The following is from the 1989 Green Chicken Math Competition between Middlebury and Williams Colleges. The aerial view of a roller-coaster is a perfect circle. Show that there are two diametrically opposed points on the roller-coaster having the same height.

**#1328:** *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

Let a sequence $\{a_n\}$ satisfy

$$a_{n+1} \;=\; \frac{(2n-1)a_n - 9(n-2)a_{n-1}}{n+1}$$

for $n \geq 1$ and set $a_1 = 1$ and $a_2 = -1$. Prove that $a_n$ is integral if $n \in \mathbb{Z}^+$. *Hint: Note the similarity of the recursion to those of Motzkin numbers, Delannoy numbers, and super-Catalan numbers, though "combinatorially" it is different.*

**#1329:** *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

There is a beautiful closed form expression for the number of ways to tile a regular hexagon with edge length of $n$ with diamonds of side length 1 and angles $60^\circ$ and $120^\circ$; it is

$$\prod_{i=0}^{n-1} \frac{(i)!(i+2n)!}{[(i+n)!]^2}.$$

This formula is a special case of MacMahon's formula which considers the more general problem of plane partitions; see the sequence A008793 in the Online Encyclopedia of Integer Sequences (OEIS) (`https://oeis.org/A008793`) or `https://oeis.org/wiki/Plane_partitions` for further information. MacMahon's formula is traditionally proved using a tricky generating function argument, and even our special case is no simpler (see `https://aquazorcarson.wordpress.com/2011/02/25/` for an excellent presentation of such an argument). Prove directly that this quantity is in fact an integer without resorting to the combinatorial interpretation. *Hint: Use Legendre's Formula, which says the largest power of a prime $p$ dividing an integer $m$ is $\sum_{\ell=1}^{\infty}\lfloor m/p^\ell \rfloor$, where $\lfloor x \rfloor$ is the greatest integer at most $x$.*

**#1330:** *Proposed by Ioana Mihăilă, Cal Poly Pomona.*

Let $ABCD$ be a quadrilateral with opposite right angles $A$ and $C$. Let $BE$ and $DG$ be the perpendiculars dropped on $AC$ from $B$ and $D$ respectively (see Figure 2). Show that $AE = GC$.

**#1331:** *Proposed by Greg Oman, University of Colorado, Colorado Springs.*

A nonempty subset $G$ of $\mathbb{R}$ is an *additive subgroup* of $\mathbb{R}$ provided for any $x, y \in G$, also $x - y \in G$. Now suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is both continuous and injective. Assume further that $f$ preserves additive subgroups of $\mathbb{R}$; that is, if $G$ is an additive subgroup of $\mathbb{R}$, then so is $f[G] := \{f(g) \colon g \in G\}$. Prove that there exists a real number $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. *Note*: it is well-known that a continuous $f\colon \mathbb{R} \to \mathbb{R}$ for which $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ is necessarily linear. The purpose of this exercise is to show that if one strengthens "continuous" to "continuous and injective", then one can weaken the assumption of additivity to preservation of subgroups.