A note on strongly Jónsson binary relational structures

Greg Oman

Abstract. Let X be a set and let R be a binary relation on X. A subset L of X is said to be a *lower set of* $\mathbf{X} := (X, R)$ provided whenever $x \in L$ and $y \in X$ with yRx, then also $y \in L$. In this note, we study binary relational structures $\mathbf{X}$ with the property that distinct lower sets of $\mathbf{X}$ have distinct cardinalities.

1. Introduction

We begin by recalling the following famous problem, which has spawned a massive literature over the years:

Problem 1.1 (Jónsson's Problem)**.** For which infinite cardinals κ does there exist an algebra $\mathbf{A}$ of size κ, with but finitely many operations (of finite arity), for which every proper subuniverse of $\mathbf{A}$ has cardinality less than κ?

Infinite algebras satisfying the above condition are known as *Jónsson algebras*. In the modern era, the theory of Jónsson algebras has proved to be a useful tool in the investigation of large cardinals. We refer the reader to [1] and [2] for a survey of the classical results on these algebras.

The canonical interpretation of the Jónsson property for posets already appears in the literature. In [3], a poset $\mathbf{P} := (P, \preceq)$ is called a *Jónsson poset* if every proper order ideal of $\mathbf{P}$ has cardinality strictly less than $|P|$. Jónsson posets were utilized therein to obtain new results on unary Jónsson algebras.

Now consider strengthening the Jónsson property but dispensing with the assumption that the binary relation is a partial order. To wit, let $\mathbf{X} := (X, R)$ be a binary relational structure, and recall that a subset $L \subseteq X$ is a *lower set of* $\mathbf{X}$ or an *R–lower subset of* X provided for all $x \in L$ and $y \in X$: if yRx, then $y \in L$. Say that $\mathbf{X}$ is *strongly Jónsson* provided distinct lower sets of $\mathbf{X}$ have distinct cardinalities.

An analogous 'strongly Jónsson' property for modules over a commutative ring was recently studied by the author. In particular, a structure theorem for such modules was presented in [4, Theorem 1]. The purpose of this note is to port over Theorem 1 to the binary relational universe; Theorem 2.7 is our principle result.

Presented by . . .
Received . . . ; accepted in final form . . .
2010 *Mathematics Subject Classification*: Primary: 06A75; Secondary: 06A06.
Key words and phrases: linear extension, lower set, reflexive closure, transitive closure, weighted poset.

2. Main results

We begin with comments on notation and terminology which will be used throughout the paper. The symbol $\leq$ will exclusively refer to the usual inclusive epsilon order on the class *Ord* of ordinal numbers; we denote the subclass of cardinal numbers by *Card*. If R is a binary relation on a set X and $R^{\neq} := R - \{(x,x) : x \in X\}$, then $\mathbf{G} := (X, R^{\neq})$ is a directed loopless graph. Recall that a directed graph $\mathbf{G}$ is *acyclic* provided $\mathbf{G}$ has no cycles. We will make use of the following well-known fact:

Fact 2.1. Let R be a binary relation on a set X. Then R admits a linear extension if and only if $(X, R^{\neq})$ is acyclic.

Sketch of Proof. Necessity is obvious. To prove sufficiency, note that the reflexive transitive closure $\overline{R}$ of R is both reflexive and transitive. If $(X, R^{\neq})$ is acyclic, it is easy to see that $\overline{R}$ is also antisymmetric. Thus $(X, \overline{R})$ is a poset, and it is well-known that every poset has a linear extension. □

We will soon show that there is a large class $\mathcal{C}$ of binary structures such that for all $\mathbf{X} \in \mathcal{C}$: if $\mathbf{X}$ is strongly Jónsson, then $\mathbf{X}$ is isomorphic to a suborder of the inclusive epsilon order on a countable cardinal. First, a lemma:

Lemma 2.2. *Let $\alpha \leq \omega$ be an ordinal and suppose that $\leq$ is an extension of a binary relation $\preceq$ on α. If $\mathbf{X} := (\alpha, \preceq)$ is strongly Jónsson, then $n \preceq n+1$ for all $n \in \omega$ such that $n+1 < \alpha$.*

