\left(-Ax^n + \left(\sqrt{x}/d\right)^n \left(\beta - \sqrt{x}/d - x\right) q_n - x\left(\sqrt{x}/d\right)^n \left(1 - \sqrt{x}/d\right) q_{n-1}\right), \tag{3.35}$$

where in the power term alone we used the fact that $\beta - d\sqrt{x} - 1 = -A$.

Now to verify (3.34) we substitute $q_{n+1} = \beta q_n - xq_{n-1}$ into the formula for u_{n+1} of Lemma 3.9. Thus obtain that the right side of the formula we want to verify (with $I + II$ as the left side) is, after a bit of algebra to combine q_n terms and q_{n-1} terms separately,

$$\left(\sqrt{x}/d\right)^n u_{n+1} = \frac{-Ax^n}{B} + \left(\sqrt{x}/d\right)^n \left(\left(\beta + \frac{A}{B}(\beta - \sqrt{x}/d) \right) q_n - x \left(1 + \frac{A}{B} \right) q_{n-1} \right). \tag{3.36}$$

Now we may verify that this last expression is in fact equal to $I + II$ of (3.35) by matching coefficients of q_n and q_{n-1} in (3.35) and (3.36). Indeed by the definitions of $A = 1 + d\sqrt{x} - \beta$ and $B = \beta - d\sqrt{x} - \sqrt{x}/d$ we verify by inspection that

$$(\beta - \sqrt{x}/d - x) = \beta(\beta - d\sqrt{x} - \sqrt{x}/d) + (1 + d\sqrt{x} - \beta)(\beta - \sqrt{x}/d) = B\beta + A(\beta - \sqrt{x}/d).$$

Hence after dividing both sides of this last relation by B, we see that the coefficients of q_n match. Second, we see that $-x(1 - \sqrt{x}/d) = -x(A + B)$. Hence after dividing both sides by B, we see that the coefficients of q_{n-1} match. Therefore the induction step (3.34) is established, so the proof is complete. □

We summarize the result of Lemma 3.8 as follows. Denote the solution to v_n of Lemma 3.9 by $\widetilde{v}_n$ when d is replaced by $1/d$, that is when the roles of p and $1 - p$ are interchanged. Since x, β, and $B = \beta - (d + 1/d)\sqrt{x}$ do not change under this procedure, then it is a simple matter to transform the formula of Lemma 3.9 as follows:

$$\widetilde{v}_n = q_n + \frac{1}{B}\left(-(d\sqrt{x})^n + d\sqrt{x}q_n - xq_{n-1}\right). \tag{3.37}$$

Proposition 3.10. *Let v_n and $\widetilde{v}_n$ be given by Lemma 3.9 and* (3.37)*, respectively. Then, for all $N \geq 1$, we have, with $k_N^\circ = hc_N / \mathbb{P}\left(E_{\mathbf{c}} \cap \{1 \leq \mathbf{H} \leq N\}\right)$, that*

$$K_N^\circ = k_N^\circ r^2 z^2 \left(\frac{pv_N + (1-p)\widetilde{v}_N}{w_N} \right).$$

Proof. The proof follows by the representation (3.24), Lemma 3.6, and Lemma 3.8 after substituting the formula for σ_n and also by determining $\widetilde{\sigma}_n$ via the interchange of p and $1-p$ in the formula for σ_n, including the trivial cases $\sigma_1 = phc_N r^2 z^2$ and $\widetilde{\sigma}_1 = (1-p)hc_N r^2 z^2$. □

4. Proofs of Theorem 1.1 and Corollary 1.4

In this section we first set up how we will attack the calculation of the limiting Fourier transforms in the statements of Theorem 1.1. The indicated transforms of this theorem, written as expectations or conditional expectations under a limit as $N \to \infty$, may be rewritten by applying certain substitutions $r = r(t, N)$ and $z = z(t, N)$ in either K_N of (2.1) or K_N° of (3.1). We introduce these substitutions as follows.

$$r(t, N) = e^{-2it/N}; \qquad z(t, N) = e^{it/N}. \tag{4.1}$$

For Theorem 1.1, case (b), by the Markov property we may calculate

$$\mathbb{E}\{e^{\frac{it}{N}\Delta_N^\circ} | \Omega^\circ\} = \mathbb{E}\{e^{\frac{it}{N}(\mathbf{L}^\circ - 2\mathbf{R}^\circ)} | E_{\mathbf{c}} \cap \{1 \leq \mathbf{H} \leq N\}\}, \tag{4.2}$$

where $\mathbf{R}^\circ$ and $\mathbf{L}^\circ$ are defined in the first paragraph of section 3. Thus, for example, after rewriting the right side of (4.2) in terms of K_N° of (3.1), the meaning of the statement of Theorem 1.1(b) is that

$$\lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\Delta_N^\circ} | \Omega^\circ\} = \lim_{N\to\infty} K_N^\circ\left(r(t, N), z(t, N)\right). \tag{4.3}$$

The expressions (4.1) will be applied for all cases of Theorem 1.1.

Our method to obtain the limiting joint characteristic functions of Theorem 1.1 rests on a trigonometric substitution that goes back to [9, p. 352]. The motivation for this is that by Propositions 2.6 and 3.10 we will ultimately apply the closed formula (2.9) for q_N, which lends itself nicely to a trigonometric formulation. Introduce θ by

$$\beta = \sqrt{4x}\cos\theta, \quad \alpha = \sqrt{\beta^2 - 4x}; \quad \beta \pm \alpha = \sqrt{4x}(\cos\theta \pm i\sin\theta) = \sqrt{4x}e^{\pm i\theta}. \tag{4.4}$$

