SOME PROBABILITY DISTRIBUTIONS AND INTEGER SEQUENCES RELATED TO ROOK PATHS

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Abstract. Let $\{\mathbf{S}_j, j \geq 0\}$ be a simple random walk with $\mathbf{S}_0 = 0$ and let $\mathbf{R}$ denote the number of runs in the first excursion path. We condition $\mathbf{R}$ on the event that the excursion is positive, there are at least 4 runs (two peaks), and the first minimum has level at least ℓ for some $\ell \geq 1$; we call $\mathbf{R}_\ell$ the corresponding random number of runs. We establish an explicit form of the probability generating function $E\{x^{\mathbf{R}_\ell}\}$. For $a > 0$ we find the limiting Laplace transform $\lim_{N\to\infty} E\{\exp(-\lambda N^{-2}\mathbf{R}_{\lfloor aN \rfloor})\} = e^{-a\sqrt{\lambda}}$. We show that $r_\ell(n) = 2^{\ell-2}3^{2n+2}P(\mathbf{R}_\ell = 2(n+2))$, $n \geq 0$, is an integer sequence with the property that r_ℓ is an ℓ–fold convolution of integer sequences based on rook numbers.

1. Introduction

Our motivation stems from an observation that connects the probability distribution of the number of *runs* in an *excursion* of simple random walk with the number $r(n)$, $n \geq 0$, of Catalan *rook* paths enumerated by $1, 1, 5, 29, 185, 1257, \ldots$, [10, A059231]. Here $r(n)$ is defined as the number of lattice paths from $(0, 0)$ to $(2n, 0)$ in $\mathbb{Z}^2$ using steps from $S = \{(k, k) : k \geq 1\} \cup \{(k, -k) : k \geq 1\}$ that never cross the x–axis. The ordinary generating function of the sequence is determined as $\sum_{n=0}^{\infty} r(n)t^n = (8t)^{-1}(1+3t-\sqrt{1-10t+9t^2})$, [2, Theorem 2.2], [6, Theorem 6]. Let $\{\mathbf{S}_j, j \geq 0\}$ be the simple random walk on the integers with $\mathbf{S}_0 = 0$. Denote $\mathbf{L} = \inf\{j \geq 2 : \mathbf{S}_j = 0\}$ as the epoch of first return of this random walk to the origin and let $\mathbf{R}$ denote the number of *runs*, or one more than the number of *turns* in the excursion lattice path $\{(j, \mathbf{S}_j), j = 0, \ldots, \mathbf{L}\}$. The probability generating function of $\mathbf{R}$ is

$$E\{x^{\mathbf{R}}\} = \frac{1}{4}(3 + x^2 - \sqrt{x^4 - 10x^2 + 9}), \tag{1.1}$$

[8, p. 2021]. The observation alluded to above, and proved by means of the Legendre function, is that

$$r(n) = \frac{1}{2}3^{2n+1}P(\mathbf{R} = 2(n+1)), \tag{1.2}$$

[8, Cor. 3, Remark 4]. We therefore have a probabilistic interpretation of $r(n)$. By [2, Theorem 2.2(b)] the rook numbers for $n \geq 1$ are given by $r(n) = \sum_{k=1}^{n} 4^{n-k}N(n, k)$ where $N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ are the Narayana numbers [10, A001263], which in turn enumerate Dyck paths of semi length n and k *peaks* that we shall describe below; see [3, 4, 11]. Thus $r(n)$ is a central figure among integer sequences related to the combinatorics of Dyck paths. In this paper we expand on the phenomenon emulated by (1.1)–(1.2). We find explicit generating functions for a sequence of probability distributions obtained by conditioning $\mathbf{R}$, along with convolution type descriptions of new integer sequences that arise from these distributions.

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To introduce our approach, we define a nearest neighbor path of *length* n in $\mathbb{Z}$ to be a finite sequence $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$, where $\Gamma_j \in \mathbb{Z}$ and $\varepsilon_j := \Gamma_j - \Gamma_{j-1}$ satisfies $|\varepsilon_j| = 1$ for all $j = 1, \ldots, n$. We connect successive lattice points $(j-1, \Gamma_{j-1})$ and (j, Γ_j) in the plane by straight line segments, and term this connected union of straight line segments the *lattice path*; see for example Figure 2. We define the number of runs along Γ as the number of inclines, either straight line ascents or descents, of maximal extent along the lattice path; in Figure 2 there are 6 runs in the lattice path Γ. An *excursion* is a nearest neighbor path that starts and ends at $m = 0$, $\Gamma_0 = \Gamma_n = 0$, but for which $\Gamma_j \neq 0$ for $1 \le j \le n-1$. A *positive* excursion is an excursion whose graph lies above the x–axis save for its endpoints. For a positive excursion path, the number of runs is just twice the number of peaks, where a peak at lattice point (j, Γ_j) corresponds to $\varepsilon_j = +1$ and $\varepsilon_{j+1} = -1$; a *valley* corresponds instead to $\varepsilon_j = -1$ and $\varepsilon_{j+1} = +1$. Alternatively, for any excursion path the number of runs is one more than the number of turns where a turn occurs at (j, Γ_j) means $\varepsilon_{j+1} = -\varepsilon_j$, that is there is a change of direction of Γ. In the positive excursion of Figure 2 there are 3 peaks, 2 valleys, and 5 turns.

We mainly consider lattice paths Γ such that $\Gamma_0 = \Gamma_n = 0$. One such type of lattice path is a Dyck path, [3, 4, 11], which satisfies in addition $\Gamma_j \ge 0$ for all $1 \le j \le n-1$. A positive excursion path is a special case of a Dyck path with zeros of the path only at the endpoints. Even though the length of any given lattice path is finite, we consider certain infinite sets of such paths by letting the length parameter n be arbitrary because we now *weight* a path of length n by the Bernoulli probability 2^{-n}. The event $\{\mathbf{R} = 2(k+1)\}$ defined above in terms of the simple random walk can be regarded as a collection of paths Γ that have exactly $2(k+1)$ runs along them and start and terminate on the x–axis. We apply the Bernoulli weight to each such path to calculate the probability: $P(\mathbf{R} = 2(k+1)) = \sum_\Gamma 2^{-|\Gamma|}$, where $|\Gamma|$ denotes the length of $\Gamma \in \{\mathbf{R} = 2(k+1)\}$. For any positive excursion path Γ with at least two peaks the path begins with an ascent followed by a descent, and at the end of this first descent we have the *initial minimum* height $\mathbf{m} = \mathbf{m}(\Gamma)$. Conditional on $\mathbf{S}_1 = +1$, the first excursion from zero of a simple random walk is a positive excursion. Conditional also that this positive excursion has at least 2 peaks, that is $\mathbf{R} \ge 4$, the initial minimum (random variable) $\mathbf{m}$ is well defined.

Definition 1.1. *We conditionally define the random variable $\mathbf{R}_\ell$ as the number of runs $\mathbf{R}$ in the first excursion of the random walk $\{\mathbf{S}_n\}$ given that $\mathbf{S}_1 = +1$, $\mathbf{R} \ge 4$, and $\mathbf{m} \ge \ell$. Denote the probability generating function of $\mathbf{R}_\ell$ by:*

$$r(x;\ell) = E\{x^{\mathbf{R}} \mid \mathbf{S}_1 = +1, \mathbf{R} \ge 4, \mathbf{m} \ge \ell\} = \sum_{n=0}^{\infty} \rho_\ell(n) x^{2(n+2)},$$

where $\rho_\ell(n) = P(\mathbf{R} = 2(n+2) \mid \mathbf{S}_1 = +1, \mathbf{R} \ge 4, \mathbf{m} \ge \ell)$.

Theorem 1.2. *Recall the probability generating function $f(x)$ of (1.1) for the number of runs $\mathbf{R}$ in an excursion of a simple random walk. For all $|x| \le 1$ the probability generating function $r(x;\ell)$ of Definition 1.1 is given by*

$$r(x;\ell) = 4x^2 12^{-\ell} f(x) \frac{(3 + x^2 + \sqrt{x^4 - 10x^2 + 9})^{2\ell-1}}{(3 - 5x^2/3 + \sqrt{x^4 - 10x^2 + 9})^{\ell}}. \tag{1.3}$$

Denote $\mathbf{T}_N = \mathbf{T}_{N,a} = N^{-2}\mathbf{R}_{\lfloor aN \rfloor}$ for $a > 0$ and $\mathbf{R}_\ell$ given by Definition 1.1. By calculating the limiting Laplace transform of $\mathbf{T}_N$ as $N \to \infty$ we obtain the following result of Theorem 1.2.

Corollary 1.3. *For each $\lambda \ge 0$ and $a > 0$ we have $\lim_{N\to\infty} E\{e^{-\lambda \mathbf{T}_{N,a}}\} = e^{-a\sqrt{\lambda}}$. Hence $\mathbf{T}_N$ converges in distribution to a limit that has probability density $\frac{a}{2\sqrt{\pi}} t^{-3/2} e^{-a^2/4t}$, $t > 0$.*

We shall obtain Theorem 1.2 by applying a recurrence relation (4.9) involving probabilities $\alpha_\ell(n)$ and $\delta(n)$ that we now introduce. We define for each $k \geq 0$ the event D_k, that is a collection of lattice paths, by

$$D_k = \bigcup_{n \in 2\mathbb{N}} \{\text{paths } \Gamma \text{ with } \varepsilon_1 = +1,\ \varepsilon_n = -1,\ \Gamma_0 = \Gamma_n = 0, \text{ and exactly } 2(k+1) \text{ runs}\}. \quad (1.4)$$

In words, for each path of D_k the first step is positive and the last step is negative and by assumption $\Gamma_0 = \Gamma_n = 0$, but the length n of the path is not fixed though it must be even; only the number of runs along the path is fixed. Notice that the paths in D_k are *not* restricted to be Dyck paths. An alternative description of the lattice paths in D_k is to rotate each path 45° counterclockwise in the plane and scale by $1/\sqrt{2}$ so that the original line segments connecting $(j-1, \Gamma_{j-1})$ and (j, Γ_j) now become vertical and horizontal segments of unit length, and the transformed paths now start and end on the line $y = x$. Further, because the paths in D_k have the first step positive, the rotated and scaled path will be a *North and East* path, starting by going vertically up (North) and finishing by going horizontally to the right (East), terminating on the line $y = x$. Since the original lattice paths of D_k are not restricted to lie on or above the x–axis, the corresponding North and East paths are not restricted to lie on or above the line $y = x$; see Figure 1. The probability $\delta(k) = P(D_k)$ is defined as the sum of the Bernoulli probability weights $2^{-|\Gamma|}$ over all paths in $\Gamma \in D_k$. There is a nice way to calculate $\delta(k)$ by following the idea of [11, p. 30]. There are exactly $k+1$ peaks and k valleys on the paths of D_k. Consider the paths of D_k that, after being transformed to North and East paths, go from $(0,0)$ to $(n+1, n+1)$ for some integer $n \geq k$. Denote the transformed valley points as $(x_1, y_1), \ldots, (x_k, y_k)$ where we have $1 \leq x_1 < x_2 < \cdots < x_k \leq n$ and $1 \leq y_1 < y_2 < \cdots < y_k \leq n$. By specifying these transformed valley points on the North and East paths we specify uniquely the lattice path. There are $\binom{n}{k}$ ways to choose the x_k's and independently the same number of ways to choose the y_k's. Therefore, by writing $n+1 = k+1+h$ for some $h \geq 0$ we have

$$\delta(k) = \sum_{h=0}^{\infty} 4^{-(h+k+1)} \binom{h+k}{k}^2. \quad (1.5)$$

For example, $\delta(1) = \sum_{h \geq 0} (h+1)^2 4^{-(h+2)} = \frac{5}{27}$. In contrast to the probability distribution $P(\mathbf{R} = 2(k+1))$, $k \geq 0$, the $\delta(k)$ do not comprise a discrete probability distribution; in fact their sum is infinite. One finds the well known sequence

$$d(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2 = 1, 5, 33, 245, 1921, \ldots, \quad (1.6)$$

[10, A084771], as the numerators of $\delta(n)$ in the following sense: $d(n) = 3^{2n+1}\delta(n)$. We show this by the proof of Remark 3.3. In Lemma 4.1 we show a recurrence relation between δ_k, $1 \leq k \leq n$, and $P(\mathbf{R} = 2(k+1))$, $1 \leq k \leq n$. In Corollary 4.2 we re–derive the probability generating function $f(x) = E\{x^{\mathbf{R}}\}$ of (1.1) by using the recurrence (4.1) and the well–known representation $d(n) = 3^n P_n(\frac{5}{3})$, where P_n is the n–th Legendre polynomial, to find the generating function of $\delta(n)$. Our proof of Corollary 4.2 serves to set the ideas used to obtain another recurrence (4.9) in the proof of Theorem 1.2.

