ELEMENTARILY λ-HOMOGENEOUS BINARY FUNCTIONS

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ABSTRACT. Let S and T be sets with S infinite, and $*: S \times S \to T$ be a function. Further, suppose that λ is a cardinal such that $\aleph_0 \leq \lambda \leq |S|$. Say that $(S, T, *)$ is *elementarily λ-homogeneous* provided $(X, T, *)$ is elementarily equivalent to $(Y, T, *)$ for all subsets X and Y of S of cardinality λ. In this note, we classify the elementarily λ-homogeneous structures $(S, T, *)$. As corollaries, we characterize certain mathematical structures $\mathfrak{S}$ which are also "elementarily λ-homogeneous" in the sense that all substructures of $\mathfrak{S}$ of cardinality λ are elementarily equivalent. Among our corollaries is a generalization of a theorem due to Manfred Droste.

1. INTRODUCTION

Let L be a first-order language with equality, and suppose that $\mathbf{M}$ is an infinite L-structure with universe M. For an infinite cardinal number $\lambda \leq |M|$, $\mathbf{M}$ is λ-*homogeneous* if any two substructures of $\mathbf{M}$ of cardinality λ are isomorphic. This notion was considered some time ago by W.R. Scott ([11]). In this paper, he characterizes the infinite abelian groups G which are $|G|$-homogeneous. In [7] and [8], the author extends Scott's results to infinite unitary modules over a commutative ring R, calling an infinite module M over R *congruent* if and only if every submodule N of M of the same cardinality as M is isomorphic to M (that is, M is $|M|$-homogeneous). Variants of the notion of homogeneity have also received attention in model theory, graph theory, group theory, and topology. For example, in [2], Gibson, Pouzet, and Woodrow characterize the relational structures $\mathbf{X} := (X, R_i : i \in I)$ for which there exists a cardinal λ with $\aleph_0 \leq \lambda \leq |X|$ such that $\mathbf{X}$ has but finitely many substructures of size λ up to isomorphism[1]. Their work generalizes previous results by Kierstead and Nyikos who in [4] determine hypergraphs $\mathbf{G}$ with but finitely many induced subgraphs (up to isomorphism) for some infinite cardinal κ. Transitioning to group theory, Robinson and Timm call a group G an *hc group* provided any two subgroups of G of finite index are isomorphic ([9]). An abelian group with this property is called *minimal*, and minimal abelian groups have also received attention in the literature (see [5] and [10]). A topological space $\mathbf{X} := (X, \mathcal{O})$ is called a *Toronto space* if $\mathbf{X}$ is homeomorphic to all of its subspaces of cardinality $|X|$. The countably infinite Toronto spaces have all been classified, but the question of whether there exists an uncountable non-discrete Hausdorff Toronto space remains open (see [12] for details).

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[1]Each R_i is a relation of finite arity on X. It is not assumed that the arities have a finite bound.

We now describe a related but more general notion. Suppose that $\mathbf{M}$ is an infinite L-structure, and let λ be an infinite cardinal such that $\lambda \leq |M|$. Say that $\mathbf{M}$ is *elementarily λ-homogeneous* if any two substructures of cardinality λ are elementarily equivalent. Manfred Droste classifies the elementarily λ-homogeneous structures (X, R), where R is a binary relation on X and $\lambda \leq |X|$ ([1], Theorem 1.1). Years later, the author determines the elementarily λ-homogeneous structures $\mathbf{A} := (A, f)$, where $f\colon A \to A$ is a function and $\aleph_0 \leq \lambda \leq |A|$.

In this paper, we consider binary functions $*\colon S \times S \to T$, where S and T are sets with S infinite and λ is an infinite cardinal such that $\lambda \leq |S|$. Then $(S, T, *)$ is λ-homogeneous if $(X, T, *)$ is isomorphic to $(Y, T, *)$ for any subsets X and Y of S of size λ and elementarily λ-homogeneous if $(X, T, *)$ is elementarily equivalent to $(Y, T, *)$ for all subsets X and Y of S of size λ. Our main result classifies the elementarily λ-homogeneous binary functions. As applications, we obtain elementarily λ-homogeneous-themed corollaries for metric spaces and directed graphs. Further, we generalize Theorem 1.1 of [1].