We apply (2.17), (4.1), and (4.4), after composing x and β with (4.1), to find $\cos(\theta)$ as a function of t, and N. In the following we understand without additional notation that the composition with (4.1) has been taken. We apply direct computation to expand this composition for $\cos(\theta)$, as follows. By $x = \xi z^2$, $\xi = p(1-p)h^2 = (\frac{1}{4} - \frac{b^2}{N^2})(1 - \frac{a^2}{2N^2})^2$, and $\beta = 1 + x(1-r^2) = 1 + \xi e^{-2it/N}(1 - e^{4it/N})$, we calculate $\cos\theta = \frac{1}{\sqrt{4\xi}} e^{-it/N}\left(1 + \xi(e^{2it/N} - e^{-2it/N})\right) = \frac{1}{\sqrt{4\xi}}\left(e^{-it/N} + (\frac{1}{4} + O(\frac{1}{N^2}))(e^{it/N} - e^{-3it/N})\right)$. Therefore, since by standard estimation $\frac{1}{\sqrt{4\xi}} = 1 + \frac{c^2}{2N^2} + O(\frac{1}{N^4})$, for $c^2 = a^2 + 4b^2$, we obtain $\cos\theta = \left(1 + \frac{c^2}{2N^2}\right)\left(e^{-it/N} + \frac{1}{4}(e^{it/N} - e^{-3it/N})\right) + O(\frac{1}{N^3})$. Thus, because in the Taylor expansions of the exponentials the term $it/N + \frac{1}{4}(it/N + 3it/N)$ vanishes, and since the second order terms of the exponential contributions sum to $\frac{t^2}{2N^2}\left(-1 + \frac{1}{4}(-1^2 + 3^2)\right) = \frac{t^2}{2N^2}$, we have the following:

$$\cos\theta = 1 + \frac{c^2 + t^2}{2N^2} + O(1/N^3), \quad \text{as } N \to \infty. \tag{4.5}$$

It follows that, by choosing a branch of θ so that $|e^{iN\theta}| > 1$ for large N, we have

$$\theta = \frac{-i\sqrt{c^2 + t^2}}{N} + O(1/N^2), \quad \text{as } N \to \infty. \tag{4.6}$$

Lemma 4.1. *Let x and β be defined by (2.17) and let $q_N = q_N(x, \beta)$ as defined by (2.7). Then*

$$q_N = \frac{2i}{\alpha}(\sqrt{x})^N \sin N\theta.$$

Proof. By (2.9) and (4.4), we have $q_N = \frac{2^{-N}}{\alpha}(\sqrt{4x})^N\left(e^{iN\theta} - e^{-iN\theta}\right)$, which reduces to the stated form. $\square$

We start with the proof of statement (b) of Theorem 1.1 as it depends on Proposition 3.10. The proofs of statements (a) and (c) that come afterward depend only on the development of Section 2.

Proof of Theorem 1.1(b). We must show that the limit on the right side of (4.3) equals the limit asserted in statement (b) of the theorem. By Proposition 3.10 and (4.3) we want to calculate the unnormalized term $r^2 z^2 \frac{pv_N + (1-p)\widetilde{v}_N}{w_N}$, where r and z are substituted by (4.1) and where $p = \frac{1}{2} + \frac{b}{N}$. It is easy to see that under (4.1) the coefficient $r^2 z^2 = 1 + O(\frac{1}{N})$. Therefore it suffices to establish an asymptotic expression for $(pv_N + (1-p)\widetilde{v}_N)/w_N$ that is of order N. Our method is to simply write the formulae for w_N, v_N, and $\widetilde{v}_N$ from (2.18), Lemma 3.9, and (3.37) into expressions involving the terms q_N and $\sqrt{x}q_{N-1}$, with certain coefficients that we will evaluate asymptotically by direct calculation. The reason for the factor $\sqrt{x}$ on q_{N-1} is that by Lemma 4.1 both q_N and $\sqrt{x}q_{N-1}$ have a common factor $(\sqrt{x})^N$. By Lemma 3.9, the formula for v_N is rewritten by taking $1/B$ in front of all terms as follows:

$$v_N = \frac{1}{B}\left(-(\sqrt{x}/d)^N + (\sqrt{x}/d + B)q_N - \sqrt{x}\left(\sqrt{x}q_{N-1}\right)\right), \tag{4.7}$$

where $B = \beta - (d + 1/d)\sqrt{x}$. The formula for $\widetilde{v}_N$ is the expression (4.7) with d replaced by $1/d$. By Lemma 4.1 and (4.7), since by (2.9) we have $w_N = q_N - \sqrt{x}\left(\sqrt{x}q_{N-1}\right)$, there is a common factor of $(\sqrt{x})^N$ in both the numerator and denominator of each of v_N/w_N and $\widetilde{v}_N/w_N$. To evaluate each of these fractions asymptotically, we have, as $N \to \infty$,

$$\begin{aligned}
&d = 1 - \frac{2b}{N} + O(\frac{1}{N^2}); \quad d + 1/d = 2 + \frac{4b^2}{N^2} + O(\frac{1}{N^3}); \quad \sqrt{x} = \frac{1}{2} + \frac{it}{2N} + O(\frac{1}{N^2}); \\
&\sqrt{x}/d = \frac{1}{2} + \frac{2b+it}{2N} + O(\frac{1}{N^2}); \quad d\sqrt{x} = \frac{1}{2} + \frac{-2b+it}{2N} + O(\frac{1}{N^2}) \\
&B = 1 + x(1-r^2) - (d+1/d)\sqrt{x} = \frac{a^2+t^2}{2N^2} + O(\frac{1}{N^3}); \quad \sqrt{x}/d + B = \sqrt{x}/d + O(\frac{1}{N^2});
\end{aligned} \tag{4.8}$$

Of the above only the asymptotics of B requires much attention. From our calculation of $1/\sqrt{4\xi}$ above it may be gleaned that $\sqrt{\xi} = \frac{1}{2}\left(1 - \frac{c^2}{2N^2}\right) + O(\frac{1}{N^3})$ and $\xi = \frac{1}{4}\left(1 - \frac{c^2}{N^2}\right) + O(\frac{1}{N^3})$. Therefore $B = 1 + \xi(e^{2it/N} - e^{-2it/N}) - (d+1/d)\sqrt{\xi}\left((e^{it/N}-1)+1\right)$ can be calculated by expanding $\frac{1}{4}(e^{2it/N} - e^{-2it/N}) - (e^{it/N}-1) = \frac{t^2}{2N^2} + O(\frac{1}{N^3})$. The rest of the main term for B arises from $1 - (d+1/d)\sqrt{\xi} = -\frac{2b^2}{N^2} + \frac{c^2}{2N^2} = \frac{a^2}{2N^2}$, to order $\frac{1}{N^2}$.