Proof. Assume that $\mathbf{X}$ is strongly Jónsson with $\leq$ an extension of $\preceq$, and let $n \in \omega$ be arbitrary. Further, suppose that $n+1 < \alpha$. If $n = 0$ and $0 \not\preceq 1$, then $\{0\}$ and $\{1\}$ are distinct lower sets of $\mathbf{X}$ of the same size, a contradiction. Assume now that $n \neq 0$. Since $\mathbf{X}$ is strongly Jónsson and $\{0, 1, \ldots, n\}$ is a $\preceq$–lower set of $\mathbf{X}$ of size $n+1$, it follows that $\{0, \ldots, n-1, n+1\}$ is *not* a lower set of $\mathbf{X}$. Thus $n \preceq n+1$, and the proof of the lemma is complete. □

Proposition 2.3. *Suppose that $\mathbf{X} := (X, R)$ is a structure such that $\mathbf{G} := (X, R^{\neq})$ is an acyclic digraph. Then $\mathbf{X}$ is strongly Jónsson if and only if there exists an ordinal $\alpha \leq \omega$ and a relation $\preceq$ on α such that $\mathbf{X} \cong (\alpha, \preceq)$, and both of the following hold:*

(1) *$\leq$ is an extension of $\preceq$ on α, and*
(2) *$n \preceq n+1$ for all n such that $n+1 < \alpha$.*

Proof. We prove only the nontrivial direction. Thus assume that $\mathbf{X}$ is strongly Jónsson and $\mathbf{G}$ is acyclic. Now let $\preceq^*$ be a linear extension of R (which exists by Fact 2.1). Then clearly $(X, \preceq^*)$ is strongly Jónsson as well. We claim that $\preceq^*$ is a well-order on X. Otherwise there exists an infinite strictly decreasing sequence $\cdots \prec^* x_3 \prec^* x_2 \prec^* x_1 \prec^* x_0$ in X. Let C be the complement of the $\preceq^*$–principal order filter $[x_0)$. Then C and $C \cup \{x_0\}$ are distinct $\preceq^*$–lower subsets of X of the same cardinality, a contradiction. Since $\preceq^*$ well orders X

and $(X, \preceq^*)$ is strongly Jónsson, it is easy to see that $(X, \preceq^*) \cong (\alpha, \leq)$ for some ordinal $\alpha \leq \omega$. Let $\varphi\colon X \to \alpha$ be an isomorphism, and set $\preceq := \{(\varphi(x), \varphi(y)) : (x, y) \in R\}$. Then clearly (1) above holds; Lemma 2.2 implies (2). □

Corollary 2.4. *Let* $\mathbf{P} := (P, \preceq)$ *be a poset. Then* $\mathbf{P}$ *is strongly Jónsson if and only if* $\mathbf{P} \cong (\alpha, \leq)$ *for some ordinal* $\alpha \leq \omega$*.*

More generally, we now consider the problem of classifying the strongly Jónsson binary relational structures. Toward this end, we will require a characterization of the countable Jónsson posets. Recall from the introduction that a poset $\mathbf{P} := (P, \preceq)$ is Jónsson provided every proper lower set of $\mathbf{P}$ has cardinality less than $|P|$.

Lemma 2.5. *Let* $\mathbf{P} := (P, \preceq)$ *be a countable poset. Then* $\mathbf{P}$ *is Jónsson if and only if* $(P, \preceq^*) \cong (|P|, \leq)$ *for every linear extension* $\preceq^*$ *of* $\preceq$*.*

Proof. If $\mathbf{P}$ is Jónsson, then the proof that every linear extension is isomorphic to $(|P|, \leq)$ is analogous to the proof of Proposition 2.3. Conversely, suppose that $\mathbf{P}$ is not Jónsson. Then P is infinite; further, $[p_0)^c$ is infinite for some $p_0 \in P$. Now set $R := \preceq \cup \{(x, p_0) : x \in [p_0)^c\}$. We claim that $\mathbf{G} := (P, R^{\neq})$ is acyclic (this is known; we include the short proof). Suppose by way of contradiction that $x_0, \ldots, x_n$ are distinct and

$$C := \{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)\} \subseteq R^{\neq} \text{ for some } n > 0. \quad (2.1)$$