To introduce the key element for the recurrence (4.9), we make a definition.

Definition 1.4. *Recall the definition of the collection of paths D_n defined by* (1.4) *and the initial minimum $\mathbf{m} = \mathbf{m}(\Gamma)$ defined just before Definition 1.1. Define for each $\ell \geq 1$*

$$\alpha_\ell(n) = P(\Gamma \in D_{n+1} : \mathbf{m}(\Gamma) \geq \ell), \quad n \geq 0.$$

The reason we put D_{n+1} instead of D_n under the probability sign in Definition 1.4, so that the paths have $n+2$ peaks, is that we require at least 2 peaks in the paths to satisfy the condition that the level $\mathbf{m}$ of the first valley point satisfies $\mathbf{m} \geq \ell$. We shall calculate $\alpha_\ell(n)$ by Lemma 2.2. To accomplish this we may use a direct probability summation method based on Lemma 2.1 that in turn follows by the classical "stars and bars" counting method. Another method is to adapt the valley points argument by introducing some surgery on the paths. Since we will use similar surgery in a constructive proof of Theorem 1.6 we will show both of these methods in the proof of Lemma 2.2. A reason to apply surgery in the valley points construction is that, unlike (1.5), we obtain a factor of $\frac{1}{3}$ in the formula (2.3).

There are binomial sums associated as numerators of the $\alpha_\ell(n)$ as follows. Define for each $\ell \geq 1$ the integer sequence

$$A_\ell(n) = \sum_{k=0}^{n} 4^k \binom{n+1-\ell}{n-k}\binom{n+\ell}{k}, \quad n \geq 0, \tag{1.7}$$

We find in Lemma 3.2 that $\alpha_\ell(n) = 4^{1-\ell}3^{-2n-3}A_\ell(n)$ as a consequence of the representation (2.3) and one of Euler's transformation formulae (3.1) for the hypergeometric function. In Proposition 3.5 we pass to the Jacobi polynomial generating function to calculate the generating function $A(x;\ell)$ of the integer sequence A_ℓ of (1.7) as defined by $A(x;\ell) = \sum_{n=0}^{\infty} A_\ell(n)x^n$. This then leads in Section 4.1 to a proof of Theorem 1.2 via generating function algebra after utilizing the recurrence (4.9).

In a parallel development of the relation (1.2), by (1.3) and Definitions 1.1 and 4.5 we find for each $\ell \geq 1$ a positive integer sequence $r_\ell = r_\ell(n)$, $n \geq 0$, that determines the probability distribution with generating function $r(x;\ell)$. We easily find $r_1(n) = r(n+1) = \sum_{k=0}^{n} 4^k \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$, $n \geq 0$, or $r_1 = (1, 5, 29, \dots)$ for $r = (1, 1, 5, 29, \dots)$, [10, A059231]; see (4.15) and Definition 4.5. By Definition 4.6 and (4.19) we have a positive integer sequence $s_0 = (1, 3, 12r(1), 12r(2), 12r(3), \dots)$ such that the terms of s_0 after the leading term 1 are written: $(3, 12r(1), 12r(2), \dots) = 3s_1$ where $s_1 = (1, 4, 20, 116, \dots)$ enumerates rook paths for positive excursions from $(0,0)$ to $(2(n+1), 0)$, [10, A082298]. By the following we have that r_ℓ is r_1 convolved with the $(\ell-1)$–fold convolution power of s_0.

Corollary 1.5. *Let r_ℓ be given by Definition 4.5. Then for each $\ell \geq 1$ we have the convolution identity: $r_{\ell+1} = r_\ell * s_0$.*

We study also $s(x;\ell) = 2r(x;\ell) - r(x;\ell+1)$, that is the probability generating function of the number of runs in simple random walk paths conditioned to be positive excursions with first local minimum *equal* to ℓ; see Definition 5.1 and (5.1). Of course $s(x;\ell)/x^2$ is the generating function of the conditional distribution of $\mathbf{R}$ given that the simple random walk starts at position ℓ at time zero and takes its first step to the right. In Section 5 we give both an algebraic proof and a path decomposition proof of the following somewhat surprising identity.

Theorem 1.6. *For all $a, b \geq 1$ we have*

$$s(x;a)s(x;b) = x^4 r(x;a+b).$$

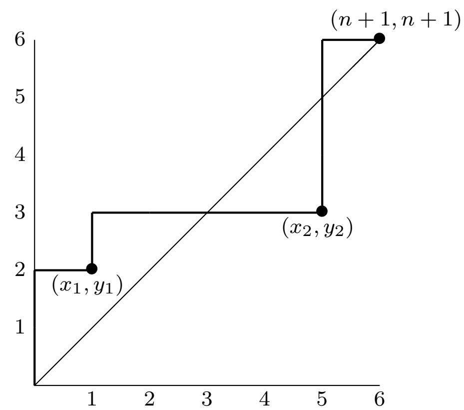

FIGURE 1. A North and East path with $k+1$ peaks and k valleys, illustrated for $k = 2$. The path starts North, ends East, and runs from $(0, 0)$ to $(n+1, n+1)$ for some $n \geq k$.

Our path decomposition proof of Theorem 1.6 involves the introduction of an arbitrary *tent* path consisting of h steps up followed immediately by h steps down with some $h \geq 1$; see Figure 3. We utilize the tent paths first in the constructive proof of Lemma 2.2.

By Definitions 5.1–5.2, for each $\ell \geq 1$ we have a sequence of positive integers s_ℓ associated to the probability generating function $s(x; \ell)$; here s_1 is as shown above by Remark 5.6. By Corollary 5.4 of Theorem 1.6 we tie the study of the integer sequences r_ℓ to those of s_ℓ by the relation $s_a * s_b = r_{a+b}$ for all $a, b \geq 1$. By Remark 5.8 we also have $s_{\ell+1} = s_\ell * s_0$.

In summary, here is an outline of the rest of the paper. In Section 2 we calculate the probabilities $\alpha_\ell(n)$; see Lemma 2.2. By Lemma 3.2 we establish A_ℓ of (1.7) as the integer sequence of "numerators" of the probabilities $\alpha_\ell(n)$, $n \geq 0$. In Proposition 3.5 we apply hypergeometric function transformation formulae to calculate the generating function $A(x; \ell)$ of the integer sequence A_ℓ. In Section 4 we develop the recurrence (4.9) and use it in Section 4.1 to prove Theorem 1.2. We prove Corollary 1.5 in Section 4.2. In Section 5 we prove Theorem 1.6. We present some corollaries of Theorem 1.6 involving the integer sequences r_ℓ and s_ℓ in Section 5.1. We prove Corollary 1.3 in Section 6.

2. CALCULATION OF $\alpha_\ell(n)$

Let m and N be positive integers with $m \leq N$. We want the number of m–tuples $\mathbf{k} = (k_1, \ldots, k_m)$ of positive integers, that is the number of $\mathbf{k} \in \mathbb{N}^m$, that satisfy (a) $k_1 + \cdots + k_m = N$. We also want to count the number of such solutions to (b) $m \leq k_1 + \cdots + k_m \leq N$. The numbers of such solutions are well known and are given as follows.

Lemma 2.1. *The number of solutions in $\mathbb{N}^m$ to the equations (a) $k_1 + \cdots + k_m = N$ and (b) $m \leq k_1 + \cdots + k_m \leq N$ are given respectively by*

(a) $\binom{N-1}{m-1}$

(b) $\binom{N}{m}$

Recall the probability $\alpha_\ell(n)$ as given by Definition 1.4; the paths under the probability sign in this definition have exactly $n + 2$ peaks. We first find a representation of $\alpha_\ell(n)$ by a direct calculation involving the lengths $a_1, \ldots, a_{n+2} \geq 1$ of the ascents and the lengths

$d_1, \ldots, d_{n+2} \geq 1$ of the descents. The calculation is a little simpler to develop if we take the symmetric condition that the final minimum ordinate on the paths is at least ℓ instead of $\mathbf{m} \geq \ell$. So we calculate $\alpha_\ell(n)$ by summing over all the ascents and descents under the conditions I: $\sum_{i=1}^{n+1}(a_i - d_i) \geq \ell$ and II: $\sum_{i=1}^{n+2}(a_i - d_i) = 0$. Note that under condition II we may rewrite the total number of steps $\sum_{i=1}^{n+2}(a_i + d_i) = 2\sum_{i=1}^{n+2} a_i$ and we may rewrite condition I by $-\ell + \sum_{i=1}^{n+1} a_i \geq \sum_{i=1}^{n+1} d_i$. Note that the value of d_{n+2} is determined by the values of $a_1, \ldots, a_{n+1}$ and $d_1, \ldots, d_{n+1}$ satisfying I, together with a free parameter $a_{n+2} \geq 1$. Thus

$$\begin{aligned}\alpha_\ell(n) &= \sum_{I,\ II} 2^{-\sum_{i=1}^{n+2}(a_i+d_i)} = \sum_{-\ell+\sum_{i=1}^{n+1} a_i \ \geq\ \sum_{i=1}^{n+1} d_i} 4^{-\sum_{i=1}^{n+1} a_i} \sum_{a_{n+2}=1}^{\infty} 4^{-a_{n+2}} \\ &= 3^{-1} \sum_{a_1,\ldots,a_{n+1}\geq 1} 4^{-\sum_{i=1}^{n+1} a_i} \sum_{\sum_{i=1}^{n+1} d_i \leq -\ell+\sum_{i=1}^{n+1} a_i} 1,\end{aligned} \tag{2.1}$$

where it is understood that in the last sum, that is the inner sum of 1's, the indices $d_1, \ldots, d_{n+1}$ vary subject to the summation condition I with $d_i \geq 1$. Put now $h = -(n+1)+a_1+\cdots+a_{n+1}$, so that $h \geq 0$. Then by Lemma 2.1(b) with $m = n+1$ and $N = h+n+1-\ell$, the inner sum of the double sum at the end of of (2.1) is written $\binom{h+n+1-\ell}{n+1}$. Now rewrite the sum over $a_1, \ldots, a_{n+1} \geq 1$ by fixing $h \geq 0$ as an index of outer summation and apply the condition $a_1 + \cdots + a_{n+1} = h + (n+1)$. Thus by Lemma 2.1(a) applied to this condition, again with $m = n+1$ but this time with $N = h+n+1$, we have

$$\alpha_\ell(n) = 3^{-1} \sum_{h=0}^{\infty} 4^{-(h+n+1)} \binom{h+n}{n} \binom{h+n+1-\ell}{n+1}. \tag{2.2}$$

Therefore, since $\binom{h+n+1-\ell}{n+1} = 0$ for $0 \leq h \leq \ell$, we replace $h-\ell$ by h in (2.2) to conclude the following:

Lemma 2.2. *Let $\ell \geq 1$. Then for all $n \geq 0$ we have*

$$\alpha_\ell(n) = 3^{-1} \sum_{h=0}^{\infty} 4^{-(h+n+1+\ell)} \binom{h+n+\ell}{n} \binom{h+n+1}{n+1} \tag{2.3}$$

We next show a proof of Lemma 2.2 by counting valley points.

Proof. For $\ell \geq 1$ consider $\widehat{\alpha}_\ell(n) = P(\Gamma \in D_{n+1} : \mathbf{m}(\Gamma) = \ell)$. We recall by the definition that D_{n+1} consists of paths with first incline an ascent from the origin, last incline a descent to the x-axis, and $n+2$ peaks. The event defining $\widehat{\alpha}_\ell(n)$ specifies that the first valley point has level exactly ℓ. We **claim** the following:

$$\widehat{\alpha}_\ell(n) = \frac{1}{3} P(\widehat{\Gamma} \in D_n : \text{ there are at least } \ell+1 \text{ steps in the first ascent of } \widehat{\Gamma}) \tag{2.4}$$

Note that the paths in the event under the probability sign in (2.4) have $n+1$ peaks. To establish the claim, we work with the original paths that leave from and return to the x–axis. Let $\widehat{\Gamma} \in D_n$ such that the first ascent of $\widehat{\Gamma}$ consists of at least $\ell+1$ steps. We cut $\widehat{\Gamma}$ at the level ℓ on the first ascent and horizontally translate the part coming after the cut point (ℓ, ℓ) to the right by $2j$ units for an arbitrary $j \geq 1$. We connect the two pieces of the cut path by inserting a $j \times j$–*tent* with base points (ℓ, ℓ) and $(\ell + 2j, \ell)$, where a $j \times j$–tent is a path with a single ascent of j steps followed by a single descent of j steps. Since we have thus added a peak, from $\widehat{\Gamma}$ we have constructed for each $j \geq 1$ a path $\Gamma = \Gamma_j \in D_{n+1}$ with $\mathbf{m}(\Gamma) = \ell$.