To asymptotically evaluate the fraction $(pv_N + (1-p)\widetilde{v}_N)/w_N$, we first focus on v_N/w_N. It follows from Lemma 4.1, (4.7), and (2.18), after dividing both numerator and denominator of $\frac{v_N}{w_N}$ by $\frac{2i}{\alpha}(\sqrt{x})^N$, that

$$\frac{v_N}{w_N} = \frac{1}{B}\frac{-\frac{\alpha}{2i}(1/d)^N + (\sqrt{x}/d + B)\sin N\theta - \sqrt{x}\sin(N-1)\theta}{\sin N\theta - \sqrt{x}\sin(N-1)\theta} = \frac{1}{B}\frac{\nu_N}{\delta_N}, \tag{4.9}$$

where $\nu_N = \frac{\alpha}{2i}(\sqrt{x})^{-N}Bv_N$ and $\delta_N = \frac{\alpha}{2i}(\sqrt{x})^{-N}w_N$. The factor $1/B$ is of order N^2. We examine the remaining fraction ν_N/δ_N in the product of (4.9). By (4.8)–(4.9) we have that

$$\frac{\nu_N}{\delta_N} = \frac{-\frac{\alpha}{2i}(\frac{1}{d})^N + \left(\frac{1}{2} + \frac{2b+it}{2N}\right)\sin N\theta - \left(\frac{1}{2} + \frac{it}{2N}\right)\sin(N-1)\theta + O(\frac{1}{N^2})}{\sin N\theta - \left(\frac{1}{2} + \frac{it}{2N}\right)\sin(N-1)\theta + O(\frac{1}{N^2})} \tag{4.10}$$

where we use implicitly that $\sin N\theta$ and $\sin(N-1)\theta$ are each $O(1)$ as $N \to \infty$ by (4.6). We claim that this numerator ν_N is of order $1/N$ and the denominator δ_N is of order 1. Since by the substitution (4.4) and (4.6) we have $\frac{\alpha}{2i} = \sqrt{x}\sin\theta = \frac{\theta}{2} + O(\frac{1}{N^2})$ is of order $1/N$, while $(1/d)^N \sim e^{2b}$, we must show how the order 1 contributions in the remaining terms of the numerator ν_N in fact cancel. We apply the angle addition formula for the sine to write $\sin(N-1)\theta = (\cos\theta)\sin N\theta - (\sin\theta)\cos N\theta$ so that the sine terms in the numerator ν_N are written

$$\begin{aligned}
&\left(\tfrac{1}{2} + \tfrac{2b+it}{2N}\right)\sin N\theta - \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)\sin(N-1)\theta \\
&= \tfrac{b}{N}\sin N\theta + \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)(1-\cos\theta)\sin N\theta + \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)(\sin\theta)\cos N\theta.
\end{aligned} \tag{4.11}$$

Now $\sin N\theta$ and $\cos N\theta$ are each of order 1 and by (4.5) we have $1 - \cos\theta = O(\frac{1}{N^2})$, and finally, by (4.6), $\sin\theta = \theta + O(\theta^3) = -\frac{i\sqrt{c^2+t^2}}{N} + O(\frac{1}{N^2})$, as $N \to \infty$. Therefore by (4.11) the numerator of (4.10) becomes

$$\nu_N = \frac{\theta}{2}\left(-e^{2b} + \cos N\theta\right) + \frac{b}{N}\sin N\theta + O(1/N^2). \tag{4.12}$$

We also have, by replacing d by $1/d$ in our calculation of ν_N of (4.9)–(4.10), by way of (4.8), that $\widetilde{\nu}_N = \frac{\alpha}{2i}(\sqrt{x})^{-N}B\widetilde{v}_N$ takes the form

$$\widetilde{\nu}_N = -\tfrac{\alpha}{2i}d^N + \left(\tfrac{1}{2} + \tfrac{-2b+it}{2N}\right)\sin N\theta - \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)\sin(N-1)\theta + O(1/N^2). \tag{4.13}$$

Therefore, by comparing ν_N of (4.10) with $\widetilde{\nu}_N$ of (4.13), we find by a wholly similar analysis upon replacing b by $-b$ in (4.11)–(4.12) that

$$\widetilde{\nu}_N = \frac{\theta}{2}\left(-e^{-2b} + \cos N\theta\right) - \frac{b}{N}\sin N\theta + O(1/N^2). \tag{4.14}$$

The denominator $\delta_N = \sin N\theta - \left(\frac{1}{2} + \frac{it}{2N}\right)\sin(N-1)\theta + O(1/N^2)$ is easily treated by expanding $\sin(N-1)\theta$ as before so that

$$\begin{aligned}\delta_N &= \left(1 - \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)\cos\theta\right)\sin N\theta + \left(\tfrac{1}{2} + \tfrac{it}{2N}\right)(\sin\theta)\cos N\theta + O(\tfrac{1}{N^2}) \\ &= \left(\tfrac{1}{2} - \tfrac{it}{2N}\right)\sin N\theta + \tfrac{1}{2}(\sin\theta)\cos N\theta + O(\tfrac{1}{N^2}),\end{aligned} \tag{4.15}$$

where we used (4.5)–(4.6). Finally, we have by (4.12), (4.14), and (4.15) that

$$\frac{\frac{1}{2}v_N + \frac{1}{2}\widetilde{v}_N}{w_N} = \frac{1}{B}\frac{\frac{1}{2}\nu_N + \frac{1}{2}\widetilde{\nu}_N}{\delta_N} = \frac{1}{B}\frac{\frac{\theta}{2}\left(\cos N\theta - \cosh(2b)\right) + O(\frac{1}{N^2})}{\frac{1}{2}\sin N\theta + O(\frac{1}{N})}, \tag{4.16}$$

