984. *Proposed by D. M. Bătinetu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $a, b, c \in (1, \infty)$ and $m, n \in (0, \infty)$. Prove that

$$\log_{b^m c^n} a + \log_{c^m a^n} b + \log_{a^m b^n} c \geq \frac{3}{m+n}.$$

985. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be an *infinite* commutative ring with identity, and let I and J be ideals of R. Recall that I and J are isomorphic (as R-modules) provided there is a bijection $\varphi\colon I \to J$ such that $\varphi(i_1 + i_2) = \varphi(i_1) + \varphi(i_2)$ for all $i_1, i_2 \in I$ and $\varphi(ri) = r\varphi(i)$ for $r \in R$ and $i \in I$. Suppose that I is isomorphic to R (in the above sense, viewing R as an ideal) for every ideal I of R of the same cardinality as R. Prove that R is a principal ideal domain, that is, R is an integral domain and every ideal of R is principal.

SOLUTIONS

Guaranteeing three function values whose product is 1

956. *Proposed by Duong Viet Thong, National Economics University, Hanoi City, Vietnam.*

Let $f\colon [0, 1] \to \mathbb{R}$ be a strictly monotonic and continuous function such that

$$\int_0^1 f(t)\,dt = 1.$$

Prove there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha < \beta < \gamma$ such that

$$f(\alpha)f(\beta)f(\gamma) = 1.$$

Solution by Eugene Herman, Grinnell College, Grinnel IA; Elias Lampakis, Kiparissia, Greece; and Alfred Witkowski, University of Technology and Life Sciences, Bydgoszcz, Poland (independently).

We prove a more general result: Suppose f is continuous on an open interval I, y is in $f(I)$, and n is a positive integer. If either f is constant on I or y is not an extreme value of f, there exist distinct $t_1, \dots, t_n \in I$ such that

$$\prod_{i=1}^{n} f(t_i) = y^n. \tag{1}$$

If f is constant, then (1) holds for any $t_1, \dots, t_n \in I$. Otherwise, $f(I)$ is an interval and y is an interior point of that interval. We claim that there exist distinct points $y_1, \dots, y_n \in f(I)$ such that $\prod_{i=1}^{n} y_i = y^n$. For example, choose $\lfloor n/2 \rfloor$ pairs of points in $f(I)$ of the form ry and $(1/r)y$, where $r > 1$ and $r \approx 1$. If n is odd, also choose y itself. Hence, there exist distinct $t_1, \dots, t_n \in I$ such that

$$\prod_{i=1}^{n} f(t_i) = \prod_{i=1}^{n} y_i = y^n.$$