Since $\preceq$ is a partial order, it follows that $C \nsubseteq \preceq$. Without loss of generality, we may assume that $(x_0, x_1) = (x, p_0)$ for some $x \notin [p_0)$. One shows by induction that $p_0 \preceq x_i$ for all i with $1 \leq i \leq n$. In particular, $p_0 \preceq x_n$. But then $(x_n, x_0) \notin \{(x, p_0) : x \in [p_0)^c\}$. It follows that $x_n \preceq x_0$, and $x = x_0 \in [p_0)$, a contradiction. Fact 2.1 now implies that there exists a linear extension $\preceq^*$ of R (hence also of $\preceq$). Then the set $\{p \in P : p \preceq^* p_0\}$ is infinite, and thus $(P, \preceq^*)$ is not isomorphic to $(|P|, \leq)$. □

We now translate the classification problem for strongly Jónsson structures to an analogous problem for weighted posets. Note that to every structure $\mathbf{X} := (X, R)$, we may associate a weighted poset as follows: let $\overline{R}$ be the reflexive transitive closure of R. Now define $\sim$ on X by $x \sim y$ if and only if $x\overline{R}y$ and $y\overline{R}x$. Setting $P := X/{\sim}$ and $\preceq := R/{\sim}$, it is well-known that $\mathbf{P} := (P, \preceq)$ is a poset. Define a weight function w on P by $w([p]) := |[p]|$.

Say that a weighted poset $\mathbf{P}_w := (P, \preceq, w)$ with $w\colon P \to \mathit{Card} - \{0\}$ is a *strongly Jónsson weighted poset* if distinct lower sets of $\mathbf{P}$ have distinct total weights (the *total weight* of a subset X of P is $w(X) := \sum_{x \in X} w(x)$). It is straightforward to verify that $\mathbf{X}$ is strongly Jónsson if and only if $\mathbf{P}_w$ is a strongly Jónsson weighted poset. Thus the classification problem has been reduced to classifying the strongly Jónsson weighted posets. We give an example of such a structure and then we present the main result of the paper.

Example 2.6. Let a be an ordinal, $P := (\omega - \{0\}) \cup \{\aleph_{i+1} : i < a\}$, and w be the identity on P. Then $(P, \leq, w)$ is a strongly Jónsson weighted poset.

Theorem 2.7. *Let* $\mathbf{P}_w := (P, \preceq, w)$ *be a weighted poset such that* $w(p) \in Card - \{0\}$ *for all* $p \in P$*. Then* $\mathbf{P}_w$ *is a strongly Jónsson weighted poset if and only if* $\mathbf{P}_w = (\mathbf{P}_0)_w \oplus (\mathbf{P}_1)_w$ *for some weighted posets* $(\mathbf{P}_0)_w$ *and* $(\mathbf{P}_1)_w$ *such that*

(1) $w(p) < \aleph_0$ *for every* $p \in P_0$*,*
(2) $w(p) \geq \aleph_0$ *for every* $p \in P_1$*,*
(3) $(\mathbf{P}_0)_w$ *is strongly Jónsson,*
(4) P_0 *is countable,*
(5) $\mathbf{P}_0$ *is Jónsson,*
(6) $(w(P_0))^+ + \aleph_0 \leq w(p)$ *for every* $p \in P_1$*, and*
(7) $\mathbf{P}_1$ *is isomorphic to an ordinal. Moreover, for every* $p \in P_1$*, we have* $\sum\{w(x) : x \in P_1, x \prec p\} < w(p)$*.*

Proof. If $\mathbf{P}_w$ decomposes into an ordinal sum of weighted posets $(\mathbf{P}_0)_w$ and $(\mathbf{P}_1)_w$ which satisfy (1)–(7), then it is straightforward to check that $\mathbf{P}_w$ is a strongly Jónsson weighted poset.

Conversely, suppose that $\mathbf{P}_w$ is a strongly Jónsson weighted poset. We first establish that

$$\text{for all } p \in P : w([p)^c) < w(p) + \aleph_0. \tag{2.2}$$

If not, then for some $p \in P$, $[p)^c$ and $[p)^c \cup \{p\}$ are distinct lower sets of $\mathbf{P}$ of the same weight, a contradiction.

Now set $P_0 := \{p \in P : w(p) < \aleph_0\}$ and $P_1 := P_0^c$. Then (1) and (2) are patent. Moreover, if $x \in P_0$ and $y \in P_1$, then by (2.2), $y \notin [x)^c$. Thus $x \leq y$, and we see that $\mathbf{P}_w = (\mathbf{P}_0)_w \oplus (\mathbf{P}_1)_w$. It follows that every lower set of $\mathbf{P}_0$ is also a lower set of $\mathbf{P}$, whence $\mathbf{P}_0 := (P_0, \preceq, w)$ is a strongly Jónsson weighted poset. This proves (3).