where we note that the sum of the terms $\frac{1}{2B}\frac{b}{N}\sin N\theta$ and $-\frac{1}{2B}\frac{b}{N}\sin N\theta$ cancel in finding the last equality. Here the higher order expansion of δ_N in (4.15) was not applied; however it will be used in the proof Theorem 1.1(a). Now put $p = \frac{1}{2} + \frac{b}{N}$ and compute

$$\frac{pv_N(1-p)\widetilde{v}_N}{w_N} = \frac{1}{B}\frac{p\nu_N + (1-p)\widetilde{\nu}_N}{\delta_N} = \frac{1}{B}\frac{\frac{1}{2}\nu_N + \frac{1}{2}\widetilde{\nu}_N}{\delta_N} + O(1), \tag{4.17}$$

where the $O(1)$ error term comes about from $\frac{1}{B}\frac{b}{N}(\nu_N - \widetilde{\nu}_N)/\delta_N$, since $\frac{1}{B}$ is of order N^2 so that by (4.12) and (4.14) each of ν_N/B and $\widetilde{\nu}_N/B$ are of order $N^2\frac{1}{N} = N$. Here by (4.6) we asymptotically evaluate that $\delta_N \sim \frac{1}{2}\sin(-i\sqrt{c^2+t^2}) = -\frac{i}{2}\sinh\sqrt{c^2+t^2}$. Note that also $\cos N\theta \sim \cosh\sqrt{c^2+t^2}$, while θ is given by (4.6) and B is given by (4.8), so that finally by (4.16)–(4.17) we have, as $N \to \infty$

$$\frac{pv_N + (1-p)\widetilde{v}_N}{w_N} \sim \frac{1}{B}\frac{\frac{1}{2}\nu_N + \frac{1}{2}\widetilde{\nu}_N}{\delta_N} \sim \frac{2\sqrt{c^2+t^2}\left(\cosh\sqrt{c^2+t^2} - \cosh(2b)\right)}{(a^2+t^2)\sinh\sqrt{c^2+t^2}}N. \tag{4.18}$$

Therefore by Proposition 3.10 and (4.18) we have that $K_N^\circ(r,z)/k_N^\circ$ is asymptotic to the right side of (4.18). Hence by $K_N^\circ[\mathbf{1}] = 1$ we have, after setting $t = 0$, that the normalization constant k_N° in Proposition 3.10 satisfies

$$1/k_N^\circ = \frac{\mathbb{P}\left(E_{\mathbf{c}} \cap \{1 \le \mathbf{H} \le N\}\right)}{hc_N} \sim \frac{2c\left(\cosh(c) - \cosh(2b)\right)}{a^2\sinh(c)}N, \quad \text{as } N \to \infty. \tag{4.19}$$

It follows by (4.19) and the definitions of h and c_N that

$$\mathbb{P}\left(E_{\mathbf{c}} \cap \{1 \le \mathbf{H} \le N\}\right) \sim \frac{c\left(\cosh(c) - \cosh(2b)\right)}{\sinh(c)}\frac{1}{N}, \quad \text{as } N \to \infty. \tag{4.20}$$

Furthermore, the limit of the right side of (4.3) is given as in the statement of Theorem 1.1(b). □

Proof of Theorem 1.1(a). We have that the number of excursions $\mathcal{M}_N$ until the last visit epoch $\mathcal{L}_N$ is a geometric variable with distribution $\mathbb{P}(\mathcal{M}_N = \ell) = \pi_0^\ell(1-\pi_0)$, $\ell = 0, 1, \dots$, where we denote $\pi_0 = \mathbb{P}(E_0 \cap \{1 \le \mathbf{H} \le N\})$. Therefore by (2.1), we have that the joint probability generating function $\mathbb{E}\left(r^{\mathcal{R}_N}z^{\mathcal{L}_N}u^{\mathcal{M}_N}\right)$ is given by

$$(1-\pi_0)\sum_{\ell=0}^{\infty}(\pi_0 u)^\ell\,\mathbb{E}\left\{r^{\mathcal{R}_N}z^{\mathcal{L}_N}|E_0 \cap \{\mathbf{H} \le N\}\right\}^\ell = \frac{1-\pi_0}{1-\pi_0 uK_N(r,z)}, \tag{4.21}$$

since, by the Markov property, $\mathbb{E}\{r^{\mathcal{R}_N}z^{\mathcal{L}_N}|\mathcal{M}_N = \ell\} = \mathbb{E}\{r^{\mathbf{R}}z^{\mathbf{L}}|E_0 \cap \{1 \le \mathbf{H} \le N\}\}^\ell$. Now we substitute (4.1) into (4.21), and we also substitute

$$u = u(t,N) = e^{it/N}. \tag{4.22}$$

Thus by Proposition 2.6, since the normalization constant k_N of K_N satisfies $\pi_0 k_N = 2\xi$, we have

$$\mathbb{E}\{e^{\frac{it}{N}\Delta_N}\} = \frac{1-\pi_0}{1-\pi_0 u\cdot K_N(r,z)} = \frac{(1-\pi_0)w_N}{w_N - 2\xi ur^2z^2q_N}, \tag{4.23}$$

where it is understood that in this last expression r, z, u, w_N and q_N are all composed with (4.1) and (4.22). We evaluate the coefficient of q_N for the term $2\xi ur^2z^2q_N$ asymptotically as follows. We have, as $N\to\infty$,

$$2\xi ur^2z^2 = 2\xi e^{-\frac{it}{N}} = \tfrac{1}{2} - \tfrac{it}{2N} + O(\tfrac{1}{N^2}). \tag{4.24}$$

