818. *Proposed by Angelo S. DiDomenico, Milford, MA.*
Let $\langle G_n \rangle$ denote a sequence defined by the recurrence relation

$$G_{n+2} = G_{n+1} + G_n,$$

where G_0 and G_1 are arbitrary real numbers.

(a) Show that for five consecutive elements in such a sequence, the difference of the fourth power of the third element and the product of the other four elements is a constant; that is,

$$G_{n+2}^4 - G_n G_{n+1} G_{n+3} G_{n+4} = K,$$

where K is a constant.

(b) Find a formula for K in terms of G_0 and G_1.

(c) Characterize the sequences for which the constant K vanishes.

819. *Proposed by Michael Andreoli, Miami-Dade College, Miami, FL.*
For $n \geq 1$, evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)\cdots(k+n)}.$$

820. *Proposed by Greg Oman, Ohio State University, Columbus, OH.*
It is known that if two elements a and b in an integral domain D are such that $a \mid b$ and $b \mid a$, then a and b are associates; that is, there exists a unit $u \in D$ such that $a = ub$. Show that the condition "D is an integral domain" is necessary; that is, find a commutative ring with identity in which there exist elements x and y such that $x \mid y$ and $y \mid x$ and x and y are not associates.

SOLUTIONS

Roots of recursively defined polynomials

787. (Corrected) *Proposed by Michel Bataille, Rouen, France.*
Let $\{P_n\}$ be the sequence of polynomials defined by $P_0(x) = 1$, $P_1(x) = x + 1$, and

$$P_{n+2}(x) = xP_{n+1}(x) - P_n(x)$$

for all non-negative integers n. Find the roots of P_n.

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Karl Havlak (jointly), Angelo State University, San Angelo, TX.

This solution follows the discussion of the zeros of the Chebyshev Polynomials which is given in Theorem 8.9 on pp. 498–499 of [**1**].

To begin, note that it is easily shown that each P_n is a polynomial of degree n. Also, we will use the following well-known trigonometric identities:

$$\sin 3A = \sin A(2\cos 2A + 1), \tag{1}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right). \tag{2}$$