Each path Γ in the event defining $\widehat{\alpha}_\ell(n)$ corresponds to exactly one pair $\widehat{\Gamma}$ and j, where j is determined as the length of the descent to the first valley point in Γ and $\widehat{\Gamma}$ results from cutting off the $j \times j$–tent containing this descent followed by reconnecting the cut path by horizontal translation to the left. Therefore, since the probability of an arbitrary tent is $\sum_{j=1}^{\infty} 4^{-j} = \frac{1}{3}$, the claim is established.

We now calculate the probability on the right side of (2.4) by counting valley points on the North and East transformed paths $\widehat{\Gamma}$; recall Figure 1. Consider North and East paths with $n+1$ peaks and n valleys that run from $(0,0)$ to $(N+1, N+1)$ and have valley points (x_i, y_i), $1 \le i \le n$, where $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$. To match the condition that there are at least $\ell+1$ steps in the first ascent of $\widehat{\Gamma}$, we assume $y_1 \ge \ell+1$. Thus we have $\ell+1 \le y_1 < \cdots < y_n \le N$ and $1 \le x_1 < \cdots < x_n \le N$ for some $N \ge n$. The collection of all such valley point sequences is in one to one correspondence with the subcollection of all paths $\widehat{\Gamma}$ under the probability sign of the right side of (2.4) determined by the condition that the length of the path is exactly $2(N+1)$. There are $\binom{N}{n}\binom{N-\ell}{n}$ many paths in this subcollection, where this expression evaluates to zero unless $N \ge n+\ell$. Therefore by (2.4) we have

$$\widehat{\alpha}_\ell(n) = \frac{1}{3}\sum_{N=n}^{\infty} 4^{-(N+1)}\binom{N}{n}\binom{N-\ell}{n}. \tag{2.5}$$

Now we apply the simple fact that $\alpha_\ell(n) = \sum_{r=\ell}^{\infty} \widehat{\alpha}_r(n)$. Correspondingly, we sum the right side of (2.5) with r in place of ℓ over all $r \ge \ell$. This is easily accomplished by noting the binomial identity $\sum_{k=a}^{A}\binom{k}{a} = \binom{A+1}{a+1}$. Therefore, applying this identity with $A = N-\ell$ and $a = n$, by (2.5) we have

$$\alpha_\ell(n) = \frac{1}{3}\sum_{N=n}^{\infty} 4^{-(N+1)}\binom{N}{n}\binom{N-\ell+1}{n+1} = \frac{1}{3}\sum_{N=n+\ell}^{\infty} 4^{-(N+1)}\binom{N}{n}\binom{N-\ell+1}{n+1}, \tag{2.6}$$

where in the last expression we simply started the sum on N from $n+\ell$ since the preceding terms are zero. Finally we take $h = N-(n+\ell) \ge 0$ to rewrite the last sum in (2.6) and thus obtain exactly the formula (2.3) obtained by our first proof. □

Recall the definition of the binomial coefficient: $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$ for any real r and nonnegative integer k, [5, (5.1)].

Example 2.3. *By direction calculation of* (2.3) *we have* $4^2 3^3 \alpha_3(0) = 1$, $4^2 3^5 \alpha_3(1) = 15$, *and* $4^2 3^7 \alpha_3(2) = 160$. *On the other hand, by* (1.7) *we find* $A_3(0) = 1$, $A_3(1) = 4^0\binom{-1}{1}\binom{4}{0} + 4^1\binom{-1}{0}\binom{4}{1} = -1+16 = 15$, *and* $A_3(2) = 4^0\binom{0}{2}\binom{5}{0} + 4^1\binom{0}{1}\binom{5}{1} + 4^2\binom{0}{0}\binom{5}{2} = 0+0+160$.

3. The sequence $A_\ell(n)$, $n \ge 0$, and its generating function.

In this section we first show the simple relation of Lemma 3.2 between the probabilities $\alpha_\ell(n)$ and the integers $A_\ell(n)$ defined by (1.7). We then find the ordinary generating function of the sequence $A_\ell(n)$, $n \ge 0$, where we regard $\ell \ge 0$ as a parameter. In the proofs of this section we make extensive use of the definition and transformation properties of the hypergeometric function ${}_2F_1$, which is defined as follows.

Definition 3.1. [1, 15.1.1] *Define*

$$ {}_2F_1[a,\ b;\ c;\ z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where we employ the Plochammer symbol $(a)_n = (a)(a+1)\cdots(a+n-1)$, *with* $(a)_0 = 1$.

Lemma 3.2. *Let* $\ell \geq 1$ *and let* $\alpha_\ell(n)$ *be given by* (2.3) *for* $n \geq 0$ *and* $A_\ell(n)$ *be defined by* (1.7)*. Then*

$$ A_\ell(n) = 4^{\ell-1} 3^{2n+3} \alpha_\ell(n). $$

Proof. We shall apply Euler's transformation formula for the hypergeometric function as follows:

$$ {}_2F_1[a,\ b;\ c;\ z] = (1-z)^{c-a-b} {}_2F_1[c-a,\ c-b;\ c;\ z], \quad |\arg(1-z)| < \pi, \tag{3.1} $$

[1, 15.3.1, 15.3.3]. To apply (3.1) we first write $\alpha_\ell(n)$ as represented by (2.3) in terms of the hypergeometric function. Use the symmetry of the binomial coefficients to rewrite (2.3) by $\alpha_\ell(n) = 3^{-1} 4^{-(n+\ell+1)} \sum_{h=0}^{\infty} 4^{-h} \binom{h+n+\ell}{h+\ell} \binom{h+n+1}{h}$. Hence, by this formula and the Plochammer symbol of Definition 3.1, we have

$$ \begin{aligned} &3^1 4^{n+\ell+1} \alpha_\ell(n) = \\ &\frac{(n+1)_\ell}{\ell!} + \frac{(n+1)_{\ell+1}}{(\ell+1)!} \frac{(n+2)}{4^1 1!} + \frac{(n+1)_{\ell+1}}{(\ell+2)!} \frac{(n+2)(n+3)}{4^2 2!} + \cdots \\ &= \binom{n+\ell}{\ell} \left(1 + \frac{(n+\ell+1)}{\ell+1} \frac{(n+2)}{4^1 1!} + \frac{(n+\ell+1)(n+\ell+2)}{(\ell+1)(\ell+2)} \frac{(n+2)(n+3)}{4^2 2!} + \cdots \right) \\ &= \binom{n+\ell}{\ell} {}_2F_1[n+\ell+1,\ n+2;\ \ell+1;\ 4^{-1}]. \end{aligned} \tag{3.2} $$

Now apply (3.1) to the hypergeometric function of the last line of (3.2). Then

$$ \binom{n+\ell}{\ell} {}_2F_1[n+\ell+1,\ n+2;\ \ell+1;\ 4^{-1}] = \left(\frac{4}{3}\right)^{2n+2} \binom{n+\ell}{\ell} {}_2F_1[-n,\ \ell-(n+1);\ \ell+1;\ 4^{-1}] \tag{3.3} $$

Note that $\binom{n+\ell}{\ell} {}_2F_1[-n,\ \ell-(n+1);\ \ell+1;\ 4^{-1}]$ has a finite expansion consisting of $n+1$ terms that we write starting from the $(n+1)$–st term as follows.

$$ \begin{aligned} &\binom{n+\ell}{\ell} {}_2F_1[-n,\ \ell-(n+1);\ \ell+1;\ 4^{-1}] \\ &= \frac{(n+1)_\ell}{\ell!} \left(\frac{(-n)_n}{(1+\ell)_n} \frac{(\ell-(n+1))_n}{4^n n!} + \frac{(-n)_{n-1}}{(1+\ell)_{n-1}} \frac{(\ell-(n+1))_{n-1}}{4^{n-1}(n-1)!} + \cdots + 1 \right) \\ &= \frac{(1)_{n+\ell}}{(1)_{n+\ell}} \frac{(-1)^n (\ell-(n+1))_n}{n!} 4^{-n} + \frac{(2)_{n+\ell-1}}{(1)_{n+\ell-1}} \frac{(-1)^{n-1}(\ell-(n+1))_{n-1}}{(n-1)!} 4^{-n+1} + \cdots + 1 \\ &= \binom{n+\ell}{0} \binom{n+1-\ell}{n} 4^{-n} + \binom{n+\ell}{1} \binom{n+1-\ell}{n-1} 4^{-n+1} + \cdots + 1 \\ &= \sum_{k=0}^{n} 4^{-n+k} \binom{n+\ell}{k} \binom{n+1-\ell}{n-k} = 4^{-n} A_\ell(n). \end{aligned} \tag{3.4} $$

Accounting finally for the various powers of 3 and 4, by (3.2)–(3.4) we have $3^1 4^{n+\ell+1} \alpha_\ell(n) = \left(\frac{4}{3}\right)^{2n+2} 4^{-n} A_\ell(n)$, as desired. □

Remark 3.3. *We have that $d(n)$ defined by (1.6) satisfies*

$$d(n) = 3^{2n+1}\delta(n), \quad n \geq 0.$$

Proof. First, write $\delta(n)$ as given by (1.5) as $\delta(n) = 4^{-n-1}\sum_{h=0}^{\infty} 4^{-h}\binom{h+n}{h}^2$. By Definition 3.1 we easily see that $\delta(n) = 4^{-n-1}{}_2F_1[1+n,\ 1+n;\ 1;\ 1/4]$. Apply the Euler transformation (3.1) to obtain $\delta(n) = 4^{-n-1}\left(\frac{4}{3}\right)^{2n+1}{}_2F_1[-n,\ -n;\ 1;\ 1/4]$, or $3^{2n+1}\delta(n) = 4^n{}_2F_1[-n,\ -n;\ 1;\ 1/4]$. Then, by writing the finite expansion of this last hypergeometic function from the last term to the first, similar as in (3.4), the remark follows in view of (1.6). □

Recall Definition 3.1. To find the generating function of the sequence $A_\ell(n)$, $n \geq 0$, we use another transformation of the hypergeometric function as well as a representation of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, [1, 22.1, 22.2.1], in terms of it. First, by [1, 15.3.1,15.3.4] we have the Euler transformation

$$ {}_2F_1[a,\ b;\ c;\ z] = (1-z)^{-a}{}_2F_1[a,\ c-b;\ c;\ z/(z-1)], \qquad |\arg(1-z)| < \pi. \tag{3.5}$$

Second, by [1, 15.4.6] we have

$$ {}_2F_1[-n,\ \alpha+1+\beta+n;\ \alpha+1;\ x] = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(1-2x). \tag{3.6}$$

Here, for nonnegative integers α, we may write $\frac{n!}{(\alpha+1)_n} = 1/\binom{n+\alpha}{n}$. Further, by [1, 22.9.1] we have that

$$\sum_{n=0}^{\infty} z^n P_n^{(\alpha,\beta)}(u) = 2^{\alpha+\beta}R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta}, \quad |z| < 1, \tag{3.7}$$

where $R = \sqrt{1-2uz+z^2}$.

Definition 3.4. *Define $A(x;\ell) = \sum_{n=0}^{\infty} x^n A_\ell(n)$ for $A_\ell(n)$ defined by (1.7).*

Proposition 3.5. *Denote $R(x) = \sqrt{1-10x+9x^2}$. For all $\ell \geq 0$ and $x \in (-1,1)$ we have*

$$A(x;\ell) = 2^{1-\ell}\frac{1}{R(x)}\frac{(1+3x+R(x))^{2\ell-1}}{(1-5x+R(x))^{\ell}}. \tag{3.8}$$

Proof. By Definition 3.4 we have

$$\begin{aligned} A(x;\ell) &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} 4^k \binom{n+1-\ell}{n-k}\binom{n+\ell}{k} = \sum_{k=0}^{\infty}(4x)^k \sum_{j=0}^{\infty} x^j \binom{j+k+1-\ell}{j}\binom{j+k+\ell}{k} \\ &= \sum_{k=0}^{\infty}(4x)^k \binom{k+\ell}{\ell}\ {}_2F_1[k+2-\ell,\ k+\ell+1;\ \ell+1;\ x], \end{aligned} \tag{3.9}$$

where we introduced $j = n-k$ to rewrite the double sum. Here, to obtain the hypergeometric function we rewrite the sum on j in the first line of (3.9) as $\sum_{j=0}^{\infty} x^j \binom{j+k+1-\ell}{j}\binom{j+k+\ell}{j+\ell} = \frac{(k+1)_\ell}{(1)_\ell}\left(1 + \frac{(k+2-\ell)_1}{1!}\frac{(k+\ell+1)_1}{(\ell+1)_1}x + \frac{(k+2-\ell)_2}{2!}\frac{(k+\ell+1)_2}{(\ell+1)_2}x^2 + \cdots\right)$. This last expression is clearly equal to $\binom{k+\ell}{\ell}\ {}_2F_1[k+2-\ell,\ k+\ell+1;\ \ell+1;\ x]$ as appears on the second line of (3.9), as desired.