As in the proof of Theorem 1.1(b) write $\delta_N = \frac{\alpha}{2i}(\sqrt{x})^{-N}w_N$, and introduce $\epsilon_N = \frac{\alpha}{2i}(\sqrt{x})^{-N}q_N$. We divide both the numerator and denominator of the last ratio in (4.23) by $\frac{2i}{\alpha}(\sqrt{x})^N$ to write

$$\mathbb{E}\{e^{\frac{it}{N}\Delta_N}\} = \frac{(1-\pi_0)\delta_N}{\delta_N - 2\xi ur^2z^2\epsilon_N}. \tag{4.25}$$

Now by (4.24) and Lemma 4.1 we have that

$$2\xi ur^2z^2\epsilon_N = \left(\tfrac{1}{2} - \tfrac{it}{2N}\right)\sin N\theta + O(1/N^2). \tag{4.26}$$

From the expressions (4.15) and (4.26) we find a cancellation of the $\left(\frac{1}{2} - \frac{it}{2N}\right)\sin N\theta$ terms such that

$$\delta_N - 2\xi ur^2z^2\epsilon_N = \frac{1}{2}(\sin\theta)\cos N\theta + O(1/N^2).$$

Therefore by (4.25) and the result of (4.15) for the numerator we have

$$\mathbb{E}\{e^{\frac{it}{N}\Delta_N}\} = \frac{(1-\pi_0)\left(\frac{1}{2}\sin N\theta + O(\frac{1}{N})\right)}{\frac{1}{2}(\sin\theta)\cos N\theta + O(1\frac{1}{N^2})}. \tag{4.27}$$

Now plug in (4.6) to (4.27) and multiply the top and bottom of the fraction by N to find that

$$\lim_{N\to\infty}\mathbb{E}\{e^{\frac{it}{N}\Delta_N}\} = \lim_{N\to\infty}\frac{N(1-\pi_0)\sinh\sqrt{c^2+t^2}}{\sqrt{c^2+t^2}\cosh\sqrt{c^2+t^2}}. \tag{4.28}$$

By setting $t=0$ we conclude by (4.28) that

$$\lim_{N\to\infty} N(1-\pi_0)\frac{\tanh(c)}{c} = 1; \quad \text{that is, } \pi_0 \sim 1 - \frac{c}{N\tanh(c)}, \quad \text{as } N\to\infty. \tag{4.29}$$

Therefore by (4.28)–(4.29) the proof of Theorem 1.1(a) is complete. □

Discussion. Since $\mathbb{P}(E_0)+\mathbb{P}(E_{\mathbf{c}})+\mathbb{P}(E_N)=1$, we have by (4.20) and (4.29) that

$$\mathbb{P}(E_N) \sim \frac{1}{N}\frac{c\cdot\cosh(2b)}{\sinh(c)}, \quad \text{as } N\to\infty. \tag{4.30}$$

Proof of Theorem 1.1(c). The event E_N consists of paths that start from $m=0$ and either stay strictly positive after the starting point until they reach level $N+1$ or else stay strictly negative after the starting point and reach level $-(N+1)$. The joint probability generating function of runs, short runs, and steps for all paths that exit one of the boundaries $\pm(N+1)$ on the first excursion attempt is zg_N. Therefore by the Markov property, we have, under the substitution (4.1), that

$$\mathbb{E}\left\{e^{\frac{it}{N}\Delta'_N}|\Omega'\right\} = zg_N. \tag{4.31}$$

Recall that by Proposition 2.3 we have $g_n = C_n rz^n/w_n$, where $C_n = w_n[\mathbf{1}]$. Therefore, by $z=\sqrt{x/\xi}$, we have

$$zg_N = C_N\frac{rz^{N+1}}{w_N} = rz\frac{C_N}{(\sqrt{\xi})^N}\frac{(\sqrt{x})^N}{w_N}. \tag{4.32}$$

Now apply (2.9) to find $w_N = q_N - \sqrt{x}(\sqrt{x}q_{N-1})$, where we have both the coefficient $\sqrt{x} = \frac{1}{2} + O(1/N)$ from (4.8), and also $rz = 1 + O(1/N)$, as $N \to \infty$. Therefore by Lemma 4.1 we have from (4.32), after dividing top and bottom of the fraction by $\frac{2i}{\alpha}(\sqrt{x})^N$, that

$$zg_N = \frac{C_N}{(\sqrt{\xi})^N} \frac{\frac{\alpha}{2i}}{\sin N\theta - \frac{1}{2}\sin(N-1)\theta + O(\frac{1}{N})}. \tag{4.33}$$

Therefore, since by (4.6) we have $\frac{\alpha}{2i} = \frac{1}{2}\sin\theta \sim \frac{1}{2}\theta \sim -\frac{i\sqrt{c^2+t^2}}{2N}$, as $N \to \infty$, and since by (4.15) we have $\sin N\theta - \frac{1}{2}\sin(N-1)\theta \sim \frac{1}{2}\sin N\theta \sim -\frac{i}{2}\sinh\sqrt{c^2+t^2}$, as $N \to \infty$, we have by (4.31)–(4.33) that

$$\lim_{N\to\infty} \mathbb{E}\left\{e^{\frac{it}{N}\Delta'_N}|\Omega'\right\} = \lim_{N\to\infty} \frac{C_N}{N(\sqrt{\xi})^N} \frac{\sqrt{c^2+t^2}}{\sinh\sqrt{c^2+t^2}}. \tag{4.34}$$

By setting $t = 0$ in (4.34) we have

$$\lim_{N\to\infty} \frac{C_N}{N(\sqrt{\xi})^N} = \frac{\sinh(c)}{c}. \tag{4.35}$$

Therefore by (4.34)–(4.35) the proof of Theorem 1.1(c) is complete. □

Proof of Theorem 1.1(d). Write $\pi' = \mathbb{P}(E_N)$ and recall that $\pi_0 = \mathbb{P}(E_0 \cap \{1 \le \mathbf{H} \le N\})$ is asymptotically evaluated by (4.29). Thus calculate by (4.29) and (4.30) that, as $N \to \infty$,

$$\mathbb{P}(\Omega') = \pi' + \pi_0\pi' + \pi_0^2\pi' + \cdots = \tfrac{\pi'}{1-\pi_0} \sim \tfrac{c\cdot\cosh(2b)}{\sinh(c)}\tfrac{\tanh(c)}{c} = \tfrac{\cosh(2b)}{\cosh(c)}; \tag{4.36}$$

that is, $\lim_{N\to\infty} \mathbb{P}(\Omega') = \frac{\cosh(2b)}{\cosh(c)}$.