We now show that P_0 is countable. Toward this end, let $\preceq^*$ be a linear extension of $\preceq$. Then one shows by contradiction as in the proof of Proposition 2.3 that $\preceq^*$ well orders P_0 (the same argument goes through, but the contradiction is the existence of distinct lower sets of $\mathbf{P}_0$ with the same total weight). This in turn implies via an analogous argument that $(P_0, \preceq^*)$ is isomorphic to $(\alpha, \leq)$ for some ordinal $\alpha \leq \omega$. We have proved (4). Invoking Lemma 2.5, (5) follows.

Lastly, we prove (6) and (7). For (6), it suffices to show that $w(P_0) < w(p)$ for all $p \in P_1$. Let $p \in P_1$ be arbitrary. Since $\mathbf{P} = \mathbf{P}_0 \dotplus \mathbf{P}_1$, we see that $P_0 \subseteq [p)^c$. Hence by (2.2), we have $w(P_0) \leq w([p)^c) < w(p) + \aleph_0 = w(p)$. As for (7), let $a, b \in P_1$ be arbitrary. If a and b are incomparable, then $a \in [b)^c$ and $b \in [a)^c$. Thus by (2.2), $w(a) < w(b) < w(a)$, which is absurd. We apply (2.2) yet again to conclude that $a \prec b$ if and only if $w(a) < w(b)$, whence $\preceq$ is a well-order on P_1. Now let $p \in P_1$. That $\sum\{w(x) : x \in P_1, x \prec p\} < w(p)$ follows immediately from (2.2). This concludes the proof. □

Remark 2.8. Though $(\mathbf{P}_1)_w$ is a strongly Jónsson weighed poset, observe from Example 2.6 that $\mathbf{P}_1$ can be (isomorphic to) *any* ordinal. Thus $\mathbf{P}_1$ need not be a Jónsson poset.

Several additional remarks are now in order. First, $\mathbf{P}_0$ is determined up to isomorphism by Lemma 2.5 and (5) of Theorem 2.7. Further, the weighted poset $(\mathbf{P}_1)_w$ is determined up to isomorphism. It remains to classify the weight function w on P_0. We address this problem with an example. Let $0 < k < \omega$, and suppose that $X := \{x_i : i < k\}$ is a set of k positive integers such that for all $Y, Z \in \mathcal{P}(X)$: if $Y \neq Z$, then $\sum_{y \in Y} Y \neq \sum_{z \in Z} Z$ (for example, $X := \{2^i : i < k\}$). Define w on $P_0 := k$ by $w(i) := x_i$ for all $i < k$. Then $(\mathbf{P}_0)_w := (P_0, =, w)$ is a countable strongly Jónsson weighted poset. Therefore the problem of classifying the weight functions on countable strongly Jónsson weighted posets (with weights in $\omega - \{0\}$) is at least as hard as determining all k-element sets of positive integers for which distinct subsets have distinct sums. The latter is a well-known combinatorial problem (the so-called *distinct subset-sum problem*) for which no complete solution is known. Thus it is likely that the classification given in Theorem 2.7 cannot be considerably strengthened.

Acknowledgment. The author thanks Keith Kearnes for reading a preliminary draft of this manuscript and providing several helpful suggestions. The author also thanks the referee (and the referee of an older draft of this manuscript) for comments which improved the paper. Finally, the author is indebted to Brian Davey for his assistance in the preparation of the final version of this paper.

References

[1] Coleman, E.: Jónsson groups, rings, and algebras. Irish Math. Soc. Bull. **36**, 34–45 (1996)

[2] Jech, T.: Set Theory (third millennium edition). Springer Monographs in Mathematics, New York (2002)

[3] Kearnes, K.; Oman, G.: Jónsson posets and unary Jónsson algebras. Algebra Universalis **69**, 101–112 (2013)

[4] Oman, G.: Strongly Jónsson and strongly HS modules. J. Pure Appl. Algebra **218**, 1385–1399 (2014)

Greg Oman

Department of Mathematics, University of Colorado, Colorado Springs CO 80918, USA

e-mail: `goman@uccs.edu`

URL: `http://www.uccs.edu/~goman/`