Next, first apply (3.5) to obtain that

$$ {}_2F_1[k+2-\ell,\ k+\ell+1;\ \ell+1;\ x] = (1-x)^{-(k+2-\ell)}{}_2F_1[k+2-\ell,\ -k;\ \ell+1;\ x/(x-1)]. \tag{3.10}$$

Note that by definition ${}_2F_1[a,\ b;\ c;\ z] = {}_2F_1[b,\ a;\ c;\ z]$, so we can rewrite the right side of (3.10) by $(1-x)^{-(k+2-\ell)}{}_2F_1[-k,\ k+2-\ell;\ \ell+1;\ x/(x-1)]$. Second, apply (3.6) to write

$$ {}_2F_1[-k,\ k+2-\ell;\ \ell+1;\ x/(x-1)] = \binom{k+\ell}{\ell}^{-1} P_k^{(\alpha,\beta)}(1-2x/(x-1)), \tag{3.11} $$

where the parameters α and β satisfy $\alpha+1+\beta+k = k+2-\ell$ and $\alpha+1 = \ell+1$. Thus, $\alpha = \ell$ and $\beta = 1-2\ell$. Therefore by (3.9)–(3.11), after canceling the term $\binom{k+\ell}{\ell}$, we have

$$ A(x;\ell) = \sum_{k=0}^{\infty} (4x)^k (1-x)^{-(k+2-\ell)} P_k^{(\ell,1-2\ell)}((1+x)/(1-x)). \tag{3.12} $$

Now apply (3.7) to the right side of (3.12) with $z = 4x/(1-x)$, $u = (1+x)/(1-x)$, $\alpha = \ell$ and $\beta = 1-2\ell$. Then $R = \sqrt{1+2uz+z^2} = (1-x)^{-1}\sqrt{(1-x)^2 - 2(1+x)4x + 16x^2} = (1-x)^{-1}\sqrt{1-10x^2+9x^2}$. And therefore, with $S = (1-x)R$,

$$ (1-x)^{\ell-2}R^{-1}(1+z+R)^{-\ell}(1-z+R)^{2\ell-1} = S^{-1}((1-x)(1-z)+S)^{-\ell}((1-x)(1+z)+S)^{2\ell-1}. $$

Hence by (3.7), (3.12), $(1-x)(1-z) = 1-5x$, $(1-x)(1+z) = 1+3x$, and $\alpha+\beta = 1-\ell$, we have

$$ A(x;\ell) = \frac{2^{1-\ell}}{\sqrt{1-10x^2+9x^2}}(1-5x+\sqrt{1-10x^2+9x^2})^{-\ell}(1+3x+\sqrt{1-10x^2+9x^2})^{2\ell-1}, $$

which matches the statement of the proposition. □

4. Proof of Theorem 1.2

In this section we first set the stage for a recurrence approach based on Lemma 4.4 to prove Theorem 1.2 in Section 4.1. We construct the framework of this approach by recovering (1.1) by the prototype recurrence (4.1). Once we establish the formula (1.3) for the probability generating function of $\mathbf{R}$ given $\mathbf{S}_1 = +1$, $\mathbf{m} \geq \ell$, and $\mathbf{R} \geq 4$, we will develop the integer sequences associated with these conditional probability distributions in Section 4.2.

Recall that $\delta(k) = P(D_k)$, where D_k as defined by (1.4) is the event that a nearest neighbor path starts on the x–axis with an ascent and ends on the x–axis with a descent and has $2(k+1)$ runs or $k+1$ peaks. We easily see directly that $\delta(0) = P(D_0) = \frac{1}{3}$, or find this by Remark 3.3. The following recurrence leads to the proof of Corollary 4.2.

Lemma 4.1. *For all $n \geq 0$ we have*

$$ \begin{aligned} \delta(n) &= \frac{3}{2}\sum_{k=1}^{n} P(\mathbf{R} = 2(k+1))\delta(n-k) \\ &\quad + \frac{1}{2}\sum_{k=0}^{n-1} P(\mathbf{R} = 2(k+1))\delta(n-k-1) + \frac{1}{2}P(\mathbf{R} = 2(n+1)). \end{aligned} \tag{4.1} $$

Proof. Recall that $\mathbf{R}$ is the number of runs in an excursion of a simple random walk on the integers, denoted $\{\mathbf{S}_n : n = 0,1,2\ldots\}$ with $\mathbf{S}_0 = 0$. Here we define σ as the epoch at which the simple random walk first returns to zero:

$$ \sigma = \inf\{n \geq 0 : \mathbf{S}_n = 0\}, $$

and the (first) excursion is the path $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_\sigma$. Thus, save for the endpoints, the paths Γ in the event $\{\mathbf{R} = 2(k+1)\}$ lie either above strictly above or strictly below the x–axis. Thus the probability that we have a positive excursion path with $2(k+1)$ runs is $\frac{1}{2}P(\mathbf{R} = 2(k+1))$. We will focus on paths of a simple random walk that have the first step positive. Consider the paths of the event D_n in the definition of $\delta(n)$ for some $n \geq 0$: for any path $\Gamma \in D_n$ we have $\Gamma_\sigma = 0$. Of course it is possible that the path $\Gamma \in D_n$ actually terminates at epoch σ. If $\Gamma \in D_n$ does not terminate at σ, then there are two possibilities: the path could either (1) continue to descend and fall below the x–axis at the next step, or (2) could turn from its descent to the x–axis and ascend above the x–axis at the next step. Define $\tau \geq \sigma$ as the first epoch at or after σ such that a path ascends to a level $\Gamma_{\tau+1} = +1$, so that $\Gamma_\tau = 0$. In the case (1) we have $\tau > \sigma$, while in case (2) we have $\tau = \sigma$. In case Γ terminates at σ we define $\tau(\Gamma) = +\infty$; this can cause no confusion because $\{0 = \mathbf{S}_\sigma, \mathbf{S}_{\sigma+1}, \mathbf{S}_{\sigma+2}, \dots\}$ is a simple random walk on the integers, so must return to zero in finite time with probability 1. Now focus on the case (1): $\tau > \sigma$. Then we have an excursion below the x–axis along the random walk path from epoch σ to epoch τ. If we now modify any given such path Γ by reflecting this negative excursion over $[\sigma, \tau]$ in the x–axis to form a positive excursion over $[\sigma, \tau]$, but otherwise leave the path Γ intact, then we will have added two runs to the final tally of runs for the full modified path, and we have that on the discrete time interval between 0 and τ we have at least one interior local minimum value 0 along the path, namely at epoch σ. So, on any event $\{\sigma = s, \tau = t\} \cap D_n$ for which $s < t$, we modify each path by reflection over $[s, t]$ to form a Dyck path on $[0, t]$ of $k+1$ peaks for some $1 \leq k \leq n$ and at least one interior zero; note that a path in D_n can not terminate at epoch τ on the event $\sigma > \tau$ (case (1)). If $k \geq 1$, define the event F_k by

$$F_k = \{\Gamma \in D_k : \Gamma \text{ is a Dyck path of } k+1 \text{ peaks and at least one interior zero}\}. \tag{4.2}$$

Then for all $k \geq 1$, $F_k = G_k \setminus E_k$ where G_k is the event of a general Dyck path of $k+1$ peaks and E_k is the event of a Dyck path with $k+1$ peaks and no interior zeros, that is a positive excursion path for the simple random walk. We can lift any path in the event G_k by adding a positive step at the beginning and a negative step at the end to make a positive excursion path of the same number $k+1$ of peaks, so that $P(G_k) = (\frac{1}{2})^{-2} P(E_k) = 4\frac{1}{2} P(\mathbf{R} = 2(k+1))$. Therefore

$$P(F_k) = P(G_k) - P(E_k) = (4-1)P(E_k) = \frac{3}{2} P(\mathbf{R} = 2(k+1)), \quad k \geq 1. \tag{4.3}$$

To obtain the recurrence (4.1) we divide the event $D = D_n$ into the three disjoint pieces $D = D^{(1)} \cup D^{(2)} \cup D^{(3)}$ as follows:

$$D^{(1)} = D \cap \{\sigma < \tau\}, \quad D^{(2)} = D \cap \{\sigma = \tau\}, \quad D^{(3)} = D \cap \{\tau = \infty\}.$$

Note that $P(F_k)$ equals the total of all weights of unmodified paths in $D^{(1)}$ that originally had exactly k peaks up to the original epoch τ, because the modification does not change these weights even though it adds one more peak over the interval $[0, \tau]$. Thus $P(D^{(1)})$ is the sum over $k \geq 1$ of the product of $P(F_k)$ with the probability that we have a path with $(n+1) + 1 - (k+1) = n - k + 1$ peaks that starts from zero with an ascent and ends by coming again to zero with a descent. Thus this second probability is $\delta(n-k)$. Hence by (4.3) we obtain that $P(D^{(1)})$ equals the first sum in (4.1).

Since the event E_k as defined above is the event of a positive excursion with $k+1$ peaks, the probability $P(D^{(2)})$ is the sum over $k \geq 0$ of the product of $P(E_k)$ with $P(D_{n-(k+1)}) = \delta(n-k-1)$. Therefore we obtain the second term of (4.1). Finally the third term of (4.1) is simply $P(D^{(3)}) = P(E_n) = \frac{1}{2} P(\mathbf{R} = 2(n+1))$. □

As a corollary of Lemma 4.1 we can recover the probability generating function of the random variable $\mathbf{R}$, that is the number of runs in an excursion from the origin of a simple random walk on the integers. We note that the joint distribution of $\mathbf{R}$ and the number of steps $\mathbf{L}$ is well known in the case of simple random walk due to the result of Narayana [9], [11]; for the corresponding joint generating function see [7], [8, p. 2014].

Corollary 4.2. [8, p. 2021]
Denote $f(x) = E\{x^{\mathbf{R}}\} = \sum_{n=0}^{\infty} x^{2(n+1)} P(\mathbf{R} = 2(n+1))$. *Then for all* $|x| < 1$ *we have that* (1.1) *holds.*

Proof. We simply construct the ordinary generating function of each side of (4.1) and solve algebraically for the probability generating function $f(x)$ of $\mathbf{R}$, as follows. First we define and calculate the generating function of $\delta(n) = P(D_n)$ while keeping track of the number of runs for the paths of D_n in the exponent of the generating function variable.

Definition 4.3. *Define*

$$G(x) = \sum_{n=0}^{\infty} x^{2(n+1)} \delta(n) = \frac{x^2}{3} + \frac{5x^4}{27} + \cdots .$$

By definition 4.3 and Lemma 4.1 we have

$$\begin{aligned} G(x) = \frac{x^2}{3} + \sum_{n=1}^{\infty} x^{2(n+1)} \delta(n) = \frac{3}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{n} x^{2(n+1)} P(\mathbf{R} = 2(k+1)) \delta(n-k) \\ + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} x^{2(n+1)} P(\mathbf{R} = 2(k+1)) \delta(n-k-1) + \frac{x^2}{3} + \frac{1}{2} \sum_{n=1}^{\infty} x^{2(n+1)} P(\mathbf{R} = 2(n+1)). \end{aligned} \tag{4.4}$$

Notice that, by $P(\mathbf{R} = 2) = \frac{2}{3}$, the term $\frac{x^2}{3}$ added to the last sum on the right side of (4.4) yields $\frac{1}{2} f(x)$. Therefore, with $j = n - k$ after changing the order of summation, by (4.4) and Definition 4.3 we have

$$\begin{aligned} G(x) - \frac{f(x)}{2} = \frac{3}{2} \sum_{k=1}^{\infty} x^{2(k+1)} P(\mathbf{R} = 2(k+1)) \sum_{j=0}^{\infty} x^{2j} \delta(j) \\ + \frac{1}{2} \sum_{k=0}^{\infty} x^{2(k+1)} P(\mathbf{R} = 2(k+1)) \sum_{j=1}^{\infty} x^{2j} \delta(j-1) = \frac{3}{2} \left(f(x) - \frac{2x^2}{3} \right) \frac{G(x)}{x^2} + \frac{1}{2} f(x) G(x). \end{aligned} \tag{4.5}$$