Now denote the limits of Theorem 1.1, parts (a)–(c), respectively as $T(t)$, $U(t)$, and $V(t)$. Then, by independence between Δ_N and $\Delta_N^\circ \cdot 1_{\Omega^\circ} + \Delta'_N \cdot 1_{\Omega'}$, we have by (4.36) and $\mathbb{P}(\Omega^\circ) = 1 - \mathbb{P}(\Omega')$ that $\lim_{N\to\infty} \mathbb{E}\{e^{\frac{it}{N}\boldsymbol{\Delta}_N}\}$ is written by

$$\begin{aligned} &T(t) \lim_{N\to\infty} \Big(\mathbb{P}(\Omega^\circ)\,\mathbb{E}\{e^{\frac{it}{N}\Delta_N^\circ}|\Omega^\circ\} + \mathbb{P}(\Omega')\,\mathbb{E}\{e^{\frac{it}{N}\Delta'_N}|\Omega'\}\Big) \\ &= T(t)\left(\tfrac{\cosh(c)-\cosh(2b)}{\cosh(c)}U(t) + \tfrac{\cosh(2b)}{\cosh(c)}V(t)\right). \end{aligned} \tag{4.37}$$

With a bit of algebra after substituting the expressions for $T(t)$, $U(t)$, and $V(t)$ from the statement of Theorem 1.1(a)–(c) into the right side of (4.37), we obtain that the limit of part (d) is $\left(\frac{a^2(\cosh\sqrt{c^2+t^2}-\cosh(2b))}{a^2+t^2} + \cosh(2b)\right)/\cosh\sqrt{c^2+t^2} = \frac{a^2\cosh\sqrt{c^2+t^2}+t^2\cosh(2b)}{(a^2+t^2)\cosh\sqrt{c^2+t^2}}$. □

4.1. **Limiting Univariate Laplace transforms with scaling by N^2.**

Proof of Corollary 1.4. We focus first on statement (b) of the corollary. We start with the runs statistic and then turn to the other two statistics. For all cases in which we study a runs statistic alone we set $r = r(\lambda, N) = e^{-\lambda/N^2}$, and $z = 1$. Define β and x composed with these substitutions according to (2.17). We define $\cos\theta$ again by (4.4) so that $\cos\theta$ is a function of $\lambda \ge 0$ and $N \ge 1$. By direct computation we have

$$\cos\theta = 1 + \frac{c^2+\lambda}{2N^2} + O(1/N^4), \quad \text{as } N \to \infty. \tag{4.38}$$

We choose a branch of θ so that $|e^{iN\theta}| > 1$ for large N, so by (4.38) we have

$$\theta = \frac{-i\sqrt{c^2+\lambda}}{N} + O(1/N^3), \quad \text{as } N \to \infty. \tag{4.39}$$

By direct computation we rewrite (4.8) in the current context as follows:

$$\begin{aligned}&B=\tfrac{a^2+\lambda}{2N^2}+O(\tfrac{1}{N^4});\quad d=1-\tfrac{2b}{N}+O(\tfrac{1}{N^2});\quad \sqrt{x}=\tfrac12+O(\tfrac{1}{N^2});\\ &\sqrt{x}/d=\tfrac12+\tfrac{b}{N}+O(\tfrac{1}{N^2});\quad d\sqrt{x}=\tfrac12-\tfrac{b}{N}+O(\tfrac{1}{N^2});\quad \sqrt{x}/d+B=\sqrt{x}/d+O(\tfrac{1}{N^2});\end{aligned} \tag{4.40}$$

Recall the formula (4.7) for v_N. By (2.9) we have $w_N = q_N - \sqrt{x}\,(\sqrt{x}q_{N-1})$. We now again compute an asymptotic expression for $v_N/w_N = \frac{1}{B}(\nu_N/\delta_N)$ of (4.9), this time under (4.38)–(4.40). We apply the angle addition formula for the sine as in (4.11) and obtain again a cancellation of order 1 terms for the sine terms $(\sqrt{x}/d + B)\sin N\theta - \sqrt{x}\sin(N-1)\theta$ of the numerator ν_N, now under (4.40), as follows: $(\frac12+\frac{b}{N})\sin N\theta - \frac12\sin(N-1)\theta + O(\frac{1}{N^2}) = \frac12\sin\theta\cos N\theta + \frac{b}{N}\sin N\theta + O(\frac{1}{N^2})$. Therefore, just as in the analysis that yields (4.12) and (4.14), these equations continue to hold verbatim in the current context since θ of (4.39) is still of order $1/N$. Hence by (4.12) and (4.14) we have

$$p\nu_N + (1-p)\widetilde{\nu}_N = \frac{\theta}{2}\left(\cos N\theta - \cosh(2b)\right) + O(1/N^2), \tag{4.41}$$

where again the contributions of $+\frac{b}{N}\sin N\theta$ and $-\frac{b}{N}\sin N\theta$ cancel in the linear combination $p\nu_N + (1-p)\widetilde{\nu}_N$ to order $1/N^2$. The denominator $\delta_N = \sin N\theta - \sqrt{x}\sin(N-1)\theta$ of (4.9) is simply given, again by the angle addition formula for the sine and (4.40), by $\delta_N = \left(1-\frac12\cos\theta\right)\sin N\theta + \frac12(\sin\theta)\cos N\theta + O(\frac{1}{N^2})$, so that by (4.38)–(4.39),

$$\delta_N = \frac12\sin N\theta + \frac12(\sin\theta)\cos N\theta + O(1/N^2). \tag{4.42}$$

