PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent either by email (preferred) to CMJproblems@maa.org as a TEX, pdf, or Word attachment or by mail to Curtis Cooper at the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problems are not under consideration by another journal and otherwise unpublished.

Solutions to the problems in this issue should be sent no later than October 15, 2016 either by email (preferred) to CMJsolutions@maa.org as a TEX, pdf, or Word attachment or by mail to Chip Curtis at the address provided above.

PROBLEMS

1076. *Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Consider a triangle with sides a, b, and c and circumradius R and $x, y, z > 0$. Prove that

$$\frac{x+y}{za^4} + \frac{y+z}{xb^4} + \frac{z+x}{yc^4} \geq \frac{2}{3R^4}.$$

1077. *Proposed by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece.*

For each integer $n \geq 1$, let $a_n = 1/(\ln(n+1) - \ln n)$. Prove that

$$\sum_{n=1}^{\infty} \frac{(a_n - n)^2}{(a_n + n)(a_n + n + 1)} < \frac{\pi}{8} - \frac{1}{3}.$$

1078. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Consider the well-ordered set $\mathbb{N}$ of natural numbers along with the usual Euclidean metric $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by $d(x, y) = |x - y|$. Clearly, whenever $x < y < z$ are natural numbers, we have $d(y, z) < d(x, z)$. Prove or disprove: There exists an

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uncountable well-ordered set $(X, <)$ (that is, $<$ is a well-order on X) and a function $f\colon X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$, if $x < y < z$, then $f(y, z) < f(x, z)$.

1079. *Proposed by Valeriy Karachik and Leonid Menikhes, South Ural State University, Chelyabinsk, Russia.*

Prove that the following limit exists and evaluate it. Here $\{a\}$ denotes the fractional part of a:

$$\lim_{n\to\infty} \left\{\sqrt{n^2+1} + \cdots + \sqrt{n^2+2n}\right\}.$$

1080. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

For $x, y, z > 0$ with $x + y + z = 3$, prove that

$$24xyz \le 8\left(\sum yz\right) \le \sum \frac{(x+y)^4}{x^2+y^2} \le 8\left(\sum x^2\right) \le 8\left(\frac{3}{xyz}\right) \le 8\left(\sum \frac{1}{x^2}\right),$$

where the sums are over all cyclic permutations of (x, y, z).

SOLUTIONS

GLB and LUB

1051. *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Find the greatest lower bound and least upper bound of

$$\frac{n - \left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right)}{1 - \left(\frac{1}{n+x_1} + \frac{1}{n+x_2} + \cdots + \frac{1}{n+x_n}\right)}$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers.

Solution by Michael Andreoli, Miami-Dade College, Miami, FL.

We claim that the least upper bound is n^2 and the greatest lower bound is n.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and

$$Q(\mathbf{x}) = \frac{n - \sum_{i=1}^{n} \frac{1}{1+x_i}}{1 - \sum_{i=1}^{n} \frac{1}{n+x_i}}$$

for $x_i > 0$. A simple algebraic manipulation shows that

$$Q(\mathbf{x}) = n \cdot \frac{\sum_{i=1}^{n} \frac{x_i}{1+x_i}}{\sum_{i=1}^{n} \frac{x_i}{n+x_i}}.$$

Letting $\mathbf{a} = (a, a, \ldots, a)$ with $a > 0$, we have

$$Q(\mathbf{a}) = \frac{n(n+a)}{1+a},$$

from which $\lim_{a\to 0^+} Q(\mathbf{a}) = n^2$ and $\lim_{a\to\infty} Q(\mathbf{a}) = n$.