From (4.5) after multiplying through by 2 we solve algebraically for f via $2G = (3f - 2x^2)G/x^2 + fG + f$ or $4G = f(1 + G + 3G/x^2)$. Hence

$$f(x) = \frac{4x^2 G(x)}{3G(x) + x^2(1 + G(x))}. \tag{4.6}$$

It only remains to calculate the exact form of $G(x)$ which we simply take from the Definition 4.3 and Remark 3.3, so that $\delta(n) = 3^{-(2n+1)} d(n)$, valid for all $n \geq 0$, for $d(n)$ given by (1.6). Now it is well known that $d(n) = 3^n P_n(\frac{5}{3})$ where P_n is the n–th Legendre polynomial, that is the special case $\alpha = \beta = 0$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. In fact this is easy to see by writing $d(n) = 3^{2n+1} \delta(n) = 3^{2n+1} 4^{-1-n} {}_2F_1[1+n,\ 1+n;\ 1;\ 1/4]$ via (1.5) and Definition 3.1, followed by applying (3.5)–(3.6). Thus

$$G(x) = \sum_{n=0}^{\infty} x^{2(n+1)} 3^{-(2n+1)} 3^n P_n(5/3) = \frac{x^2}{3} \sum_{n=0}^{\infty} x^{2n} 3^{-n} P_n(5/3).$$

But by (3.7) we have $\sum_{n=0}^{\infty} x^{2n}3^{-n}P_n(5/3) = R^{-1}$ for $R = \sqrt{1-2(5/3)x^2/3+(x^2/3)^2} = \frac{1}{3}\sqrt{9-10x^2+x^4}$. Therefore we have

$$G(x) = \frac{x^2}{\sqrt{9-10x^2+x^4}}. \tag{4.7}$$

Hence by (4.6)–(4.7) we easily find that $f(x) = \frac{4x^2}{\sqrt{9-10x^2+x^4}+x^2+3} = \frac{x^2+3-\sqrt{9-10x^2+x^4}}{4}$. □

4.1. **Positive excursion runs statistic conditioned by the first local minimum.** Recall that $\{\mathbf{S}_n\}$ denotes a simple random walk on the integers with $\mathbf{S}_0 = 0$. Conditional on $\mathbf{S}_1 = +1$, the first excursion from zero of the random walk is a positive excursion. Any positive excursion path Γ of at least 4 runs, that is at least 2 peaks, has an initial local minimum value $\mathbf{m} = \mathbf{m}(\Gamma)$. We use the recurrence method via the following Lemma 4.4 that is proved in a similar manner as Lemma 4.1 to establish (1.3). Recall the definition of $\rho_\ell(n)$ along with the probability generating function $r(x;\ell)$ of $\mathbf{R}_\ell$ in Definition 1.1. Now denote the normalization constant in the definition of the conditional distribution of $\mathbf{R}_\ell$ by $C(\ell) = P(\mathbf{S}_1 = +1,\ \mathbf{R} \geq 4,\ \mathbf{m} \geq \ell)$, so that for all $n \geq 0$, $C(\ell)\rho_\ell(n) = P(\mathbf{S}_1 = +1,\ \mathbf{R} = 2(n+2),\ \mathbf{m} \geq \ell)$. Note that the paths under $\alpha_\ell(n)$ of Definition 1.4 may cross the x–axis, while the paths under $C(\ell)\rho_\ell(n)$ are positive excursions. As in the proof of Lemma 4.1, for all $j \geq 0$ we define

$$\begin{aligned} G_j &= \{\Gamma \in D_j : \Gamma \text{ is a Dyck path of } j+1 \text{ peaks}\}; \\ E_j &= \{\Gamma \in D_j : \Gamma \text{ is a positive excursion of } j+1 \text{ peaks}\}. \end{aligned} \tag{4.8}$$

Note that $P(G_0) = P(E_0)$, but for $j \geq 1$ we have $P(G_j) = 4P(E_j) = 2P(\mathbf{R} = 2(j+1))$.

Lemma 4.4. *For each $\ell \geq 1$ we have*

$$\alpha_\ell(n)/C(\ell) = \sum_{k=0}^{n-1} \sum_{i,j\geq 0,\ i+j=k} \rho_\ell(i)P(G_j)\delta(n-k-1) + \sum_{k=0}^{n-1} \rho_\ell(k)\delta(n-k-1) + \rho_\ell(n), \quad n \geq 0. \tag{4.9}$$

Proof. For simplicity we prove the relation after both sides have been multiplied by $C(\ell)$ so that (4.9) becomes a recurrence between probabilities instead of between conditional probabilities. We follow the method of proof of Lemma 4.1. We define σ and τ exactly as before. Also as before, on the event $\{\tau > \sigma\}$ we modify the paths by reflecting the part of the path over the time interval $[\sigma, \tau]$ (which originally was an upside down Dyck path) across the x–axis. However, whereas in Lemma 4.1 we treated the distribution of $\mathbf{R}$ and therefore $P(G_j)$ as unknown, due to the condition $\mathbf{m} \geq \ell$ we will now bootstrap the probability $P(G_j)$ in the summation term corresponding to the case $\tau > \sigma$. That is, the condition $\mathbf{m} \geq \ell$ is now carried by the term $\rho_\ell(i)$ for some $i \geq 0$, where by definition of $\rho_\ell(i)$ this means we have $i+2$ peaks in the path until σ. Still in the case $\tau > \sigma$, following these $i+2$ peaks we have in the modified path over $[\sigma, \tau]$ a general Dyck path of $j+1$ peaks for some $j \geq 0$. Since by $i+j=k$ we have at this point $k+3$ peaks and require $(n+2)+1 = n+3$ peaks for the final modified paths, and since again there is one more peak than the index m in the paths under the probability $\delta(m)$, we have a final term $\delta(n-k-1)$ in the right side of the first summation. The remaining two terms, corresponding to $\tau = \sigma$ and $\tau = \infty$, respectively, are handled as before. □

By Proposition 3.5 and Lemma 4.4 we can now prove the explicit expression (1.3) for $r(x;\ell)$ of Definition 1.1.

Proof of Theorem 1.2. Define

$$H_\ell(x)=\sum_{n=0}^{\infty}\alpha_\ell(n)x^{2(n+2)}, \tag{4.10}$$

where $\alpha_\ell(n)$ is given by Definition 1.4. By the definition (4.10) and Lemma 4.4 we now write $H_\ell(x)/C(\ell)$ as a sum of three generating functions given respectively by the three expressions on the right side of (4.9). The first of these generating functions, corresponding to the double sum expression of (4.9), that is

$$I=\sum_{n=0}^{\infty}x^{2(n+2)}\sum_{k=0}^{n-1}\sum_{i,j\geq 0,\ i+j=k}\rho_\ell(i)P(G_j)\delta(n-k-1),$$

may be rewritten by a change of order of summation with indices $j\geq 0$, $i=k-j\geq 0$, and $m=n-k-1\geq 0$, so that $(j+1)+(i+1)+(m+1)=n+2$. Thus in all, by Lemma 4.4 we have

$$\begin{aligned} H_\ell(x)/C(\ell)&=\sum_{j=0}^{\infty}P(G_j)x^{2(j+1)}\sum_{i=0}^{\infty}\rho_\ell(i)x^{2(i+1)}\sum_{m=0}^{\infty}x^{2(m+1)}\delta(m)\\ &\quad+\sum_{k=0}^{\infty}\rho_\ell(k)x^{2(k+2)}\sum_{m=0}^{\infty}x^{2(m+1)}\delta(m)+\sum_{n=0}^{\infty}\rho_\ell(n)x^{2(n+2)}=I+II+III. \end{aligned} \tag{4.11}$$

Here, by (4.11), $I=\left(P(G_0)x^2+\sum_{j=1}^{\infty}2P(\mathbf{R}=2(j+1))x^{2(j+1)}\right)x^{-2}r(x;\ell)G(x)$, where we applied the evaluation of $P(G_j)$ after (4.8) and we used Definitions 4.3 and 1.1. Now $P(G_0)x^2+\sum_{j=1}^{\infty}2P(\mathbf{R}=2(j+1))x^{2(j+1)}=\frac{x^2}{3}+2(f(x)-\frac{2x^2}{3})=2f(x)-x^2$. Therefore

$$I=(2f(x)-x^2)\frac{r(x;\ell)}{x^2}G(x).$$

Again by Definitions 4.3 and 1.1 we have $II=r(x;\ell)G(x)$ and $III=r(x;\ell)$. Hence by (4.11), and by substitution of the explicit formulae for $f(x)$ and $G(x)$ from (1.1) and (4.7) we have

$$\begin{aligned}\frac{H_\ell(x)}{C(\ell)r(x;\ell)}&=(2f(x)-x^2)\frac{G(x)}{x^2}+G(x)+1\\ &=\frac{3-x^2-\sqrt{9-10x^2+x^4}}{2\sqrt{9-10x^2+x^4}}+\frac{x^2}{\sqrt{9-10x^2+x^4}}+1=\frac{3+x^2+\sqrt{9-10x^2+x^4}}{2\sqrt{9-10x^2+x^4}}\end{aligned} \tag{4.12}$$

Here, $C(\ell)=P(\mathbf{S}_1=+1,\ \mathbf{R}\geq 4,\ \mathbf{m}\geq\ell)=\frac{1}{2}\sum_{s=1}^{\infty}\sum_{r\geq s+\ell}2^{-(r+s)}=2^{-\ell}\sum_{s=1}^{\infty}4^{-s}=2^{-\ell}/3$. Another way to compute $C(\ell)$ is to use (4.12)–(4.13) together with the requirement that $r(1;\ell)=1$. Because by Lemma 3.2 we have $A_\ell(n)=4^{\ell-1}3^{2n+3}\alpha_\ell(n)$, by (4.10) and Proposition 3.5 we **claim**:

$$H_\ell(x)=\sum_{n=0}^{\infty}x^{2(n+2)}4^{1-\ell}3^{-2n-3}A_\ell(n)=\frac{72x^4}{27}\frac{2^{-3\ell}3^{-\ell}}{\sqrt{9-10x^2+x^4}}\frac{(3+x^2+\sqrt{x^4-10x^2+9})^{2\ell-1}}{(3-5x^2/3+\sqrt{x^4-10x^2+9})^{\ell}}. \tag{4.13}$$

To verify (4.13) we simply note by Definition 3.4 and Proposition 3.5 that

$$H_\ell(x)=\frac{4^{1-\ell}x^4}{27}A(x^2/9;\ell)=\frac{4^{1-\ell}x^4 2^{1-\ell}}{27\sqrt{1-10x^2/9+x^4/9}}\frac{(1+3x^2/9+\sqrt{1-10x^2/9+x^4/9})^{2\ell-1}}{(1-5x^2/9+\sqrt{1-10x^2/9+x^4/9})^{\ell}}$$

Therefore after a bit of simplification the claim (4.13) is established. Finally via (4.12)–(4.13) we compute

$$\begin{aligned} r(x;\ell) &= \frac{H_\ell(x)}{C(\ell)} \frac{2\sqrt{9-10x^2+x^4}}{3+x^2+\sqrt{9-10x^2+x^4}} \\ &= \frac{8x^4 2^{-3\ell}3^{-\ell}}{2^{-\ell}} \frac{2}{3+x^2+\sqrt{9-10x^2+x^4}} \frac{(3+x^2+\sqrt{x^4-10x^2+9})^{2\ell-1}}{(3-5x^2/3+\sqrt{x^4-10x^2+9})^{\ell}}. \end{aligned} \tag{4.14}$$

Since $\frac{1}{3+x^2+\sqrt{9-10x^2+x^4}} = \frac{f(x)}{4x^2}$, by (4.14) the proof is complete. □

4.2. **Integer sequences arising from the probability generating functions $r(x;\ell)$.** In this section we establish for each $\ell \geq 1$ an integer sequence associated in a natural way with the generating function $r(x;\ell) = \sum_{n=0}^\infty \rho_\ell(n) x^{2(n+2)}$ of Definition 1.1. First, either by direct computation from the formula of Theorem 1.2, or by elementary considerations after the Definition 1.1, we note that $r(x;1) = 3f(x) - 2x^2$, where $f(x)$ is given by (1.1). Indeed we have that the condition $\mathbf{m} \geq 1$ is automatic under the condition that $\mathbf{R} \geq 4$, and therefore $r(x;1)$ is the (conditional) probability generating function of $\mathbf{R}$ given $\mathbf{R} \geq 4$, which is in turn $\frac{f(x)-\frac{2x^2}{3}}{P(\mathbf{R}\geq 4)} = 3(f(x) - \frac{2x^2}{3})$. By (1.1)–(1.2) we have $f(x) = 2\sum_{n=0}^\infty 3^{-2n-1} r(n) x^{2(n+1)}$ for $r(n) = 1, 1, 5, 29, 185, \ldots$ [10, A059231]. Therefore, $r(x;1) = 3f(x) - 2x^2 = 2\sum_{m=0}^\infty 3^{-2(m+1)} r(m+1)x^{2(m+2)}$, that is

$$\rho_1(n) = 2\cdot 3^{-2n-2} r(n+1), \quad n = 0, 1, 2, \ldots. \tag{4.15}$$

In particular, $\rho_1(n) = \frac{2}{9}\cdot 1, \frac{2}{81}\cdot 5, \frac{2}{729}\cdot 29, \ldots$, for $n = 0, 1, 2, \ldots$. We find by Taylor series expansions of the explicit formulae for $r(x;2)$ and $r(x;3)$ given by (1.3) that

$$r(x;2) = \frac{x^4}{9} + \frac{8x^6}{81} + \frac{56x^8}{729} + \frac{392x^{10}}{6561} + \cdots; \;\; r(x;3) = \frac{x^4}{18} + \frac{11x^6}{162} + \frac{46x^8}{729} + \frac{358x^{10}}{6561} + \cdots. \tag{4.16}$$