Therefore by Proposition 3.10, (4.41), this expression for δ_N, and the asymptotical expansions of θ and B in (4.39)–(4.40), we have that $\lim_{N\to\infty}\mathbb{E}\{e^{-\lambda\frac{1}{N^2}\mathcal{R}_N^\circ} \mid \Omega^\circ\}$ is given as

$$\lim_{N\to\infty} k_N^\circ \frac{1}{B}\frac{p\nu_N + (1-p)\widetilde{\nu}_N}{\delta_N} = C_{a,b}\frac{\sqrt{c^2+\lambda}\left(\cosh\sqrt{c^2+\lambda} - \cosh(2b)\right)}{(a^2+\lambda)\sinh\sqrt{c^2+\lambda}}, \tag{4.43}$$

since by (4.19) we have $\lim_{N\to\infty} N k_N^\circ = C_{a,b}$. Therefore the proof of part (b) of the corollary is complete for the case of the runs statistic.

We now briefly discuss the case of the steps statistic $\mathcal{L}_N^\circ$ for part (b) of the corollary. For this case we put $r=1$ and $z = e^{-\frac12\lambda/N^2}$. Even though now β and x composed with these values of r and z are no longer the same as for the case of the runs statistic, it turns out that there is only a difference starting from the order $1/N^2$ term in the expansions of these quantities. Also the value of B matches the case of the runs statistic through order $1/N^2$, and only differs starting from order $1/N^4$. In fact (4.40) continues to hold verbatim for the steps statistic. Moreover we have that (4.38)–(4.39) hold. Therefore, by the same lines of proof as for the case of runs, the proof of part (b) is complete.

We turn to the proof of statement (a). For the runs statistic, by the poof of part (a) of Theorem 1.1, by rewriting (4.23) with $u=1$ and w_N and q_N composed with $r = e^{-\lambda/N^2}$ and $z=1$ we have that $\lim_{N\to\infty}\mathbb{E}\{e^{-\lambda\mathcal{R}/N^2}\}$ is computed as

$$\lim_{N\to\infty}\frac{1-\pi_0}{1-\pi_0 K_N} = \lim_{N\to\infty}\frac{(1-\pi_0)w_N}{w_N - 2\xi r^2 z^2 q_N} = \lim_{N\to\infty}\frac{(1-\pi_0)\delta_N}{\delta_N - 2\xi r^2 z^2 \epsilon_N}, \tag{4.44}$$

with $\epsilon_N = \frac{\alpha}{2i}(\sqrt{x})^{-N} q_N$. For either the runs or steps statistics of the corollary, due to scaling by N^2 and $u=1$, instead of (4.24) we now have

$$2\xi u r^2 z^2 = \tfrac12 + O(1/N^2). \tag{4.45}$$

Therefore by (4.42) and (4.45) we have that the denominator of the last limit in (4.44), namely $\delta_N - 2\xi r^2 z^2 \epsilon_N$, is given by

$$\tfrac{1}{2}\sin N\theta + \tfrac{1}{2}(\sin\theta)\cos N\theta - \tfrac{1}{2}\sin\theta + O(\tfrac{1}{N^2}) = \tfrac{1}{2}(\sin\theta)\cos N\theta + O(1/N^2). \tag{4.46}$$

Now plug in $\delta_N \sim \frac{1}{2}\sin N\theta$ in the numerator and $\delta_N - 2\xi r^2 z^2 \epsilon_N \sim \frac{1}{2}\sin\theta\cosh N\theta$ for the denominator of (4.44). Here, as shown in (4.29), $(1-\pi_0)$ has order $1/N$ to match the order $1/N$ of $\sin\theta$. Therefore by way of (4.39) and (4.46) plugged into (4.44), the proof of part (a) is complete.

The proof of part (c) follows exactly as in Theorem 1.1(c) since we merely take the limit as $N\to\infty$ in (4.33), where the denominator of that display is by (4.40) asymptotically $\frac{1}{2}\sin N\theta \sim -i\sinh\sqrt{c^2+\lambda}$. The proof of part (d) follows by algebra as in the proof of Theorem 1.1(d). $\square$

4.2. **Example for Theorem 1.1.** In this section we compute in Example 4.2 the limiting measures for the statistics of Theorem 1.1 parts (a) and (b). Consider $\frac{1}{N}\Delta_N^\circ$ given Ω°. By Theorem 1.1(b), after writing $\cosh(2b) = 1 - 1 + \cosh(2b)$, and separating terms we have that

$$\lim_{N\to\infty}\mathbb{E}\{e^{\frac{it}{N}\Delta_N^\circ} \mid \Omega^\circ\} = \tfrac{C_{a,b}\sqrt{c^2+t^2}}{a^2+t^2}\left(\tanh\tfrac{1}{2}\sqrt{c^2+t^2} + \tfrac{(1-\cosh(2b))}{\sinh\sqrt{c^2+t^2}}\right). \tag{4.47}$$

Here we have applied the trigonometric identity for $\tanh\frac{\varphi}{2}$ of Remark 1.2 and $C_{a,b}$ is defined in Theorem 1.1. By the uniqueness and continuity theorems, [3, Sect. 26], there is a unique probability measure $\mu_{a,b}$ such that its characteristic function $\widehat{\mu}_{a,b}(t) = \int\limits_{-\infty}^{\infty} e^{itx}\,\mu_{a,b}(dx)$ is given by (4.47). Recall the following well known Mittag–Leffler expansions:

$$\frac{\tanh(u)}{u} = \sum_{n=0}^{\infty}\frac{8}{(2n+1)^2\pi^2+4u^2}; \quad \frac{u}{\sinh(u)} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n u^2}{n^2\pi^2+u^2}. \tag{4.48}$$