2. Main Results

We begin by fixing an infinite set S, a set T disjoint from S, and a function $*\colon S \times S \to T$. The associated parameters for the structure $\mathfrak{U} := (S, T, *)$ are as follows: equality, a single binary function symbol $\boldsymbol{*}$, and for each $t \in T$, a constant c_t. Interpretation of the parameters in $\mathfrak{U}$ is canonical; the universe of $\mathfrak{U}$ is $S \cup T$, $\boldsymbol{*}$ names $*$, and each constant c_t names $t \in T$. Further, all variable assignments take on *only* values in S (and quantification of variables is solely over S as well). To streamline notation (since no confusion shall result), we denote $\boldsymbol{*}$ by $*$ and each constant c_t simply by t. Moreover, for $s_1, s_2 \in S$, we shall denote $*((s_1, s_2))$ by the more compact $s_1 * s_2$.

Now let X and Y be subsets of S, and consider the structures $(X, T, *)$ and $(Y, T, *)$. Then $(X, T, *)$ is *isomorphic* to $(Y, T, *)$, denoted $(X, T, *) \cong (Y, T, *)$, if there is a bijection $f\colon X \to Y$ such that $a * b = f(a) * f(b)$ for all $a, b \in X$. More generally, $(X, T, *)$ is *elementarily equivalent* to $(Y, T, *)$ provided for any sentence φ in the language outlined above, $(X, T, *) \models \varphi$ if and only if $(Y, T, *) \models \varphi$.

Remark 1. *Suppose that X and Y are subsets of S such that $(X, T, *) \cong (Y, T, *)$. Then also $(X, T, *) \equiv (Y, T, *)$. Thus for any infinite $\lambda \leq |S|$: if $(S, T, *)$ is λ-homogeneous, then $(S, T, *)$ is elementarily λ-homogeneous.*

Throughout this section, we shall make repeated use of the following lemma.

Lemma 1. *Suppose S is an infinite set of size κ and $*\colon S \times S \to T$ is a function such that $(S, T, *)$ is elementarily κ-homogeneous. Assume further that $(S, T, *) \models \exists x \varphi(x)$ for some formula $\varphi(x)$ (in which at most the variable x occurs free). Then there exists $\{s_i \colon i < \kappa\} \subseteq S$ such that $(S \backslash \{s_i \colon i < j\}, T, *) \models \varphi(s_j)$ for all $j < \kappa$.*

Proof. We suppose that $(S,T,*) \models \exists x\varphi(x)$. Since $(S,T,*)$ is elementarily κ-homogeneous, it follows that $(X,T,*) \models \exists x\varphi(x)$ for every $X \subseteq S$ of cardinality κ. Hence for such an X, there is $x_0 \in X$ such that $(X,T,*) \models \varphi(x_0)$; set $X_\varphi := \{x_0 \in X\colon (X,T,*) \models \varphi(x_0)\}$. Now let π be a choice function for $\{X_\varphi\colon X \subseteq S \text{ and } |X| = \kappa\}$, and pick $z \notin S$ arbitrarily. Transfinite Recursion furnishes us with a unique function F with domain ORD (the class of ordinal numbers) such that for any $j \in ORD$:

(2.1)
$$F(j) := \begin{cases} \pi(S\backslash\{F(i)\colon i<j\}) & \text{if } \{F(i)\colon i<j\} \subseteq S \text{ and } |S\backslash\{F(i)\colon i<j\}| = \kappa; \\ z & \text{otherwise.} \end{cases}$$

It is clear that F is injective on $\{k \in ORD\colon F(k) \in S\}$; hence (as S is a set) $z \in \operatorname{ran}(F)$. Let $\alpha \in ORD$ be least such that $F(\alpha) = z$. Observe that $\alpha \geq \kappa$. Finally, for $i < \kappa$, set $s_i := F(i)$. Then $\{s_i\colon i < \kappa\}$ yields the required set. □

We now present the main result of this note.

Theorem 1. *Let S and T be sets with S infinite of cardinality κ, and suppose that $*\colon S\times S \to T$ is a function. Then $(S,T,*)$ is elementarily κ-homogeneous if and only if there exists a (strict) well-order $<$ on S and elements $a,b,c \in T$ (not necessarily distinct) such that*

(1) *$(S,<)$ has the order type of a cardinal, and*
(2) *for all $s_1, s_2 \in S$:*

$$s_1 * s_2 = \begin{cases} a & \text{if } s_1 = s_2, \\ b & \text{if } s_1 < s_2, \text{ and} \\ c & \text{if } s_1 > s_2. \end{cases} \tag{2.2}$$

Proof. Suppose first that $<$ and $(S,T,*)$ satisfy (1) and (2) above. Now assume that $X \subseteq S$ and that $|X| = |S|$. Then $(S,<) \cong (X,<)$. Let $f\colon S \to X$ be an order isomorphism, and let R_a, R_b, and R_c denote the relations $=$, $<$, and $>$ on S, respectively. Now let $i \in \{a,b,c\}$ be arbitrary. Then for any $s_1, s_2 \in S$, we have $s_1*s_2 = i$ if and only if $s_1R_is_2$ if and only if $f(s_1)R_if(s_2)$ if and only if $f(s_1)*f(s_2) = i$. Thus $(S,T,*) \cong (X,T,*)$, and $(S,T,*)$ is elementarily κ-homogeneous by Remark 1.

