1118. *Proposed by Greg Oman and Logan Robinson, University of Colorado Colorado Springs, CO.*

Let set $D(\mathbb{R})$ of differentiable functions $f:\mathbb{R}\to\mathbb{R}$ is a semigroup under the usual composition of functions. Find all finite subsemigroups of $D(\mathbb{R})$ (up to algebraic isomorphism) which do not contain a constant function.

1119. *Proposed by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece.*

For any number x with $0<x<1$, prove that

$$\sum_{n=1}^{\infty}\frac{x^n}{1+x+x^2+\cdots+x^n}<\log\left(\frac{1}{1-x}\right).$$

1120. *Proposed by Johannas Winterink, Carlsbad, NM.*

The three circles with equations $x^2+y^2=3^2$, $(x-28)^2+y^2=25^2$, and $(x+8)^2+(y-15)^2=14^2$ are mutually tangent. The two nonintersecting circles tangent to all three given circles are called their Soddy circles. Determine the line containing the inner center S and outer center S' of these Soddy circles.

SOLUTIONS

Two-by-two integer matrices

1091. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $A\in\mathcal{M}_2(\mathbb{Z})$, the set of two by two integer matrices. Prove that $\sin A\in\mathcal{M}_2(\mathbb{Z})$ if and only if A^2 is the zero matrix.

Solution by Tommy Goebeler, The Episcopal Academy, Newtown Square, PA.

If $A^2=0$, then $\sin A=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}A^{2n+1}=A\in\mathcal{M}_{\in}(\mathbb{Z})$.

Now suppose A has Jordan canonical form J, where $A=PJP^{-1}$ for some invertible matrix P. We consider two cases.

If A is diagonalizable, then $J=\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$ for some λ_1,λ_2, in which case

$$\sin A=\sin(PJP^{-1})=P(\sin J)P^{-1}=P\begin{bmatrix}\sin\lambda_1&0\\0&\sin\lambda_2\end{bmatrix}P^{-1}.$$

Because $\det A=\det J$, we have $\sin\lambda_1\sin\lambda_2\in\mathbb{Z}$. Because λ_1 and λ_2 are not transcendental (since they are the roots of a polynomial with integer coefficients), at least one of them must be zero. Because trace is invariant under cyclic permutations of products, trace $J=$ trace $A\in\mathbb{Z}$, and so $\sin\lambda_1+\sin\lambda_2\in\mathbb{Z}$. From this we see that both λ_1 and λ_2 are zero. It follows that A, and thus A^2, is the zero matrix.

If A is not diagonalizable, then its Jordan canonical form consists of a single Jordan block, $J=\begin{bmatrix}\lambda&1\\0&\lambda\end{bmatrix}$. Note that $J^n=\begin{bmatrix}\lambda^n&n\lambda^{n-1}\\0&\lambda^n\end{bmatrix}$. So

$$\sin A=P\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\begin{bmatrix}\lambda^{2n+1}&(2n+1)\lambda^{2n}\\0&\lambda^{2n+1}\end{bmatrix}\right)P^{-1}=P\begin{bmatrix}\sin\lambda&\cos\lambda\\0&\sin\lambda\end{bmatrix}P^{-1}.$$