A short proof of the Bolzano-Weierstrass Theorem

Article *in* The College Mathematics Journal · January 2018

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ABSTRACT. We present a short proof of the Bolzano-Weierstrass Theorem on the real line which avoids monotonic subsequences, Cantor's Intersection Theorem, and the Heine-Borel Theorem.

1. INTRODUCTION

A fundamental tool used in the analysis of the real line is the well-known Bolzano-Weierstrass Theorem[1]:

Theorem 1 (Bolzano-Weierstrass Theorem, Version 1). *Every bounded sequence of real numbers has a convergent subsequence.*

To mention but two applications, the theorem can be used to show that if $[a, b]$ is a closed, bounded interval and $f\colon [a, b] \to \mathbb{R}$ is continous, then f is bounded. One may also invoke the result to establish Cantor's Intersection Theorem: if $\{C_n\colon n \in \mathbb{N}\}$ is a nested sequence of closed bounded intervals, then there is a real number belonging to every C_i. One would be hard-pressed to find a book on elementary real analysis which does not include the statement of Theorem 1 along with a proof.

A sketch of one of the most popular proofs proceeds as follows: let (x_n) be a bounded sequence of real numbers. Call a member x_n of the sequence a "peak" if $x_m \leq x_n$ for every $m \geq n$. If (x_n) has but finitely many peaks, then one shows that (x_n) has a monotone increasing subsequence. Otherwise, it can be argued that (x_n) has a monotone decreasing subsequence. In any case, there exists a monotone subsequence (x_{n_k}) of (x_n). Then (x_{n_k}) converges to its sup or inf according to whether it is increasing or decreasing, respectively. Such an approach can be found in the books Bartle [1], Pugh [7], and Ross [8], among many others.

Another well-known proof begins by noting that since (x_n) is bounded, there exist $a_0, b_0 \in \mathbb{R}$ such that $\{x_n\colon n \in \mathbb{N}\} \subseteq [a_0, b_0]$. Let c_0 be the midpoint of a_0 and b_0. Then either there are infinitely many n for which $a_n \in [a_0, c_0]$ or there are infinitely many n for which $a_n \in [c_0, b_0]$; say the former holds. Then take the midpoint c_1 of $[a_0, c_0]$ and repeat the above argument. Continuing recursively, one obtains a nested sequence of closed bounded intervals whose lengths tend to 0. By Cantor's Intersection Theorem, there is a (unique) real number x^* which lies in every interval. It is then straightforward to obtain a subsequence (x_{n_k}) of (x_n) which converges to x^*. One can find this proof in Hoffman & Marsden [5] and Morry & Protter [6], for example.

Still other texts state the Bolzano-Weierstrass Theorem in a slightly different form, namely:

Theorem 2 (Bolzano-Weierstrass Theorem, Version 2). *Every bounded, infinite set of real numbers has a limit point.*

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[1]This theorem was originally proved by Bolzano in 1817. It was reproved by Weierstrass in the latter half of the 19th century.

Multiple proofs of the second version also appear in the literature. Specifically, Brand ([3]), Gaughan ([4]), and Watson ([9]) prove the result via the bisection method outlined above. On p. 323 of Bartle & Sherbert [1], the authors present Theorem 2 as an exercise and instruct the reader to use the Heine-Borel Theorem.

It is easy to deduce either form of the Bolzano-Weierstrass Theorem from the other. We give an outline of an argument proving the first version from the second. Suppose (x_n) is a bounded sequence. If (x_n) has but finitely many distinct terms, then (x_n) has a constant subsequence, which trivially converges. Otherwise, $\{x_n \colon n \in \mathbb{N}\}$ is infinite; let L be a limit point. It is not difficult to recursively construct a subsequence of (x_n) converging to L.

2. Short proof

The purpose of this note is to give a short proof of the second version of the Bolzano-Weierstrass Theorem. Our proof hinges upon a set-theoretic observation of the German mathematician Paul Stäckel[2] dating back to 1907. It was during this time that set theory was rapidly evolving into what would ultimately become ZFC (as the reader may recall, Zermelo introduced his famous list of axioms in 1908). One important question at the time was how to formulate a rigorous mathematical definition of "finite set." Many famous mathematicians contributed answers to this question, including both Dedekind and Tarski (the so-called Dedekind-finite and Tarski-finite sets now bear their names). Stäckel proposed to call a set S finite if there exists a linear order $\leq$ on S for which every nonempty subset of S has both an $\leq$-least element and a $\leq$-greatest element; in other words, both $\leq$ and $\geq$ are well-orders on S. A set with such a property is said to be *Stäckel-finite* (Bohnet [2]). In modern set theory (namely, ZFC), one can prove that a Stäckel finite set is finite.

Lemma 1. *Let S be a set, and let $\leq$ be a linear order on S. If for every nonempty $X \subseteq S$, both $\inf(X)$ and $\sup(X)$ exist and belong to X, then S is finite.*

Proof. (By contradiction) Suppose S is a set and $\leq$ is a linear order on S satisfying the above property, yet S is infinite. For $n \geq 0$, we recursively define $x_n := \inf(S \backslash \{x_i \colon i < n\})$, and set $X := \{x_n \colon n \geq 0\}$. Then (x_n) is a strictly increasing infinite sequence of members of S. Therefore, $\sup(X) \notin X$, a contradiction. $\qquad\square$

Finally, we present our proof of the Bolzano-Weierstrass Theorem.

Proof. (By contraposition) Let S be a bounded subset of $\mathbb{R}$, and assume S has no limit point. Suppose $X \subseteq S$ is nonempty. Then $\inf(X) \in X$, lest $\inf(X)$ be a limit point of X, hence also of S. Analogously, $\sup(X) \in X$. Lemma 1 implies that S is finite. $\qquad\square$

References

[1] R. Bartle, D. Sherbert, *Introduction to real analysis. Third edition.* John Wiley & Sons, Inc., New York, 2000.

[2] T. Bohnet, *Leben und Werk des Mathematikers Paul Stäckel*, Staatsexamensarbeit, Karlsruhe University, 1993.

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[2]It was Stäckel who first coined the term *twin primes*. Interestingly, Stäckel obtained his Ph.D. under the direction of none other than Karl Weierstrass.

[3] L. Brand, *Advanced calculus. An introduction to classical analysis.* John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1955.

[4] E. Gaughan, *Introduction to analysis. Fifth edition.* Brooks/Cole Publishing Company, Pacific Grove, 2009.

[5] M. Hoffman, J. Marsden, *Elementary classical analysis. Second edition.* W.H. Freeman and Company, San Francisco, 1993.

[6] C. Morrey, M. Protter, *A first course in real analysis. Second edition.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1991.

[7] C. Pugh, *Real mathematical analysis. Second edition.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2015.

[8] K. Ross, *Elementary analysis. The theory of calculus. Second edition.* In collaboration with Jorge M. Lopez. Undergraduate Texts in Mathematics. Springer, New York, 2013.

[9] G.N. Watson, E.T. Whittaker, *A course of modern analysis. Fourth edition.* Cambridge University Press, New York, 1927.