11813. *Proposed by Greg Oman, University of Colorado-Colorado Springs, Colorado Springs, CO.* Let X be a set, and let $*$ be a binary operation on X (that is, a function from $X \times X$ to X). Prove or disprove: there exists an uncountable set X and a binary operation $*$ on X such that for any subsets Y and Z of X that are closed under $*$, either $Y \subseteq Z$ or $Z \subseteq Y$.

11814. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* Let ϕ be a continuously differentiable function from $[0, 1]$ into $\mathbb{R}$, with $\phi(0) = 0$ and $\phi(1) = 1$, and suppose that $\phi'(x) \neq 0$ for $0 \leq x \leq 1$. Let f be a continuous function from $[0, 1]$ into $\mathbb{R}$ such that $\int_0^1 f(x)\,dx = \int_0^1 \phi(x) f(x)\,dx$. Show that there exists t with $0 < t < 1$ such that $\int_0^t \phi(x) f(x)\,dx = 0$.

11815. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let x, y, and z be positive numbers such that $x + y + z = 3$. Prove that

$$\frac{x^4 + x^2 + 1}{x^2 + x + 1} + \frac{y^4 + y^2 + 1}{y^2 + y + 1} + \frac{z^4 + z^2 + 1}{z^2 + z + 1} \geq 3xyz.$$

11816. *Proposed by Sabin Tabirca, University College Cork, Cork, Ireland.* Let ABC be an acute triangle, and let B_1 and C_1 be the points where the altitudes from B and C intersect the circumcircle. Let X be a point on arc BC, and let B_2 and C_2 denote the intersections of XB_1 with AC and XC_1 with AB, respectively. Prove that the line B_2C_2 contains the orthocenter of ABC.

SOLUTIONS

If the Sum of the Squares is the Square of the Sum, . . .

11671 [2012, 800]. *Proposed by Sam Northshield, SUNY-Plattsburgh, Plattsburgh, NY.* Show that if relatively prime integers a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2,$$

then $|a + b + c|$ can be written as $m^2 - mn + n^2$ for some integers m and n.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. Note that $m^2 - mn + n^2$ is the norm of $m + n\omega$ in the number ring $\mathbb{Z}[\omega]$. This ring is a unique factorization domain. The primes that split in this number ring are 3 and all primes congruent to 1 modulo 3. Thus a positive integer can be written in the form $m^2 - mn + n^2$ if and only if every prime congruent to 2 modulo 3 divides it an even number of times.

Let $g = \gcd(a + b + c, a + b + d, a + c + d, b + c + d)$. Now $(a + b + d) + (a + c + d) + (b + c + d) - 2(a + b + c) = 3d$ and symmetrically, and since $\gcd(a, b, c, d) = 1$, g is a divisor of 3.

Thus for any prime p congruent to 2 modulo 3 that divides $a + b + c$, we can choose one of $a + b + d$, $a + c + d$, and $b + c + d$ that is not divisible by p. Rewriting the given equality as

$$(a + b + d)(a + b + c) = a^2 - a(-b) + (-b)^2,$$

we see that p divides the right side with even multiplicity and hence divides $a + b + c$ with even multiplicity. By the remarks above, $a + b + c$ can be written in the form $m^2 - mn + n^2$ for some integers m and n.