A CHARACTERIZATION OF LARGE DEDEKIND DOMAINS

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ABSTRACT. Let D be a commutative domain with identity, and let $\mathcal{L}(D)$ be the lattice of nonzero ideals of D. Say that D is *ideal upper finite* provided $\mathcal{L}(D)$ is upper finite, that is, every nonzero ideal of D is contained in but finitely many ideals of D. Now let $\kappa > 2^{\aleph_0}$ be a cardinal. We show that a domain D of cardinality κ is ideal upper finite if and only if D is a Dedekind domain. We also show (in ZFC) that this result is sharp in the sense that if κ is a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, then there is an ideal upper finite domain of cardinality κ which is not Dedekind.

1. INTRODUCTION

As is well-known, in 1847, Gabrielle Lamé claimed to have a proof of Fermat's Last Theorem. Unfortunately, he made the error of assuming that $\mathbb{Z}[\zeta_n]$ is always a UFD, where ζ_n is a primitive nth root of unity. Ernst Kummer showed in 1844 that this was not always the case; in particular, he showed that $\mathbb{Z}[\zeta_{23}]$ is not a UFD. However, Kummer was able to prove Fermat's Last Theorem for a large collection of positive integers n using so-called "ideal numbers". Richard Dedekind subsequently gave the modern definition of "ideal" which enabled one to specify a class of domains now known as *Dedekind domains*, which we state below.

Definition 1. Let D be a commutative domain with identity. Then D is called a *Dedekind domain* if every proper, nonzero ideal of D is a finite product of prime ideals (which is necessarily unique up to the order of factors).

Dedekind domains abound in number theory, algebraic geometry, and commutative ideal theory. For example, if K is a number field, then the domain $\mathcal{O}_K$ of algebraic integers of K is a Dedekind domain. Moreover, there is an abundance of equivalent formulations of the notion of "Dedekind domain". The following is an amalgam of Theorem 37.1, Theorem 37.8, Theorem 38.1, and Theorem 38.5 of [2]:

Fact 1. Let D be an integral domain with identity which is not a field. Then the following are equivalent:

(1) D is a Dedekind domain.
(2) D is one-dimensional (Krull dimension), Noetherian, and integrally closed.
(3) Every proper homomorphic image of D is a principal ideal ring.
(4) D is Noetherian and for every maximal ideal J of D, the localization D_J is a discrete valuation ring (DVR).

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(5) D is Noetherian, and for every maximal ideal J of D, there are no ideals of D properly between J and J^2.

The impetus for this note stems from the following elementary observation:

Observation 1. Every nonzero integer has but finitely many integer divisors.

As a principal ideal domain (PID) has the property that every nonzero nonunit has only finitely many divisors up to units, it follows that a PID R has the property that every nonzero ideal of R is contained in but finitely many ideals of R. Using Cohen's structure theorems for complete local rings, it is easy to prove that every Dedekind domain also shares this property.

This leads us to a natural question:

Question Does the property, "every nonzero ideal of an integral domain D (with identity) is contained in but finitely many ideals of D" distinguish the Dedekind domains within the class of integral domains?

It is not hard to show that the answer to this question is "no". However, we prove that this is indeed the case for Dedekind domains of cardinality $\kappa > 2^{\aleph_0}$, that is, if D is a domain of cardinality $\kappa > 2^{\aleph_0}$, then D is Dedekind if and only if every nonzero ideal of D is contained in but finitely many ideals of D. However, we demonstrate (in ZFC) that if κ is a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, then there is an integral domain D of cardinality κ which is not Dedekind, yet every nonzero ideal of D is contained in but finitely many ideals of D.

2. Preliminaries

Throughout this paper, all rings are assumed to be commutative with identity $1 \neq 0$ unless specified otherwise. Moreover, subrings and homomorphisms are assumed to be unital.

If D is a domain, then the set $\mathcal{L}(D)$ of nonzero ideals of D is a partially ordered set (a lattice, though generally not bounded) via set-theoretic conclusion. Next, recall that a partially ordered set $(P, \leq)$ is *upper finite* provided that for every $p \in P$, there are but finitely many $y \in P$ for which $p \leq y$. For brevity, let us agree to call a domain D *ideal upper finite* provided the poset $(\mathcal{L}(D), \subseteq)$ is upper finite. We may now rephrase the question above more briefly as follows (as stated above, we shall soon give a negative answer):

Question 1. Let D be a domain. Is it the case that D is Dedekind if and only if D is ideal upper finite?

To begin building the machinery to prove the results mentioned in the introduction, we recall the following fact (see Theorem 3.3 of [4]):

Fact 2. (Cohen) Every local Artinian principal ideal ring is the proper homomorphic image of a discrete valuation ring $(V, \mathfrak{m})$.[1]

[1]Recall that a *discrete valuation ring* is a principal ideal domain V with a unique nonzero prime ideal $\mathfrak{m}$.

We shall soon use this fact to demonstrate that every Dedekind domain is ideal upper finite.

Next, say that a ring R is *residually finite* provided R/I is a finite ring for every nonzero ideal I of R. Chew and Lawn proved the following theorem that we shall use later in this note:

Fact 3. ([1], Theorem 2.3) Let R be an infinite ring. Then R is residually finite if and only if R is Noetherian and R/P is finite for every nonzero prime ideal P of R.

We pause to make the following trivial observation, which will be useful shortly.

Observation 2. If R is a residually finite ring, then R is ideal upper finite.