Definition 4.5. *Recall the conditional probability sequences $\rho_\ell(n)$, $n = 0, 1, 2, \ldots$ written by Definition 1.1. For each $\ell \geq 1$ we define the sequence $r_\ell = r_\ell(n), n = 0, 1, 2, \ldots$ by*

$$r_\ell(n) = 2^{\ell-2} 3^{2n+2} \rho_\ell(n).$$

We have just seen that $r_1(n) = r(n+1) = 1, 5, 29, 185, \ldots$ for the rook number sequence $r = r(n)$, $n \geq 0$. By Definition 4.5 and our experimental results (4.16) we also have $r_2(n) = 1, 8, 56, 392, \ldots$ and $r_3(n) = 1, 11, 92, 716, \ldots$ though we have not yet proved that these are fully integer sequences. The fact that r_ℓ is indeed an integer sequence follows immediately from our proof of Corollary 1.5 below; see Remark 4.7. Our proof follows by simply observing a multiplication relation between the generating functions $r(x;\ell+1)$ and $r(x;\ell)$ for $\ell \geq 1$. In fact by Theorem 1.2 we have

$$r(x;\ell+1) = r(x;\ell))\theta(x), \quad \theta(x) := \frac{1}{12}\,\frac{(3+x^2+\sqrt{x^4-10x^2+9})^2}{3-5x^2/3+\sqrt{x^4-10x^2+9}}. \tag{4.17}$$

Furthermore, for $\theta(x)$ defined by (4.17) we can algebraically show that

$$\theta(x) = f(x) + \frac{1}{2} - \frac{x^2}{2}, \tag{4.18}$$

where again $f(x)$ is given by Corollary 4.2. The verification of (4.18) is easily established by multiplying numerator and denominator of $\theta(x)$ by $3 - 5x^2/3 - \sqrt{x^4-10x^2+9}$ and thus find $\theta(x) = \frac{(3+x^2+\sqrt{x^4-10x^2+9})^2(3-5x^2/3-\sqrt{x^4-10x^2+9})}{12(16x^4/9)} = \frac{(3+x^2+\sqrt{x^4-10x^2+9})(3-x^2-\sqrt{x^4-10x^2+9})}{8x^2} = \frac{1}{4}(5 - x^2 - \sqrt{9-10x^2+x^4})$, which is seen to agree with (4.18) via (1.1).

Definition 4.6. *Define* $s_0(n) = 2 \cdot 3^{2n}[x^{2n}]\theta(x)$, $n = 0, 1, 2\ldots$, *for* $\theta(x)$ *given by* (4.17).

We identify the terms of the sequence s_0 of Definition 4.6 as follows. By (1.1)–(1.2) and (4.18) we have $2\theta(x) = 1 - x^2 + \frac{4}{3}x^2 + \sum_{n=1}^{\infty} \frac{4r(n)}{3^{2n+1}} x^{2n+2} = 1 + \frac{3}{9}x^2 + \sum_{n=1}^{\infty} \frac{12r(n)}{3^{2n+2}} x^{2n+2}$. Therefore by Definition 4.6 we have

$$s_0 = (1, 3, 12r(1), 12r(2), \cdots). \tag{4.19}$$

Proof of Corollary 1.5. By (4.17)–(4.18), we have

$$r(x; \ell + 1) = \frac{1}{2} r(x; \ell)(2\theta(x)). \tag{4.20}$$

By Definitions 1.1, 4.5, and 4.6 together with (4.19), we have

$$r(x; \ell) = 2^{2-\ell} \sum_{n=0}^{\infty} r_\ell(n) 3^{-2n-2} x^{2(n+2)}; \quad 2\theta(x) = \sum_{n=0}^{\infty} s_0(n) 3^{-2n} x^{2n}. \tag{4.21}$$

Hence $\frac{1}{2} r(x; \ell)(2\theta(x)) = 2^{2-(\ell+1)} \sum_{k=0}^{\infty} r_\ell(k) 3^{-2k-2} x^{2(k+2)} \sum_{j=0}^{\infty} s_0(j) 3^{-2j} x^{2j}$. Therefore by (4.17) we have $r(x; \ell + 1) = 2^{2-(\ell+1)} \sum_{n=0}^{\infty} (\sum_{j+k=n} r_\ell(k) s_0(j)) 3^{-2n-2} x^{2(n+2)}$. Hence by replacing ℓ by $\ell + 1$ in the formula for $r(x; \ell)$ of (4.21), and by matching coefficients of powers $x^{2(n+2)}$, the proof is complete. □

Remark 4.7. *It is now clear by induction and Corollary 1.5 that for each* $\ell \geq 1$ *the sequence* $r_\ell(n)$, $n \geq 0$, *is a sequence of positive integers starting with* $r_\ell(0) = 1$.

Example 4.8. *We apply Corollary 1.5 to obtain the first few terms of the sequences* $r_2(n)$ *and* $r_3(n)$ *which were shown instead via Taylor expansion and Definition 4.5 in* (4.16). *Indeed,*

$$\begin{aligned} r_2 &= r_1 * s_0 \\ &= (1, 5, 29, 185, \cdots) * (1, 3, 12, 60, \ldots) = (1,\ 5 + 3,\ 29 + 5 \cdot 3 + 12, \ldots) = (1, 8, 56, 392, \ldots) \end{aligned}$$

Next, plugging in the result of r_2 *to compute* r_3, *we have*

$$r_3 = r_2 * s_0 = (1,\ 8 + 3,\ 56 + 8 \cdot 3 + 12,\ 392 + 56 \cdot 3 + 8 \cdot 12 + 60, \ldots) = (1, 11, 92, 716, \ldots).$$

Remark 4.9. *By Theorem 1.2 and Definitions 1.1 and 4.5 we easily find the ordinary generating function of the integer sequence* r_ℓ *by*

$$\sum_{n=0}^{\infty} r_\ell(n) x^n = \frac{2^{-\ell-2}}{x} (1 + 3x - \sqrt{1 - 10x + 9x^2}) \frac{(1 + 3x + \sqrt{1 - 10x + 9x^2})^{2\ell-1}}{(1 - 5x + \sqrt{1 - 10x + 9x^2})^\ell}.$$

5. Proof of Theorem 1.6

In this section, besides the conditional distribution of $\mathbf{R}$ given $\mathbf{m} \geq \ell$, we consider the conditional distribution of $\mathbf{R}$ given $\mathbf{m} = \ell$ and $\mathbf{S}_1 = +1$. We then show how the two distributions relate by our proof of Theorem 1.6.

Definition 5.1. *We denote the probability generating function of* $\sigma_\ell(n)$, $n \geq 0$, *by*

$$s(x; \ell) = E\{x^{\mathbf{R}} \mid \mathbf{m} = \ell, \mathbf{R} \geq 4, \mathbf{S}_1 = +1\} = \sum_{n=0}^{\infty} x^{2(n+2)} \sigma_\ell(n),$$

where $\sigma_\ell(n) = P(\mathbf{R} = 2(n + 2) \mid \mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} = \ell)$.

It is easy to see that for each $\ell \geq 1$ we have $P(\mathbf{m} \geq \ell+1, \mathbf{R} \geq 4) = \frac{1}{2}P(\mathbf{m} \geq \ell, \mathbf{R} \geq 4)$, and both of these equal $P(\mathbf{m} = \ell, \mathbf{R} \geq 4)$. Thus, after taking into account that $P(\mathbf{m} = \ell, \mathbf{R} = 2(n+2)) = P(\mathbf{m} \geq \ell, \mathbf{R} = 2(n+2)) - P(\mathbf{m} \geq \ell+1, \mathbf{R} = 2(n+2))$, we have that

$$\begin{aligned}\sigma_\ell(n) &= \frac{P(\mathbf{m} = \ell, \mathbf{R} = 2(n+2))}{P(\mathbf{m} = \ell, \mathbf{R} \geq 4)} = \frac{2P(\mathbf{m} \geq \ell, \mathbf{R} = 2(n+2))}{P(\mathbf{m} \geq \ell, \mathbf{R} \geq 4)} - \frac{P(\mathbf{m} \geq \ell+1, \mathbf{R} = 2(n+2))}{P(\mathbf{m} \geq \ell+1, \mathbf{R} \geq 4)} \\ &= 2\rho_\ell(n) - \rho_{\ell+1}(n); \qquad s(x;\ell) = 2r(x;\ell) - r(x;\ell+1).\end{aligned} \tag{5.1}$$

Definition 5.2. *For each $\ell \geq 1$ we define the sequence $s_\ell(n), n = 0, 1, 2, \ldots$ by*

$$s_\ell(n) = 2^{\ell-1}3^{2n+1}\sigma_\ell(n).$$

Remark 5.3. *By Definitions 4.5 and 5.1 we calculate that $s_\ell(n) = \frac{4}{3}r_\ell(n) - \frac{1}{3}r_{\ell+1}(n)$. By Corollary 1.5 and (4.19) we have that $4r_\ell(n) - r_{\ell+1}(n)$ is divisible by 3. Therefore since by Definition 5.2 we have that $s_\ell(n)$ is positive, we indeed have that $s_\ell(n)$, $n \geq 0$, is a positive integer sequence starting with $s_\ell(0) = 1$.*

We shall give two proofs of Theorem 1.6. The first proof is a short, purely algebraic calculation involving the formula for $r(x;\ell)$ given by Theorem 1.2. To help explain the result we give a second proof involving surgery on collections of paths.

Proof of Theorem 1.6. Denote $p = p(x) = 3 + x^2 + \sqrt{x^4 - 10x^2 + 9}$ and $q = q(x) = 3 - 5x^2/3 + \sqrt{x^4 - 10x^2 + 9}$. By Theorem 1.2 we have that

$$r(x;\ell) = 4x^2 12^{-\ell} f\ \frac{p^{2\ell-1}}{q^\ell} \tag{5.2}$$

Therefore, by (5.1)–(5.2) we have

$$\begin{aligned}&s(x;a)s(x;b) - x^4 r(x;a+b) \\ &= 4^2x^4 12^{-(a+b)} f \left\{ f\left(\frac{2p^{2a-1}}{q^a} - \frac{p^{2a+1}}{12q^{a+1}}\right)\left(\frac{2p^{2b-1}}{q^b} - \frac{p^{2b+1}}{12q^{b+1}}\right) - \frac{x^2}{4}\frac{p^{2(a+b)-1}}{q^{a+b}}\right\} \\ &= 4^2x^4 12^{-(a+b)} f\ \frac{p^{2(a+b)-1}}{q^{a+b}} \left\{ fp\left(\frac{4}{p^2} - \frac{4}{12q} + \frac{p^2}{12^2q^2}\right) - \frac{x^2}{4}\right\}.\end{aligned} \tag{5.3}$$

Now $fp = 4x^2$ and $\frac{4}{p^2} - \frac{4}{12q} + \frac{p^2}{12^2q^2} = \left(\frac{2}{p} - \frac{p}{12q}\right)^2 = \left(\frac{1}{4}\right)^2$. Therefore the last expression in curly brackets in (5.3) is seen to be zero. Therefore we have completed an algebraic proof of the theorem.