Conversely, we assume that $(S,T,*)$ is elementarily κ-homogeneous. Pick $s \in S$, and set $s*s := a \in T$. Then of course, $(S,T,*) \models \exists x(x*x = a)$. Lemma 1 supplies us with a set $\{s_i\colon i<\kappa\} \subseteq S$ such that $(S\backslash\{s_i\colon i<j\},T,*) \models s_j * s_j = a$ for all $j < \kappa$. Setting $X := \{s_i\colon i<\kappa\}$, we have $(X,T,*) \models \forall x(x*x = a)$. As $(S,T,*) \equiv (X,T,*)$,

$$s * s = a \text{ for all } s \in S. \tag{2.3}$$

Next, choose distinct $\alpha, \beta \in S$ and set $\alpha * \beta := b \in T$. For $s \in S$, let $L_b(s) := \{x \in S\backslash\{s\}\colon x * s = b\}$ and $R_b(s) := \{x \in S\backslash\{s\}\colon s * x = b\}$. We now prove that

$$\text{for all } s \in S\colon\ |L_b(s)| = \kappa \text{ or } |R_b(s)| = \kappa. \tag{2.4}$$

Suppose not, and let $s \in S$ be such that $|L_b(s)| < \kappa$ and $|R_b(s)| < \kappa$. Now set $X := S \backslash (L_b(s) \cup R_b(s))$. Then observe that $|X| = \kappa$ and $(X, T, *) \models \exists x \forall y (y \neq x \Rightarrow (x * y \neq b) \wedge (y * x \neq b))$. By homogeneity[2],

$$(S, T, *) \models \exists x \forall y (y \neq x \Rightarrow (x * y \neq b) \wedge (y * x \neq b)). \tag{2.5}$$

Invoking Lemma 1 again, we obtain a set $\{s_i \colon i < \kappa\} \subseteq S$ such that $(S \backslash \{s_i \colon i < j\}, T, *) \models \forall y (y \neq s_j \Rightarrow (s_j * y \neq b) \wedge (y * s_j \neq b))$ for all $j < \kappa$. Setting $X := \{s_i \colon i < \kappa\}$, we see that

$$(X, T, *) \models \forall x \forall y (x \neq y \Rightarrow x * y \neq b). \tag{2.6}$$

The homogeneity of $(S, T, *)$ and (2.6) gives

$$(S, T, *) \models \forall x \forall y (x \neq y \Rightarrow x * y \neq b), \tag{2.7}$$

which contradicts the fact that $\alpha * \beta = b$. This verifies (2.4).

Now pick $s \in S$ arbitrarily. We may assume without loss of generality that

$$|R_b(s)| = \kappa, \tag{2.8}$$

as a symmetric argument can be used to deal with the case $|L_b(s)| = \kappa$. Applying the homogeneity of $(S, T, *)$, we deduce from (2.8) that

$$(S, T, *) \models \exists x \forall y (x \neq y \Rightarrow x * y = b). \tag{2.9}$$

Another application of Lemma 1 provides us with a set $\overline{S} := \{\overline{s}_i \colon i < \kappa\} \subseteq S$ with the property that

$$\overline{s}_i * \overline{s}_j = b \text{ for all } i, j \text{ such that } i < j < \kappa. \tag{2.10}$$

Next suppose that, conversely,

$$\overline{s}_j * \overline{s}_i = b \text{ for all } i, j \text{ such that } i < j < \kappa. \tag{2.11}$$

We conclude from (2.3), (2.10), (2.11), and the homogeneity of $(S, T, *)$ that

$$(S, T, *) \models (\forall x (x * x = a)) \wedge (\forall y \forall z (y \neq z \Rightarrow (y * z = b))). \tag{2.12}$$

[2] Throughout the article, "homogeneity" shall denote the property of being either elementarily λ-homogeneous or λ-homogeneous. Which is meant (and what λ is) should be clear from context.

Now simply choose any well-order $<$ with $(S,<) \cong (\kappa, \in)$. Then it immediate that $(S,T,*,<)$ satisfies (2.2) (with $c = b$).

