1134. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

The usual Euclidean metric d on the natural numbers defined by $d(x, y) = |x - y|$ has the property that, for natural numbers x, y, z, if $x < y < z$, then $d(x, y) < d(x, z)$. Prove or disprove: There exists an uncountable well-ordered set $(X, <)$ (that is, $<$ is a well-order on X) and a function $f : X \times X \to \mathbb{R}$ such that, for $x, y, z \in X$, if $x < y < z$, then $f(x, y) < f(x, z)$.

1135. *Proposed by Alan Loper, The Ohio State University at Newark, OH.*

Prove or disprove: There exists an infinite noncommutative ring R with identity 1_R such that every *proper* unital subring S of R (that is, S is a subring of R such that $1_R \in S$) is commutative.

SOLUTIONS

Groups with intersecting generating sets

1106. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*

Let G be a group and let $S \subseteq G$. Further, let $S^{-1} = \{s^{-1} \mid s \in S\}$. Recall that S is a *generating set* for G if every member of G is a finite product of elements, each of which is a member of S or S^{-1}. Find all nontrivial groups G with the property that any two generating sets of G have nonempty intersection.

Solution by the Anthony Bevalacqua, University of North Dakota, Grand Forks, ND.

We prove that the only such groups are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the cyclic group of order two.

Let G be a nontrivial group with the property that any two generating sets have nonempty intersection.

We first note that every nontrivial subgroup H has the same property: Assume to the contrary that H has disjoint generating sets S and T. Since $|H| > 1$, every coset of H in G contains at least two elements. So we can expand S to S^*, a generating sets for G, by choosing one element in each coset of H in G other than from the coset H itself. Similarly, we can expand T to T^* by choosing a different element in each such coset than the one we choose earlier. Now S^* and T^* are disjoint generating sets for G, a contradiction.

For any $x \in G$, both $\{x\}$ and $\{x^{-1}\}$ generate $\langle x \rangle$, so $x = x^{-1}$ by the last paragraph. Thus, each element of G has order at most two. Hence, G is abelian and we can regard G as an $\mathbb{F}_2$-vector space.

Finally, the dimension of G over $\mathbb{F}_2$ must be at most two. Otherwise there would be linearly independent $a, b, c \in G$ and the subgroup $H = \langle a, b, c \rangle$ would have disjoint generating sets $\{a, b, c\}$ and $\{a + b, a + c, a + b + c\}$. Thus, G must be isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Conversely, these two groups have the desired property: Any generating set for $\mathbb{Z}_2$ must contain the nonidentity element while any generating set for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ must contain two of the three nonidentity elements.

Also solved by Paul Budney, Sunderland, MA; Missouri State U. Problem Solving Group; Lucas Stefanic (student), Rochester Inst. Tech.; and the proposer.

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