We next give another proof by path surgery arguments. Taking into account the normalizing terms implicit in the Definitions 1.1 and 5.1 of the conditional probabilities $\rho_\ell(n)$ and $\sigma_\ell(n)$, to show the statement of the proposition we must verify that for all $a, b \geq 1$ and all $k \geq 0$ we have

$$\begin{aligned}&P(\mathbf{S}_1 = +1, \mathbf{R} = 2(k+2), \mathbf{m} \geq a+b)\frac{P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} = a)P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} = b)}{P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} \geq a+b)} \\ &= \sum_{i+j=k} P(\mathbf{S}_1 = +1, \mathbf{R} = 2(i+2), \mathbf{m} = a)P(\mathbf{S}_1 = +1, \mathbf{R} = 2(j+2), \mathbf{m} = b)\end{aligned} \tag{5.4}$$

Here the fraction involving the normalization terms for $\sigma_\ell(n)$ and $\rho_\ell(n)$ in the first line turns out to be independent of a and b. Indeed, $P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} = a) = \frac{1}{2}\sum_{s=1}^{\infty} 2^{-(2s+a)} = \frac{1}{6}2^{-a}$,

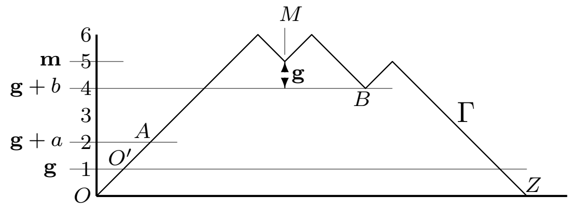

FIGURE 2. A path $\Gamma \in \{\mathbf{S}_1 = +1, \mathbf{m} \geq 4, \mathbf{R} = 6\}$. Here $\mathbf{m} = \mathbf{m}(\Gamma) = 5$ with gap $\mathbf{g} = \mathbf{m} - 4 = 1$. We take $a = 1$ and $b = 3$. O' is the translation of the origin O by $\mathbf{g}$ positive steps. A is the point of intersection of the first ascent of Γ with $y = \mathbf{g} + a$. M is the point where the first minimum $\mathbf{m}$ is achieved. B is the first point after M on Γ that touches the line $y = \mathbf{g}+b$. Z is the terminus.

while $P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} \geq a+b) = \frac{1}{2}\sum_{\ell=a+b}^{\infty}\sum_{s=1}^{\infty} 2^{-(2s+\ell)} = \frac{1}{3}2^{-(a+b)}$. Therefore the fraction in the first line of (5.4) is $\frac{(1/6)^2 2^{-a}2^{-b}}{(1/3)2^{-(a+b)}} = \frac{1}{12}$. Hence, we must show

$$\begin{aligned}&\frac{1}{12}P(\mathbf{S}_1 = +1, \mathbf{R} = 2(k+2), \mathbf{m} \geq a+b)\\&= \sum_{i+j=k} P(\mathbf{S}_1 = +1, \mathbf{R} = 2(i+2), \mathbf{m} = a)P(\mathbf{S}_1 = +1, \mathbf{R} = 2(j+2), \mathbf{m} = b).\end{aligned} \tag{5.5}$$

Notice that the factor x^4 in the statement of the theorem will come out in the wash after verifying (5.5) since there are $2(i+j+4) = 2(k+4) = 2(k+2)+4$ runs in total, that is $(k+2)+2 = k+4$ peaks, represented from two independent positive excursions on the right side of this probability equation. The way we will verify (5.5) is to fix $k \geq 0$ and $a, b \geq 1$ and modify the collection of paths $\Lambda = \{\mathbf{S}_1 = +1, \mathbf{R} = 2(k+2), \mathbf{m} \geq a+b\}$ by inserting two tent paths in each path $\Gamma \in \Lambda$. Recall that an $h \times h$–tent for any $h \geq 1$ is a path consisting of h steps up immediately followed by h steps down. Note that any peak on a positive excursion path has a 1×1–tent embedded at its apex. We identify three points denoted A, M, B on each such path Γ; see Figure 2 for an illustration with the initial minimum $\mathbf{m} = 5$ and $\mathbf{R} = 6$ (or $k = 1$), and $a = 1$, $b = 2$. The point M is the point on Γ where the initial minimum is achieved. We define $\mathbf{g} = \mathbf{m} - (a+b)$ to be the *gap* between the actual initial minimum and the level $a+b$. There must be an initial ascent of at least $\mathbf{m}+1 = \mathbf{g}+(a+b)+1$ steps on Γ. Define the point A on this initial ascent of Γ after exactly $\mathbf{g}+a$ positive steps from the origin O. The point B is the point following the point M where the path Γ first touches the line $y = \mathbf{g}+b$; see Figure 2. The other points in this Figure are: O', the point after exactly $\mathbf{g}$ positive steps from the origin, and Z, the point of termination of Γ.

We take a two step approach to verifying (5.5). As Step 1 we shall obtain an event Λ^+ based on Λ with probability $P(\Lambda^+) = \frac{1}{12}P(\Lambda)$. To accomplish this we will both lengthen each path $\Gamma \in \Lambda$ as well as increase the number of paths of Λ. We first cut out the subpath BZ without removing the point B from Γ and set it aside to be pasted after appropriate modifications. Next, we cut out the initial segment OO' and translate it (horizontally and vertically) so that O matches B and we paste this segment onto Γ, calling the translated segment of exactly $\mathbf{g}$ positive steps now BB'. If $\mathbf{g} = 0$ then we simply take $B' = B$. We now insert an $h \times h$–tent of arbitrary size $h \geq 1$ at the point A; that is we generate for each $h \geq 1$ a path $\Gamma_h \in \Lambda^+$ by the insertion at A. All the following points M, B, B' at this stage will of course be translated horizontally by $2h$ units from their original positions in Γ to their positions in Γ_h but we do not change their names via this operation. We insert a 1×1–tent at the endpoint B', calling

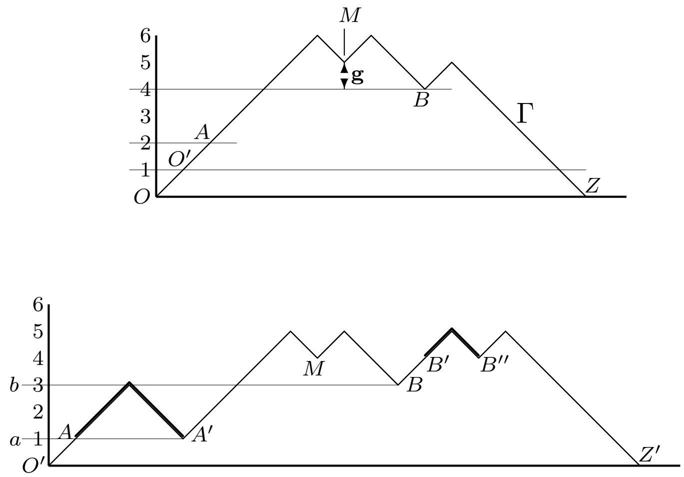

FIGURE 3. Step 1: Revise the path Γ of Figure 2 shown at top. Cut out the subpath BZ to be pasted later; leave the point B as the right endpoint of the subpath OB of Γ. Cut OO' and paste O at B, calling the result BB'. Insert an arbitrary $h \times h$–tent AA' at A; for this example $h = 2$. Insert a 1×1–tent $B'B''$. Paste the left endpoint B of BZ at B''; call the result $B''Z'$. Translate the whole revised path to the new origin O'.

this tent now $B'B''$. We now paste the segment BZ that we had removed earlier by placing the endpoint B at B''. The right endpoint Z of the pasted subpath BZ now sits at the end of the modified path exactly $\mathbf{g}$ units above the x–axis at the same level as O'. Finally we translate the whole modified path Γ_h, running from O' onwards, now back to the origin and rename the origin as O' and rename the terminus as Z'; see Figure 3. Notice that by the two insertions of tents, we have indeed constructed positive excursions in Λ^+ that have $k+4$ peaks instead of $k+2$ peaks. Since an arbitrary $h \times h$–tent has probability $\frac{1}{3}$, and a single 1×1–tent has probability $\frac{1}{4}$, we indeed have $P(\Lambda^+) = \frac{1}{12}P(\Lambda)$. This completes Step 1.

As Step 2 it remains to show how we get the right side of (5.5) from $P(\Lambda^+)$. In Step 2 we will simply rearrange the path produced in Step 1. Technically we note that Λ^+ decomposes into disjoint events $\Lambda^+_{i,j}$ where $i + j = k$ for some $0 \le i, j \le k$. In the bottom frame of Figure 4 the rearrangement of the outcome from Step 1 consists of two successive positive excursions: an a–excursion and a b–excursion respectively. In our illustration, the a–excursion has $i = 0$ for $i + 2 = 2$ peaks ultimately because there is just one peak between M and B in the original path Γ of Figure 2; there must be one such peak but in our illustration we have taken the bare minimum number of peaks between M and B for this requirement. For the b–excursion we have $j = 1$, for $j + 2 = 3$ peaks, because there is exactly one peak between B and Z in the original path Γ; in general there need not be any peaks after the point B, and j is simply the number of such peaks after B in the original path Γ. The description of the rearrangement to form the bottom frame of Figure 4 is as follows. From Step 1, cut out pieces $A'M$ and MB, that is divide the subpath $A'MB$ in two pieces, each to be pasted, yet leave the labeled endpoints A' and B at their original positions in the path of $\Lambda^+_{i,j}$ where i and j have been determined by the original path Γ. Switch the order of the divided pieces, by first translating the piece MB to the left with its translated left endpoint M now called M' pasted to A', and

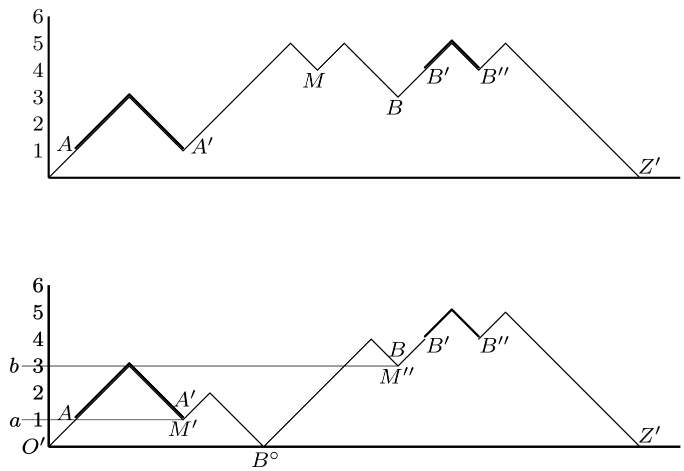

FIGURE 4. Step 2: Rearrange the path from Step 1 (shown at top). Cut out pieces $A'M$ and MB; leave the labeled endpoints A' and B at their original positions. Paste the cut piece MB with M at A'; call the pasted piece $M'B^\circ$; it completes a first excursion $O'M'B^\circ$. Paste the cut piece $A'M$ with A' at B°. The translated right endpoint M (shown as M'') coincides with the point B.

call the translated piece now $M'B^\circ$, so as to complete the a–excursion $O'M'B^\circ$. Here we note that the initial minimum for this a–excursion is exactly a because by construction the level of A' at the end of Step 1 is $(\mathbf{g}+a)-\mathbf{g}=a$. Next, paste the original piece $A'M$, to now serve as the first leg of the b–excursion, by translating A' to B°. The translated right endpoint M now coincides with the point B of the endif Step 1 and is illustrated with the additional name M''. Thus the point M at the end of Step 1, that originally marked the first valley after the initial peak on Γ, is translated in Step 2 in two ways: to the left to become the first valley M' of the a–excursion and to the right to become the first valley M'' of the b–excursion. This completes Step 2.

To summarize, we may reverse the Steps 1 and 2 to create a path Γ. Assume we have two successive positive excursions, the first an a–excusion with initial minimum exactly a and $i+2$ peaks and the second a b–excursion with initial minimum exactly b and $j+2$ peaks. Call $A'=M'$ the first valley point of the a–excursion, call B° the right endpoint of the a–excursion that is also the left endpoint of the b–excursion, and call $B=M''$ the first valley point of the b–excursion. Cut out the segments $B^\circ M''$ and $M'B^\circ$ but do not remove the labels A' and B where they originally stood. That is divide $M'B^\circ M''$ in two at B°. Now switch their left to right order by translations in the plane by both pasting the right endpoint B° of the cut segment $M'B^\circ$ now at B and pasting the left endpoint B° of the cut segment $B^\circ M''$ now at A'. Because B° was a common point of the two segments it is easy to see that the two repositioned segments will meet at a single point M to form a single excursion where the height of M currently equals $a+b$. Next we must cut the 1×1–tent at the top of the first peak after point B (such a peak must exist by $j+2\geq 2$); call this tent $B'B''$ and leave the tent base points B' and B'' on the two sides of the disconnected path after the cut. Before rejoining the path we must also cut the segment BB' of positive steps (if any) that remains, that is the maximal upward segment immediately preceding the 1×1–tent; do not remove the label B. We define $\mathbf{g} \geq 0$ as the number of positive steps in this maximal upward segment. We translate every point that remains of the subpath from the origin to B by $\mathbf{g}$ positive steps in order to relocate the removed upward segment BB' as the first $\mathbf{g}$ steps of the initial ascent of this subpath from the origin. We do not rename the points A', M, and B after this translation. At this stage the point A' lies at height $\mathbf{g} + a$, M lies at height $\mathbf{g} + a + b$, and B lies at height $\mathbf{g} + b$ and coincides with the previous position of B'. We now rejoin the remainder of the path by translating B'' and the path after it to the left by 2 units so that B'' is pasted at B. We now have a single excursion path again but with one peak removed that originally was the second peak of the b–excursion. Finally, there must be some maximal $h \times h$–tent forming the apex of the initial peak of the whole path as it stands (the initial peak of what had been the a–excursion) such that the right base point of the tent is positioned at the first valley point A'; call the left base point of this tent as A. We remove this peak by cutting off this tent and by rejoining the path by horizontally translating the entire path that lies to the right of A' by $2h$ units left to the point A. Thus we have removed one peak from what had been the first peak of the a–excursion. This completes the reversal of the Steps 1 and 2. Thus (5.5) has been verified and the second proof of the theorem is complete. □

5.1. **Corollaries of Theorem 1.6.** Recall the Definitions 4.5 and 5.2. By Remarks 4.7 and 5.3 both r_ℓ and s_ℓ are positive integer sequences.