For the remainder of the proof, we assume

$$\overline{s}_{j_0} * \overline{s}_{i_0} := c \neq b \text{ for some } i_0 < j_0 < \kappa. \tag{2.13}$$

Recall from (2.10) that if $i_0 < j < \kappa$, then $\overline{s}_{i_0} * \overline{s}_j = b$. We deduce that $|R_c(\overline{s}_{i_0}) \cap \overline{S}| < \kappa$. Applying (2.4) to the elementarily κ-homogeneous structure $(\overline{S}, T, *)$ and arguing as in (2.9), it follows that $(\overline{S}, T, *) \models \exists x \forall y (y \neq x \Rightarrow y * x = c)$. An argument analogous to the one used to establish the existence of $\overline{S}$ yields a set $\overline{\overline{S}} := \{\overline{\overline{s}}_i \colon i < \kappa\} \subseteq \overline{S}$ such that

$$\overline{\overline{s}}_j * \overline{\overline{s}}_i = c \text{ for all } i, j \text{ such that } i < j < \kappa. \tag{2.14}$$

We claim that

$$\overline{\overline{s}}_j = \overline{s}_j \text{ for all } j < \kappa. \tag{2.15}$$

To prove (2.15), fix $j < \kappa$ and suppose that (2.15) holds for all $i < j$. Since $\overline{\overline{S}} \subseteq \overline{S}$, we have

$$\overline{\overline{s}}_j = \overline{s}_{j'} \text{ for some } j' < \kappa. \tag{2.16}$$

It suffices to show that $j' = j$. Suppose first that $j' < j$. Then the inductive hypothesis along with (2.16) yields $\overline{\overline{s}}_j = \overline{s}_{j'} = \overline{\overline{s}}_{j'}$, and this is absurd. Now suppose that $j < j'$. By construction of $\overline{\overline{S}}$, we have $\overline{s} * \overline{\overline{s}}_j = c$ for all $\overline{s} \in \overline{S} \backslash \{\overline{\overline{s}}_i \colon i \leq j\} =$ (by the inductive hypothesis) $\overline{S} \backslash (\{\overline{s}_i \colon i < j\} \cup \{\overline{\overline{s}}_j\}) =$ (by (2.16))$\{\overline{s}_k \colon k \geq j,\ k \neq j'\}$. Summarizing,

$$\overline{s}_k * \overline{\overline{s}}_j = c \text{ for all } k \geq j, k \neq j'. \tag{2.17}$$

In particular, $\overline{s}_j * \overline{\overline{s}}_j = \overline{s}_j * \overline{s}_{j'} = c$. But $j < j'$, and so (2.10) implies that $\overline{s}_j * \overline{s}_{j'} = b$. This contradicts the fact that b and c are distinct, and the proof of (2.15) is complete.

Upon collecting (2.3), (2.10), (2.14), and (2.15), we see that $\overline{S} = \{\overline{s}_i \colon i < \kappa\}$ satisfies the following for all $\overline{s}_i, \overline{s}_j \in \overline{S}$:

$$\overline{s}_i * \overline{s}_j = \begin{cases} a & \text{if } i = j, \\ b & \text{if } i < j, \text{ and} \\ c \neq b & \text{if } i > j. \end{cases} \tag{2.18}$$

Now define a relation $<$ on S as follows:

(2.19) $$x < y \text{ if and only if } x \neq y \text{ and } x * y = b.$$

It is immediate that $<$ is irreflexive. Moreover, (2.18) implies that $(\overline{S}, T, *) \models \forall x \forall y \forall z(((x \neq y \wedge x * y = b) \wedge (y \neq z \wedge y * z = b)) \Rightarrow (x \neq z \wedge x * z = b))$. Since $(S, T, *)$ is elementarily κ-homogeneous, we see that $<$ is a transitive relation on S. By an analogous argument, it is easy to show that any two distinct elements of S compare under $<$. Thus $<$ is a total order on S.

Next, we prove that $<$ is a well-order on S. Toward this end, we begin by proving that

(2.20) $$\text{for any } s \in S,\ seg(s) := \{x \in S \colon x < s\} \text{ has cardinality less than } \kappa.$$

Suppose not, and let $s \in S$ be such that $|seg(s)| = \kappa$. Then $(seg(s) \cup \{s\}, T, *) \models \exists x \forall y(y \neq x \Rightarrow y * x = b)$. Homogeneity implies that also

(2.21) $$(\overline{S}, T, *) \models \exists x \forall y(y \neq x \Rightarrow y * x = b).$$

