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SHORT NOTE

# A Note on the $n$-Generator Property for Commutative Monoids

**Greg Oman**

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## Abstract

Let $M$ be a cancellative, commutative monoid with integral closure $\overline{M}$. Borrowing from ring theory, we say that $M$ has the $n$-generator property iff every finitely generated ideal of $M$ can be generated by $n$ elements, and we say $M$ has rank $n$ iff every ideal of $M$ can be generated by $n$ elements. We investigate the integral closure of such monoids. We show, in particular, that if $M$ has the $n$-generator property, then $\overline{M}$ is a valuation monoid, and if $M$ has rank $n$, then $\overline{M}$ is a principal ideal monoid.


In this note, all monoids are assumed to be cancellative and commutative.

We first recall some basic definitions. Let $M$ be a monoid. A nonempty subset $I \subseteq M$ is called an *ideal* of $M$ iff $I+M \subseteq I$. It is easy to see that if $S$ is a nonempty subset of a monoid $M$, then the set $S+M := \{s+m : s \in S, m \in M\}$ is an ideal of $M$. Further, any ideal of $M$ containing $S$ must clearly contain $S+M$. We call $S+M$ *the ideal of $M$ generated by $S$*. If $I$ is an ideal of $M$ and $X \subseteq M$, then $X$ is called a *generating set* for $I$ iff $I = X + M$. An ideal $I$ of $M$ is said to be *$n$-generated* iff there exists a generating set for $I$ with at most $n$ elements. $M$ is said to have the *$n$-generator property* iff every finitely generated ideal is $n$-generated, and to have *rank $n$* iff every ideal can be generated by at most $n$ elements. $M$ is said to be a *valuation monoid* iff for any two ideals $I$ and $J$ of $M$, either $I \subseteq J$ or $J \subseteq I$. It is easy to see that $M$ is a valuation monoid iff for any elements $a, b \in M$, either $a - b \in M$ or $b - a \in M$ ($a - b$ and $b - a$ are elements of the quotient group $G$ of $M$). If $M \subseteq N$ are monoids, then $x \in N$ is said to be *integral over $M$* iff $nx \in M$ for some positive integer $n$. The least such $n$ is the *degree* of $x$ over $M$. $N$ is an *integral extension* of $M$ iff every element of $N$ is integral over $M$. The *integral closure* of $M$ is defined to be the collection of all elements in the quotient group $G$ of $M$ which are integral over $M$. An *overmonoid* of $M$ is a monoid between $M$ and $G$.

We begin with two trivial but useful lemmas:

**Lemma 1.** *Let $I$ be an $n$-generated ideal of the monoid $M$. If $X$ is any generating set for $I$, then there exist $x_1, \ldots, x_n \in X$ such that $I = \{x_1, \ldots, x_n\} + M$.*

**Proof.** Suppose that $I$ can be $n$-generated, and let $X$ be any generating set for $I$. Let $i_1, \ldots, i_n$ be a set of generators for $I$. Then since $X$ generates $I$, we have that for each $k : 1 \leq k \leq n$, there exist $x_k \in X$ and $m_k \in M$ with $i_k = x_k + m_k$. We claim that $I = \{x_1, \ldots, x_n\} + M$. To see this, let $y \in I$ be arbitrary. Then since $i_1, \ldots, i_n$ generate $I$, we have that $y = i_k + m$ for some $k$ and for some $m \in M$. But we have that $i_k = x_k + m_k$, and so we get $y = x_k + (m_k + m)$. Hence $y \in \{x_1, \ldots, x_n\} + M$ and the proof is complete. ∎

**Lemma 2.** *Let $M$ be a monoid. Then $M$ has the $n$-generator property iff for every collection $\{m_1, \ldots, m_{n+1}\}$ of elements of $M$, there exist $i \neq j$ such that $m_i - m_j \in M$.*

**Proof.** Suppose first that the monoid $M$ has the $n$-generator property. Consider elements $m_1, \ldots, m_{n+1}$ of $M$. Let $I$ be the ideal of $M$ generated by these elements. Then since $M$ has the $n$-generator property, it follows from Lemma 1 that there exist $n$ elements from $\{m_1, \ldots, m_{n+1}\}$ that generate $I$. We may suppose (relabelling if necessary) that these elements are $m_1, \ldots, m_n$. Then in particular, we have that $m_{n+1} \in \{m_1, \ldots, m_n\} + M$. Thus $m_{n+1} - m_i \in M$ for some $i$. ∎

The converse follows easily from the condition on $M$.

