12088. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Let k be a positive integer with $k \geq 2$, and let $f : [0, 1] \to \mathbb{R}$ be a function with continuous kth derivative. Suppose $f^{(k)}(x) \geq 0$ for all $x \in [0, 1]$, and suppose $f^{(i)}(0) = 0$ for all $i \in \{0, 1, \ldots, k-2\}$. Prove

$$\int_0^1 x^{k-1} f(1-x)\,dx \leq \frac{(k-1)!k!}{(2k-1)!} \int_0^1 f(x)\,dx.$$

12089. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO, and Adam Salminen, University of Evansville, Evansville, IN.* All rings in this problem are assumed to be commutative with a nonzero multiplicative identity. A *homomorphism* from a ring R to a ring S is an identity-preserving map $\phi : R \to S$ such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$. Consider the following two properties of a ring R:

(1) For every proper ideal I of R, there is an injective homomorphism $\phi : R/I \to R$.
(2) For every proper ideal I of R, there is an injective homomorphism $\phi : R \to R/I$.

(a) Must a ring that enjoys property (1) be a field?
(b) Must a ring that enjoys property (2) be a field?
(c) Must a ring that enjoys properties (1) and (2) be a field?

SOLUTIONS

A Trigonometric Integral

11961 [2017, 180]. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Evaluate

$$\int_0^{\pi/2} \frac{\sin x}{1+\sqrt{\sin(2x)}}\,dx.$$

Solution by Koopa Tak Lun Koo, Beacon College, Hong Kong, China. The integral equals $(\pi/2) - 1$. To see this, denote the integral by I. The substitution $x \mapsto (\pi/2) - x$ yields

$$I = \int_0^{\pi/2} \frac{\sin x}{1+\sqrt{\sin 2x}}\,dx = \int_0^{\pi/2} \frac{\cos x}{1+\sqrt{\sin 2x}}\,dx.$$

Adding these two integrals, substituting $u = \cos x - \sin x$, and noting that $u^2 = \cos^2 x - 2\sin x \cos x + \sin^2 x = 1 - \sin 2x$ gives

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{1+\sqrt{\sin 2x}}\,dx = \int_{-1}^1 \frac{1}{1+\sqrt{1-u^2}}\,du = 2\int_0^1 \frac{1}{1+\sqrt{1-u^2}}\,du.$$

To compute this integral, substitute $u = \sin\theta$ to obtain

$$\int_0^1 \frac{1}{1+\sqrt{1-u^2}}\,du = \int_0^{\pi/2} \frac{\cos\theta}{1+\cos\theta}\,d\theta = \left[\theta - \tan\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

Editorial comment. Several solvers noted a more general result, with essentially the same proof: If f is continuous on $[0, \pi/2]$, then

$$\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\cos^2\theta)\cos\theta\,d\theta.$$