Now, since κ is an infinite cardinal, it follows that κ is a limit ordinal. Hence for any $i < \kappa$, also $i + 1 < \kappa$. Moreover, for any $i < \kappa$, (2.18) gives $\overline{s}_{i+1} * \overline{s}_i = c \neq b$. But then $(\overline{S}, T, *) \models \forall x \exists y(y \neq x \wedge y * x \neq b)$. This contradicts (2.21), and (2.20) is established. We now show that $<$ well-orders S. To see this, let $S' \subseteq S$ be nonempty, and choose $s_0 \in S'$. The fact that κ is a limit along with (2.18) implies that $(\overline{S}, T, *) \models \varphi := \forall x \exists y(x \neq y \wedge x * y = b)$. Thus also $(S, T, *) \models \varphi$, and so

(2.22) $$\text{there is } s_1 \in S \text{ such that } s_0 < s_1.$$

It is immediate from (2.18) that $(\overline{S}, T, *) \models \exists x \forall y(x \neq y \Rightarrow x * y = b)$. Further, (2.20) implies that $|S' \cup (S \backslash seg(s_1))| = \kappa$. Homogeneity yields the following:

(2.23) $$\text{there exists } x \in S' \cup (S \backslash seg(s_1)) \text{ with } x \leq y \text{ for all } y \in S' \cup (S \backslash seg(s_1)).$$

In particular, $x \leq s_0 < s_1$. Therefore, $x \in seg(s_1)$. It now follows from (2.23) that $x \in S'$, and we have shown that $<$ is a well-order on S. Moreover, (2.20) implies that $(S, <)$ has the order type of a cardinal.

At long last, we are ready to verify (2.2). To wit, let $s_1, s_2 \in S$ be arbitrary. If $s_1 = s_2$, then $s_1 * s_2 = a$ follows immediately from (2.3). Suppose now that $s_1 < s_2$. Then $s_1 * s_2 = b$ by definition of $<$. Now assume that $s_1 > s_2$. Observe from (2.18) that $(\overline{S}, T, *) \models \varphi := \forall x \forall y((x \neq y \wedge x * y = b) \Rightarrow y * x = c)$. Therefore, also $(S, T, *) \models \varphi$, and we deduce that $s_1 * s_2 = c$. The proof is concluded. $\square$

Recall that a function $* \colon S \times S \to T$ is *commutative* provided $s_1 * s_2 = s_2 * s_1$ for all $s_1, s_2 \in S$. The following corollary is immediately deduced from Theorem 1.

Corollary 1. *Let S and T be sets with S of infinite cardinality κ, and let $*\colon S\times S \to T$ be a commutative binary function. Then $(S, T, *)$ is elementarily κ-homogeneous if and only if there exist $a, b \in T$ such that for all $s_1, s_2 \in S$:*

$$s_1 * s_2 = \begin{cases} a & \text{if } s_1 = s_2, \text{ and} \\ b & \text{if } s_1 \neq s_2. \end{cases} \tag{2.24}$$

The next natural problem is to study elementarily λ-homogeneous structures $(S, T, *)$ where $\aleph_0 \leq \lambda < \kappa$. As we shall prove, every such structure is elementarily κ-homogeneous, but the converse holds if and only if $*$ is commutative.

Corollary 2. *Let S and T be sets with S infinite of cardinality κ, and suppose that $*\colon S \times S \to T$ is a function. Further, assume that $\aleph_0 \leq \lambda < \kappa$. Then $(S, T, *)$ is elementarily λ-homogeneous if and only if there exist $a, b \in T$ such that for all $s_1, s_2 \in S$:*

$$s_1 * s_2 = \begin{cases} a & \text{if } s_1 = s_2, \text{ and} \\ b & \text{if } s_1 \neq s_2. \end{cases} \tag{2.25}$$

Proof. Consider a structure $(S, T, *)$ which satisfies (2.25), and let X, Y be subsets of S of cardinality λ. Further, let $f\colon X \to Y$ be a bijection. Then it is clear that for any $x_1, x_2 \in X$, $x_1 * x_2 = f(x_1) * f(x_2)$. Therefore, $(X, T, *) \equiv (Y, T, *)$, and $(S, T, *)$ is elementarily λ-homogeneous.

Conversely, let $(S, T, *)$ be elementarily λ-homogeneous, where $\lambda < \kappa = |S|$. Now let $S_\lambda \subseteq S$ have size λ. Theorem 1 implies that there exist $a, b, c \in T$ and a well-order $<$ on S_λ such that

(1) $(S_\lambda, <)$ has the order type of a cardinal, and
(2) for all $s_1, s_2 \in S_\lambda$:

$$s_1 * s_2 = \begin{cases} a & \text{if } s_1 = s_2, \\ b & \text{if } s_1 < s_2, \text{ and} \\ c & \text{if } s_1 > s_2. \end{cases} \tag{2.26}$$

If we can show that $b = c$, then

$$(S_\lambda, T, *) \models \varphi := (\forall x(x * x = a) \wedge \forall x \forall y(x \neq y \Rightarrow x * y = b)).$$

In this case, the elementarily λ-homogeneity of $(S, T, *)$ implies that $(S, T, *) \models \varphi$ as well. Thus it suffices to prove that $b = c$.