**Theorem 1.** *Let $M$ be a monoid with the $n$-generator property with integral closure $\overline{M}$. Then:*

(1) *Every overmonoid of $M$ has the $n$-generator property.*

(2) *If $a, b \in M$, then there exists a positive integer $k \leq n$ such that either $k(a-b) \in M$ or $k(b-a) \in M$.*

(3) *Every element of $\overline{M}$ has degree $\leq n$ over $M$.*

(4) *$\overline{M}$ is a valuation monoid.*

(5) *$M$ is a bounded integral extension of a valuation monoid.*

**Proof.** (1) Let $S$ be an overmonoid of $M$. Consider elements $a_1 - b_1, \ldots, a_{n+1} - b_{n+1} \in S$, where each $a_i, b_i \in M$. Let $x := b_1 + \cdots + b_{n+1}$. Then note trivially that for each $i$, $a_i - b_i + x \in M$. Since $M$ has the $n$-generator property, it follows from Lemma 2 that there exist $i \neq j$ with $(a_i - b_i + x) - (a_j - b_j + x) \in M$. But then $(a_i - b_i) - (a_j - b_j) \in M \subseteq S$. By Lemma 2, $S$ has the $n$-generator property.

(2) Let $a$ and $b$ be arbitrary elements of $M$. Consider the set $\{ia+(n-i)b : 0 \leq i \leq n\}$. Then by Lemma 2, there exist $i \neq j$ with $(ia+(n-i)b)-(ja+(n-j)b) \in M$. Simplifying this expression yields that $(i-j)a+(j-i)b \in M$. If $i > j$, then $(i-j)(a-b) \in M$, and if $j > i$, then $(j-i)(b-a) \in M$. This completes the proof of (2).

(3) Consider an element $a-b$ of the integral closure of $M$, say of degree $r$. By (2), there exists a positive integer $k \leq n$ such that either $k(a-b) \in M$ or $k(b-a) \in M$. Now if $k(a-b) \in M$ then clearly $a-b$ has degree $\leq n$. Otherwise $k(b-a) \in M$. Suppose by way of contradiction that $r > n$. Then we have that $r(a-b) \in M$ and $k(b-a) \in M$. But this implies that $(r-k)(a-b) \in M$. This contradicts that $a-b$ is of degree $r$ over $M$ and completes the proof of (3).

(4) By (1), $\overline{M}$ has the $n$-generator property. Let $a, b \in \overline{M}$. By (2), there exists a positive integer $k$ such that either $k(a-b) \in \overline{M}$ or $k(b-a) \in \overline{M}$. As $\overline{M}$ is integrally closed, we get that $a-b \in \overline{M}$ or $b-a \in \overline{M}$, and thus $\overline{M}$ is a valuation monoid.

(5) By (3), we have that $n!\overline{M} \subseteq M$. As $n!\overline{M} \cong \overline{M}$, we see from (4) that $n!\overline{M}$ is a valuation monoid. It is trivial that every element of $M$ is of degree $\leq n!$ over $n!\overline{M}$. ∎

Next we prove a similar theorem for rank $n$ monoids:

**Theorem 2.** *Let $M$ be a rank $n$ monoid. Then the integral closure $\overline{M}$ of $M$ is a principal ideal monoid.*

**Proof.** Let $I$ be any ideal of $\overline{M}$. We show that $I$ is $n$-generated. As $M$ has the $n$-generator property, it follows from (3) of Theorem 1 that $n!I \subseteq M$. Let $I'$ be the ideal of $M$ generated by $n!I$. Then as $M$ has rank $n$, it follows that $I'$ can be generated by $n$ elements. By Lemma 1, we see that $I' = \{n!i_1, \ldots, n!i_n\}+M$ for some $i_1, \ldots, i_n \in I$. We claim that $I = \{i_1, \ldots, i_n\}+\overline{M}$. To see this, let $i \in I$. Then $n!i \in I'$. Hence we see that $n!i = n!i_k + m$ for some $k$ and some $m \in M$. But then $n!(i-i_k) \in M \subseteq \overline{M}$. As $\overline{M}$ is integrally closed, it follows that $i - i_k \in \overline{M}$, and so $i = i_k + x$ for some $x \in \overline{M}$. This shows that $I$ is $n$-generated.

We've shown that $\overline{M}$ has rank $n$. But by (4) of Theorem 1, we have that $\overline{M}$ is a valuation monoid. It is easy to see that every finitely generated ideal of a valuation monoid is principal (it suffices by induction to show that every ideal generated by two elements is principal, and this verification is trivial). As every ideal of $\overline{M}$ is finitely generated, it follows that $\overline{M}$ is a principal ideal monoid. This completes the proof. ∎

There is a fairly extensive literature on the ideal theory of commutative monoids. The interested reader is encouraged to consult the bibliography of [1] or [2] for a long list of references on this topic.

## References

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Department of Mathematics
The Ohio State University
231 W. 18th Ave.
Columbus, OH 43210
USA
oman@math.ohio-state.edu

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