Corollary 5.4. *For any $a, b \geq 1$ we have $(s_a * s_b)(n) = r_{a+b}(n)$, for all $n \geq 0$.*

Proof. By Definitions 1.1, 4.5, 5.1, and 5.2, we have

$$\begin{aligned} x^4 r(x; a+b) &= \frac{4}{2^{a+b}} x^4 \left(\frac{r_{a+b}(0)}{3^2} x^4 + \frac{r_{a+b}(1)}{3^4} x^6 + \frac{r_{a+b}(2)}{3^6} x^8 + \cdots \right); \\ s(x;a)s(x;b) &= \frac{2}{2^a}\frac{2}{2^b} \left(\frac{s_a(0)}{3^1} x^4 + \frac{s_a(1)}{3^3} x^6 + \cdots \right) \left(\frac{s_b(0)}{3^1} x^4 + \frac{s_b(1)}{3^3} x^6 + \cdots \right). \end{aligned} \tag{5.6}$$

Therefore, since by Theorem 1.6 the two formula in (5.6) are equal, by matching the coefficients of powers of x^{2k+8} for $k = 0, 1, 2, \ldots$, we have $\sum_{i+j=k} \frac{s_a(i) s_b(j)}{3^{2i+1} 3^{2j+1}} = \frac{r_{a+b}(k)}{3^{2k+2}}$, which immediately gives the result. □

We want to have a nice formula for $s(x; 1)$. Recall the definition $f(x) = E\{x^{\mathbf{R}}\}$ appearing in Corollary 4.2. Also recall $s(x; 1)$ as given by Definition 5.1 for $\ell = 1$.

Lemma 5.5. *We have*

$$s(x; 1) = 2x^2 f(x) - x^4.$$

Proof. We give a simple path construction based on introducing an arbitrary $h \times h$–tent after the first step of a positive excursion assuming there are at least two peaks in such an excursion. Thus, if $n \geq 1$, so that $n + 1 \geq 2$, then, because the probability of an arbitrary tent is $\frac{1}{3}$, we have

$$\frac{1}{3} P(\mathbf{S}_1 = +1, \mathbf{R} = 2(n+1)) = P(\mathbf{S}_1 = +1, \mathbf{R} = 2(n+2), \mathbf{m} = 1). \tag{5.7}$$

But the right side of (5.7) is the numerator of the conditional probability expression for $\sigma_1(n)$ as given by Definition 5.1 for $\ell = 1$ and the normalization probability is $P(\mathbf{S}_1 = +1, \mathbf{R} \geq 4, \mathbf{m} = 1) = \frac{1}{12}$. Therefore since the left side of (5.7) may be rewritten by symmetry as $\frac{1}{6} P(\mathbf{R} = 2(n+1))$, we have for all $n \geq 1$ that $P(\mathbf{R} = 2(n+1)) = \frac{6}{12}\sigma_1(n)$. Therefore $f(x) = \sum_{n=0}^{\infty} P(\mathbf{R} = 2(n+1)) x^{2(n+1)} = \frac{2}{3}x^2 + \frac{6}{12}\sum_{n=1}^{\infty} \sigma_1(n) x^{2(n+1)}$. But we easily see that $\sigma_1(0) = (\frac{1}{6})^2 / \frac{1}{12} = \frac{1}{3}$. So, taking account of the definition $s(x; 1) = \sum_{n=0}^{\infty} \sigma_1(n) x^{2(n+2)}$, we have $f(x) = \frac{2}{3}x^2 + \frac{1}{2}\left(s(x; 1) - \frac{1}{3}x^4\right)/x^2 = \frac{1}{2}x^2 + \frac{1}{2}s(x; 1)$, or $2x^2 f(x) = x^4 + s(x; 1)$. □

Recall the generating functions $r(x;\ell)$ and $s(x;\ell)$ of Definitions 1.1 and 5.1 and integer sequences r_ℓ and s_ℓ of Definitions 4.5 and 5.2. By (1.1)–(1.2) we have $f(x)=\sum_{n=0}^\infty \frac{2r(n)}{3^{2n+1}}x^{2(n+1)}$ for the rook number sequence $r(n)=1,1,5,29,185,\ldots$, [10, A059231]. Therefore we have:

Remark 5.6. *By Lemma 5.5 and Definitions 5.1–5.2 we have that* $s_1(0)=1$ *and* $s_1(n)=4r(n)$ *for all* $n\geq 1$, *or* $s_1(n)=1,4,20,116,740,\ldots$ *[10, A082298].*

The fact that $s_1(n)$ gives the number of rook paths for positive excursions of semi length $n+1$ follows from the fact that $r(n)$ gives the number of rook paths corresponding to general Dyck paths (zeros allowed) of semi length n, [2],[6]. Indeed we may simply raise the Dyck paths by one step at the beginning and one at the end to introduce the factor of 4 in the number of rook paths; see [2, Lemma 2.1].

Corollary 5.7. *For all* $\ell\geq 1$ *we have* $s(x;\ell)f(x)=x^2r(x;\ell)$. *Consequently we have the convolution relation* $s_\ell * r=r_\ell$.

Proof. By Lemma 5.5 we have $s(x;\ell)f(x)=s(x;\ell)(x^4+s(x;1))/(2x^2)$. We rewrite this last expression and then apply Theorem 1.6 to find $s(x;\ell)f(x)=\frac{x^2}{2}\left(s(x;\ell)+s(x;1)s(x;\ell)/x^4\right)=\frac{x^2}{2}\left(s(x;\ell)+r(x;\ell+1)\right)$. But as shown in (5.1) we have $s(x;\ell)=2r(x;\ell)-r(x;\ell+1)$. Therefore $s(x;\ell)f(x)=\frac{x^2}{2}(2r(x;\ell)-r(x;\ell+1)+r(x;\ell+1))=x^2r(x;\ell)$. The stated convolution relation follows from the series formulation of the identity $s(x;\ell)f(x)=x^2r(x;\ell)$ as follows

$$\frac{2}{2^\ell}\sum_{n=0}^\infty \frac{s_\ell(n)}{3^{2n+1}}x^{2(n+2)}\sum_{n=0}^\infty \frac{2r(n)}{3^{2n+1}}x^{2(n+1)}=x^2 2^{2-\ell}\sum_{n=0}^\infty \frac{r_\ell(n)}{3^{2n+2}}x^{2(n+2)}.$$

□

Remark 5.8. *For each* $\ell\geq 1$ *we have* $s_{\ell+1}=s_\ell * s_0$, *for* s_0 *written by* (4.19).

Proof. By (4.17) and (5.1) we have $s(x;\ell+1)=s(x;\ell)\theta(x)$. Thus the result follows by Definitions 4.6, 5.1, and 5.2. A second proof goes by convolution algebra as follows. By Corollary 5.7 we have $s_1 * r=r_1$. Since $r_1(n)=r(n+1)$ we therefore have that $0=r_1(n)-r_1(n)=r(n+1)-(s_1*r)(n)$, where s_1 is given by Remark 5.6. Hence the convolution inverse r^{-1} of the rook sequence r is: $r^{-1}(n)=1$ if $n=0$ and $r^{-1}(n)=-s_1(n-1)$ if $n\geq 1$. Thus $r^{-1}=(1,-1,-4,-20,-116,-740,...)$. By Remark 5.3 we have $s_\ell=(4r_\ell-r_{\ell+1})/3$. Thus by Corollary 5.7 we have $s_{\ell+1}*r=r_{\ell+1}=4r_\ell-3s_\ell$. Hence $s_{\ell+1}=(4r_\ell-3s_\ell)*r^{-1}=4s_\ell-3s_\ell*r^{-1}$. Denote the convolution identity $\delta_0=(1,0,0,\ldots)$. Observe that $4\delta_0-3r^{-1}=(1,4,12,60,484,\ldots)=(1,3s_1(0),3s_1(1),3s_1(2),\ldots)=s_0$. Thus $s_{\ell+1}=s_\ell*s_0$. □

Example 5.9. *By the explicit formula for* $r(x;\ell)$ *of Theorem 1.2 and the formula* $s(x;2)=2r(x;2)-r(x;3)$ *of* (5.1), *we have* $s(x;2)=\frac{1}{6}x^4+\frac{7}{54}x^6+\frac{22}{243}x^8+\frac{142}{2187}x^{10}+\frac{958}{19683}x^{12}+\cdots$. *Thus, by Definitions 5.1–5.2,* $s_2(n)=(1,7,44,284,\ldots)$. *By Remark 5.8 we have* $s_2=s_1*s_0=(1,4,20,116,740,\ldots)*(1,3,12,60,348,\ldots)=(1,7,44,284,1916,\ldots)$.

6. Proof of Corollary 1.3

Proof of Corollary 1.3. Let $a>0$ and $\lambda>0$. We will compute the limiting Laplace transform of the distribution of $\mathbf{T}_N=\mathbf{T}_{N,a}=N^{-2}\mathbf{R}_{\lfloor aN\rfloor}$ as $N\to\infty$. Define

$$\beta=\frac{1}{4}(5-x^2);\qquad \alpha=\sqrt{\beta^2-1}=\frac{1}{4}\sqrt{x^4-10x^2+9}. \tag{6.1}$$

By (6.1) we have

$$2-\beta+\alpha=\frac{1}{4}\left(3+x^2+\sqrt{x^4-10x^2+9}\right);\quad -4+5\beta-3\alpha=\frac{1}{4}\left(9-5x^2+3\sqrt{x^4-10x^2+9}\right) \tag{6.2}$$

By (6.2) the formula for $r(x;\ell)$ of Theorem 1.2 may be written

$$r(x;\ell)=x^2f(x)\frac{(2-\beta+\alpha)^{2\ell-1}}{(-4+5\beta+3\alpha)^{\ell}} \tag{6.3}$$

Finally put $x=e^{-\lambda/N^2}$ and $\ell=\lfloor aN\rfloor$. By the exponential series $x^2=e^{-2\lambda/N^2}=1-\frac{2\lambda}{N^2}+\frac{2\lambda^2}{N^4}+\cdots$, we easily find by (6.1) that

$$\beta=1+\frac{\lambda}{2N^2}-\frac{\lambda^2}{2N^4}+\cdots;\quad \beta^2-1=\frac{\lambda}{N^2}-\frac{3\lambda^2}{4N^4}+\cdots;\quad \alpha=\frac{\sqrt{\lambda}}{N}+O(N^{-3}),\quad N\to\infty. \tag{6.4}$$

By (6.4) we have

$$2-\beta+\alpha=1+\frac{\sqrt{\lambda}}{N}+O(N^{-2});\qquad -4+5\beta+3\alpha=1+\frac{3\sqrt{\lambda}}{N}+O(N^{-2}). \tag{6.5}$$

Since $\ell\sim aN$, as $N\to\infty$, and $\lim_{x\to 1}x^2f(x)=1$, by (6.3) and (6.5) we have

$$\lim_{N\to\infty}r(e^{-\lambda/N^2};\lfloor aN\rfloor)=e^{2a\sqrt{\lambda}}/e^{3a\sqrt{\lambda}}=e^{-a\sqrt{\lambda}}.$$

Thus $\mathbf{T}_N$ converges in distribution, where the Laplace transform of the limit distribution is $e^{-a\sqrt{\lambda}}$. Therefore the limit distribution has a density given by the inverse Laplace transform as follows: $\frac{a}{2\sqrt{\pi}}t^{-3/2}e^{-a^2/4t}$, $t>0$. □

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