Suppose by way of contradiction that $b \neq c$. Analogous to (2.19), we define a relation $\prec$ on S by $x \prec y$ if and only if $x \neq y$ and $x * y = b$. We argue that $\prec$ is a total order on S. Let X_λ be an arbitrary subset of S of size λ. It suffices to prove that $\prec$ is a total order on X_λ. But this follows immediately from the proof of

Theorem 1, (2.26), our assumption that $b \neq c$, and the elementarily λ-homogeneity of $(S, T, *)$. It is not hard to prove that, in fact, $\prec$ is a well-order on S. To do so, it suffices to show that every nonempty, countable subset of S has a $\prec$-least element. As λ is infinite and $X_\lambda \subseteq S$ of size λ is arbitrary, it is sufficient to establish that every nonempty subset $Y \subseteq X_\lambda$ has a $\prec$-least element. Again, the proof unfolds as in the proof of (2.20) and (2.23) above.

We have established that $\prec$ is a well-order on S. Now, for $s \in S$, we let $seg_\prec(s) := \{x \in S \colon x \prec s\}$. Since $|S| = \kappa > \lambda$, there is $s_0 \in S$ such that $|seg_\prec(s_0)| = \lambda$. Set $Y := seg_\prec(s_0) \cup \{s_0\}$. Observe that $|Y| = \lambda$ and that $(Y, T, *) \models \varphi := \exists x \forall y (y \neq x \Rightarrow y * x = b)$. Then by homogeneity, also $S_\lambda \models \varphi$, which by (2.26) is clearly impossible. This contradiction shows that $b = c$, and the proof is concluded. □

We conclude the section with another corollary, whose short proof we suppress.

Corollary 3. *Let S and T be sets with S infinite, and suppose that $* \colon S \times S \to T$ is a function. Further, let $\aleph_0 \leq \lambda \leq |S|$. Then $(S, T, *)$ is elementarily λ-homogeneous if and only if $(S, T, *)$ is λ-homogeneous.*

3. Some Consequences

Our first application is a characterization of the discrete metrics. Recall that if S is a nonempty set, then a metric $d \colon S \times S \to \mathbb{R}$ is called *discrete* if there exists some $r > 0$ such that for any $s_1, s_2 \in S$:

$$d(s_1, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2, \text{ and} \\ r & \text{if } s_1 \neq s_2. \end{cases}$$

Recall further that metric spaces (X, d) and (Y, ρ) are *isometric* if there is a surjective $f \colon X \to Y$ (a so-called *isometry*) such that for all $x_1, x_2 \in X$, $d(x_1, x_2) = \rho(f(x_1), f(x_2))$ (such an f is automatically one-to-one). We now state our next result.

Proposition 1. *For an infinite metric space (S, d), the following are equivalent:*

1. *(S, d) is elementarily λ-homogeneous for every infinite $\lambda \leq |S|$.*
2. *There is some infinite $\lambda \leq |S|$ such that (S, d) is elementarily λ-homogeneous.*
3. *The metric d is discrete.*
4. *Any two infinite subspaces of S of the same cardinality are isometric.*
5. *There is some infinite $\lambda \leq |S|$ such that any two subspaces of S of size λ are isometric.*

Proof. Let (S, d) be an infinite metric space.

(1) ⇒ (2) Clear.

(2) ⇒ (3) Immediate from Corollary 1, Corollary 2, and the fact that d is commutative.

(3) ⇒ (4) Trivial.

(4) ⇒ (5) Trivial.

(5) $\Rightarrow$ (1) By Corollary 3, (S, d) is elementarily λ-homogeneous. Invoking Corollary 1 and Corollary 2, we see that d is a discrete metric on S. But then (S, d) is α-homogeneous for every infinite $\alpha \leq |S|$. Corollary 3 applies again, and (S, d) is elementarily α-homogeneous for every infinite $\alpha \leq |S|$. □

We now shift our focus to graphs, specifically, directed graphs. If $\mathbf{G} := (V(G), E)$ is an infinite[3] undirected simple graph which is $|V(G)|$-homogeneous (that is, $\mathbf{G} \cong \mathbf{H}$ for every induced subgraph $\mathbf{H}$ of $\mathbf{G}$ such that $|V(H)| = |V(G)|$), then it is not hard to prove that $\mathbf{G}$ is either empty or complete. This result is noted in [4], p. 699, for example. We apply the results of the previous section to determine the elementarily λ-homogeneous directed graphs $\mathbf{G}$. *In what follows, we allow an arbitrary (possibly infinite) number of loops and multiple directed edges.*