Therefore by (4.47)–(4.48) we have that

$$\widehat{\mu}_{a,b}(t) = \frac{C_{a,b}}{a^2+t^2}\left(\sum_{n=0}^{\infty}\frac{4(c^2+t^2)}{(2n+1)^2\pi^2+c^2+t^2} + (1-\cosh(2b))\sum_{n=-\infty}^{\infty}\frac{(-1)^n(c^2+t^2)}{n^2\pi^2+c^2+t^2}\right). \tag{4.49}$$

Now define

$$s_k(x) = \frac{\frac{4b^2}{a}e^{-a|x|} + \frac{k^2\pi^2}{\sqrt{c^2+k^2\pi^2}}e^{-|x|\sqrt{c^2+k^2\pi^2}}}{4b^2+k^2\pi^2}, \quad \text{for all } -\infty < x < \infty,\ k\in\mathbb{Z}. \tag{4.50}$$

Here $s_k(x)$ has been chosen such that, by direct calculation,

$$\widehat{s}_k(t) = \int_{-\infty}^{\infty} e^{itx}s_k(x)\,dx = \frac{2(c^2+t^2)}{(a^2+t^2)(k^2\pi^2+c^2+t^2)}. \tag{4.51}$$

By the monotone convergence theorem and (4.51) with $t=0$, we have that $\sum\limits_{k=0}^{\infty} s_k(x)$ is integrable on $\mathbb{R}$. Hence $\int\limits_{-\infty}^{\infty} e^{itx}\sum\limits_{n=0}^{\infty} s_{2n+1}(x)\,dx = \sum\limits_{n=0}^{\infty}\widehat{s}_{2n+1}(x)\,dx$. By the dominated convergence theorem we justify also the term by term calculation of $\int\limits_{-\infty}^{\infty} e^{itx}\sum\limits_{n=-\infty}^{\infty}(-1)^n s_n(x)\,dx$.

Example 4.2. *Let $s_k(x)$ be defined by (4.50). Define $f_{a,b}(x)$ as follows.*

$$f_{a,b}(x) = 2C_{a,b}\sum_{n=0}^{\infty}s_{2n+1}(x) + \frac{1}{2}C_{a,b}(1-\cosh(2b))\sum_{n=-\infty}^{\infty}(-1)^n s_n(x). \tag{4.52}$$

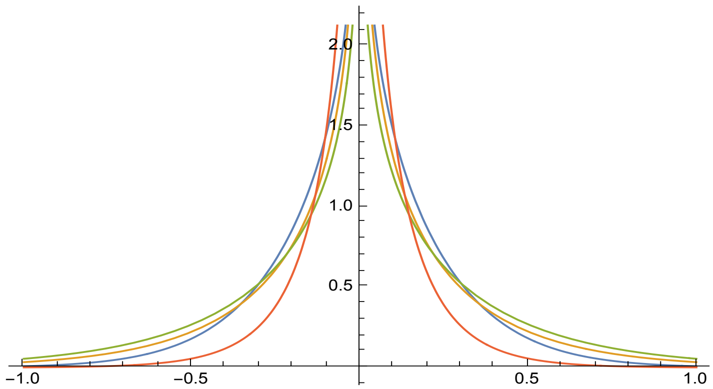

FIGURE 4. The densities $f_{a,b}(x)$ of Example 4.2 such that the characteristic function $\widehat{f}_{a,b}(x) = \int_{-\infty}^{\infty} e^{itx} f_{a,b}(x)\, dx$ is given by (4.47), with $c^2 = a^2 + 4b^2$. Green: $f_{0,0}(x) = \frac{4}{\pi}\text{arctanh}(e^{-\pi|x|})$; Gold: $f_{2,0}(x)$; Blue: $f_{2,3}(x)$; Red: $f_{\sqrt{40},0}(x)$.

By (4.50)–(4.51) *and the discussion following these displays we have shown that indeed the characteristic function* $\widehat{f}_{a,b}(t) = \int\limits_{-\infty}^{\infty} e^{itx} f_{a,b}(x)\, dx$ *is given by the right side of* (4.49) *and so by the form* (4.47) *of Theorem 1.1(b). Therefore the probability measure* $\mu_{a,b}$ *is absolutely continuous and satisfies* $\mu_{a,b}(dx) = f_{a,b}(x)\, dx$. *In case* $b = 0$, *the formula* (4.52) *reduces to* $f_{a,0}(x) = 2C_{a,0} \sum\limits_{n=0}^{\infty} \frac{e^{-|x|\sqrt{a^2+(2n+1)^2\pi^2}}}{\sqrt{a^2+(2n+1)^2\pi^2}}$. *If in addition* $a = 0$, *then we simply obtain* $f_{0,0}(x) = \frac{4}{\pi} \sum\limits_{n=0}^{\infty} \frac{e^{-(2n+1)\pi|x|}}{2n+1} = \frac{4}{\pi}\text{arctanh}(e^{-\pi|x|})$. *It is a curious fact that the density* $g(x)$ *on the half–line defined by* $g(x) = f_{0,0}(x/2)$, $x > 0$, *is its own inverse. Hence there is a logarithmic singularity at* $x = 0$ *for the green curve* $f_{0,0}$ *in Figure 4. Let* μ *be the probability measure with the limiting characteristic function* $\widehat{\mu}(t) = \frac{c}{\tanh(c)} \frac{\tanh\sqrt{c^2+t^2}}{\sqrt{c^2+t^2}}$ *of* (1.3)*(a). By direct calculation we have that* $\int\limits_{-\infty}^{\infty} e^{itx} \cdot \frac{e^{-\frac{1}{2}|x|\sqrt{(2n+1)^2\pi^2+4c^2}}}{\sqrt{(2n+1)^2\pi^2+4c^2}}\, dx = \frac{4}{(2n+1)^2\pi^2+4(c^2+t^2)}$. *Therefore by the dominated convergence theorem and* (4.48) *we have that*

$$\widehat{\mu}(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} f_{2c,0}(x/2)\, dx; \quad \text{that is, } \mu(dx) = \frac{1}{2} f_{2c,0}(x/2)\, dx.$$

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