Proposition 2. *Let $\mathbf{G} := (V(G), E)$ be an infinite directed graph, and let $\lambda \leq |V(G)|$ be an infinite cardinal. Then $\mathbf{G}$ is λ-homogeneous if and only if there exists a well-order $<$ on $V(G)$ with order type $|V(G)|$ and cardinals κ_1, κ_2, and κ_3 such that the following hold:*

(1) *there are κ_1 loops at each vertex $v \in V(G)$,*
(2) *there are κ_2 directed edges from v_1 to v_2 for $v_1 < v_2$, and*
(3) *there are κ_3 directed edges from v_2 to v_1 for $v_1 > v_2$.*

Moreover, if $\lambda < |V(G)|$, then $\kappa_2 = \kappa_3$.

Proof. Suppose that $\mathbf{G} := (V(G), E)$ is an infinite directed graph and that $\lambda \leq |V(G)|$ is an infinite cardinal. To begin, we must define what we mean by "elementarily λ-homogeneous" in this context. Observe that we may represent $\mathbf{G}$ as follows: $\mathbf{G} := (V(G), *)$, where $*\colon V \times V \to CARD$ (the class of cardinal numbers) is the function defined by $v_1 * v_2 :=$ the cardinal number of the set of directed edges from v_1 to v_2. We now let T be the range of $*$ (note that T is a set by the replacement axiom). The result now follows from Theorem 1 and Corollary 2. □

To conclude this note, we generalize a theorem of Manfred Droste. In Theorem 1.1 of [1], it is shown that if X is an infinite set, R is a nonempty binary relation on X, and λ is a cardinal with $\aleph_0 \leq \lambda \leq |X|$, then (X, R) is elementarily λ-homogeneous (here, the associated language has equality and a single binary relation symbol $\mathbf{R}$) if and only if R belongs to one of seven families of relations, three of which are singletons. If one allows the empty binary relation on X, then one obtains the following:

Proposition 3 ([1], Corollary 2.5)**.** *For each infinite cardinal κ, there are, up to isomorphism, precisely 8 binary relational structures (X, R) such that $|X| = \kappa$ and (X, R) is κ-homogeneous.*

We invoke Theorem 1 to determine all structures $(X, R_i\colon i \in I)$ which are elementarily λ-homogeneous, where $\aleph_0 \leq \lambda \leq |X|$ and each R_i is a binary relation on X

[3] That is, the set $V(G)$ of vertices of $\mathbf{G}$ is infinite.

(the associated language contains equality and a binary relation symbol $\mathbf{R}_i$ for every $i \in I$). Moreover, our argument should give the reader a sense for why the number 8 appears in the previous proposition.

Recall from the introduction that Gibson, Pouzet, and Woodrow ([2]) characterize the relational structures $\mathbf{X} := (X, R_i \colon i \in I)$ (where there is no bound assumed on the arities of the relations R_i) for which there exists a cardinal λ with $\aleph_0 \leq \lambda \leq |X|$ such that $\mathbf{X}$ has but finitely many substructures of size λ up to isomorphism. Their classification is a very deep structural result which is stated model-theoretically in terms of free definability. Our purpose is to apply Theorem 1 to give an explicit determination of the elementarily λ-homogeneous binary relational structures $(S, R_i \colon i \in I)$. It is with this determination that we conclude the paper.

Proposition 4. *Let S be an infinite set and suppose that $\{R_i \colon i \in I\}$ is a collection of binary relations on S. Further, let $\lambda \leq |S|$ be an infinite cardinal. Then $(S, R_i \colon i \in I)$ is elementarily λ-homogeneous if and only if there is a well order $<$ on S with order type of a cardinal such that every R_i is equal to one of the following relations:*

(1) $\varnothing$,
(2) $S \times S$,
(3) $=$,
(4) $\neq$,
(5) $<$,
(6) $\leq$,
(7) $>$, *or*
(8) $\geq$.

Moreover, if $\lambda < |S|$, then each R_i is one of the relations $(1) - (4)$ above.

Proof. Assume that $\aleph_0 \leq \lambda \leq |S|$. Suppose first that $<$ is a well-order on S with the order type of a cardinal and that $\{R_i \colon i \in I\}$ is a collection of binary relations on S such that each R_i is one of (1)–(8) above subject to the restriction that if $\lambda < |S|$, then each R_i is one of (1)–(4). It suffices to prove that $(S, R_i \colon i \in I)$ is λ-homogeneous. Toward this end, let S_1 and S_2 be subsets of S of size λ.

Case 1 $\lambda < |S|$. Let $f \colon S_1 \to S_2$ be a bijection. As each R_i is among (1)–(4), it is clear that f is an isomorphism between $(S_1, R_i \cap S_1^2 \colon i \in I)$ and $(S_2, R_i \cap {S_2}^2 \colon i \in I)$.

Case 2 $\lambda = |S|$. In this case, $(S, <) \cong (S_1, <)$ since $<$ has the order type of a cardinal. Let $f \colon S \to S_1$ be an isomorphism between the structures $(S, <)$ and $(S_1, <)$. Then it is easy to see that f is also an isomorphism between $(S, R_i \colon i \in I)$ and $(S_1, R_i \cap {S_1}^2 \colon i \in I)$.

Conversely, suppose that $(S, R_i \colon i \in I)$ is elementarily λ-homogeneous. Clearly we may assume that $I \neq \varnothing$. Now let $\mathcal{F}$ be a nonempty, finite subset of $\{R_i \colon i \in I\}$. Modifying the index set if necessary, we may assume that $\mathcal{F} = \{R_i \colon 1 \leq i \leq n\}$.

Observe that $(S, \mathcal{F})$ is also elementarily λ-homogeneous. For notational brevity in what follows, we set $[n] := \{1, \ldots, n\}$. Now define $*\colon S \times S \to \mathcal{P}([n])$ by

$$s_1 * s_2 := \{i \in [n] \colon s_1 R_i s_2\}. \tag{3.1}$$

We claim that

$$(S, \mathcal{P}([n]), *) \text{ is elementarily } \lambda\text{-homogeneous.} \tag{3.2}$$

To see this, let X and Y be subsets of S of size λ. We shall prove that $(X, \mathcal{P}([n]), *) \equiv (Y, \mathcal{P}([n]), *)$. Toward this end, suppose that φ is a sentence (in the language and sytax defined in the previous section) such that $(X, \mathcal{P}([n]), *) \models \varphi$. There is a natural interpretation $\overline{\varphi}$ of φ in the associated language of $(X, R_i \colon i \in [n])$ (which, we recall, contains equality and a binary relation symbol $\mathbf{R}_i$ for each $i \in [n]$) defined as follows: for any variables x and y (which need not be distinct) and $A \subseteq [n]$, replace every occurrence of $x * y = A$ in φ with $\bigwedge\limits_{i \in [n]} \chi_i$, where $\chi_i := x R_i y$ if $i \in A$ and $\chi_i := \neg x R_i y$ otherwise. As $(X, \mathcal{P}([n]), *) \models \varphi$, it is obvious that $(X, R_i \colon i \in [n]) \models \overline{\varphi}$. Since $(X, R_i \colon i \in I)$ is elementarily λ-homogeneous, also $(Y, R_i \colon i \in I) \models \overline{\varphi}$. Translating back, we see that $(Y, \mathcal{P}([n]), *) \models \varphi$, thereby establishing (3.2).

Case 1 $\lambda < |S|$. Then Corollary 2 yields the existence of $A, B \subseteq [n]$ such that for all $s_1, s_2 \in S$:

$$s_1 * s_2 = \begin{cases} A & \text{if } s_1 = s_2, \text{ and} \\ B & \text{if } s_1 \neq s_2. \end{cases} \tag{3.3}$$

Now, let $i \in [n]$ be arbitrary. If $i \in (A \cup B)^c$, then it is clear from (3.3) that $R_i = \varnothing$. At the other extreme, if $i \in A \cap B$, then $R_i = S \times S$. Finally, if $i \in A \backslash B$, then R_i is the equality relation on S; if $i \in B \backslash A$, then R_i is the inequality relation on S.

Case 2 $\lambda = |S|$. Then Theorem 1 supplies us with $A, B, C \subseteq [n]$ and a well-order $<$ on S with order type of a cardinal such that for all $s_1, s_2 \in S$:

$$s_1 * s_2 = \begin{cases} A & \text{if } s_1 = s_2, \\ B & \text{if } s_1 < s_2, \text{ and} \\ C & \text{if } s_1 > s_2. \end{cases} \tag{3.4}$$

Again, let $i \in [n]$ be arbitrary. Then of course $i \in A \cup A^c$, $i \in B \cup B^c$, and $i \in C \cup C^c$. Considering all eight possibilities (the arguments proceed as in Case 1) shows that R_i belongs to one of the relations (1)-(8) given in the statement of the proposition. We have now proved the proposition for any finite $\mathcal{F} \subseteq \{R_i \colon i \in I\}$. But then $|I| \leq 8$, and the proposition is now established. $\square$

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