We conclude this section by stating that fundamentals of commutative ideal theory are assumed; we refer the reader to the standard references [2] and [3] for further details.

3. Results

We begin with a useful (and well-known) proposition. We include a proof for completeness.

Lemma 1. *Let D be a domain which is not a field, and let K be the field of fractions of D. There is no ring R such that $D \subsetneq R \subsetneq K$ (that is, no proper nontrivial overring of D) if and only if D is a one-dimensional valuation ring (though not necessarily discrete).*

Proof. If $(V, \mathfrak{M})$ is a one-dimensional valuation ring, then the only prime ideals of V are $\{0\}$ and $\mathfrak{M}$. Since every overring of V is a localization of V, the only overrings of V are $V_{\{0\}} = K$ and $V_{\mathfrak{M}} = V$ (recall that every element in the complement of M is a unit since V is local).

Conversely, suppose that D is a domain which is not a field with quotient field K. Suppose there are no rings properly between D and K. Let $\overline{D}$ be the integral closure of D (in K). Then $\overline{D} = D$ or $\overline{D} = K$. The latter is impossible since then D would be a field. Hence $\overline{D} = D$, and D is integrally closed. It follows that D is the intersection of valuation overrings. The only possible such overrings are D and K. Since $D \neq K$, not every one of the valuation overrings is equal to K. It follows that D is a valuation ring (which is not a field). It remains to show that $D := V$ is one-dimensional. Let P be a nonzero prime ideal of V. Then V_P is a nontrivial overring of V. By the condition on V, we have $V_P = K$. If $P \subsetneq Q$ for some prime ideal Q of V, then $V \subsetneq V_Q \subsetneq V_P = K$, contradicting our assumption on V. Hence P is maximal, and V is one-dimensional. □

Next, recall that if F is a field, then $F[[X]]$ denotes the ring of formal power series in X with coefficients in F. It is well-known (and easy to prove) that $F[[X]]$ is a DVR. Next, introduce the following subring of $F[[X]]$:

Definition 2. Let F be a field. Define $F[[X^2, X^3]] := \{\sum_{n=0}^{\infty} a_n X^n \colon a_1 = 0\}$.

It is routine to verify that $F[[X^2, X^3]]$ is a subring of $F[[X]]$. We now prove a lemma which will be useful throughout this note.

Lemma 2. *Let F be a field. Then $F[[X^2, X^3]]$ is a one-dimensional Noetherian local domain. Moreover, $F[[X]]$ is the unique proper valuation overring of $F[[X^2, X^3]]$.*

Proof. $F[[X]]$ is integral over $F[[X^2, X^3]]$ and hence both rings have Krull dimension one. It is well-known that $F[[X, Y]]$, the power series ring in X and Y, is Noetherian, and clearly $F[[X^2, X^3]]$ is a homomorphic image of $F[[X, Y]]$, thus also Noetherian. Next, let K be the quotient field of $F[[X^2, X^3]]$ and let $(V, \mathfrak{M})$ be a proper valuation overring of $F[[X^2, X^3]]$. Now, $F[[X^2, X^3]] \subseteq V \subsetneq K$ and $X \in K$ is integral over $F[[X^2, X^3]]$. Because V is integrally closed, $F[[X]] \subseteq V \subsetneq K$. Lemma 1 implies that $V = F[[X]]$. This proves that $F[[X^2, X^3]]$ is a one-dimensional Noetherian domain with unique proper valuation overring $F[[X]]$. Finally, let P be a nonzero prime ideal of $F[[X^2, X^3]]$. Then $(F[[X]], XF[[X]])$ has center P on $F[[X^2, X^3]]$, that is, $XF[[X]] \cap F[[X^2, X^3]] = P$. It follows that P is the unique nonzero prime ideal of $F[[X^2, X^3]]$; hence $F[[X^2, X^3]]$ is local, and the proof is complete. □

We are now equipped to prove our first theorem.

Theorem 1. *Let D be a Dedekind domain. Then D is ideal upper finite. However, there are ideal upper finite domains which are not Dedekind.*

Proof. Suppose first that D is a Dedekind domain. If D is a field, then D is obviously ideal upper finite. Thus suppose that D is not a field, and let I be an arbitrary nonzero proper ideal of D. We shall prove that I is contained in but finitely many ideals of D. From (2) of Fact 1, D is a one-dimensional Noetherian ring; hence D/I is a zero-dimensional Noetherian ring, and so Artinian. We deduce that $D/I = R_1 \times \cdots \times R_n$ for some Artinian local rings $R_1, \ldots, R_n$. Applying (3) of Fact 1, we see that D/I is a principal ideal ring. This implies that each R_i is an Artinian local principal ideal ring. Invoking Fact 2, R_i is a proper homomorphic of a DVR $(V, \mathfrak{m})$. Recall that the nonzero ideals of V are precisely the ideals $\mathfrak{m}^k$, where k ranges over the non-negative integers. Thus $R_i \cong V/\langle \mathfrak{m}^{n_i} \rangle$ for some positive integer n_i. The only ideals of V which contain $\mathfrak{m}^{n_i}$ are the ideals $\mathfrak{m}^{n_i}, \mathfrak{m}^{n_i - 1} \cdots, \mathfrak{m}^0 := V$. Hence, there are but finitely many ideals of $V/\langle \mathfrak{m}^{n_i} \rangle \cong R_i$. Now, D/I is a finite product of rings, all of which have but finitely many ideals. It follows that D/I has but finitely many ideals. Therefore, there are only finitely many ideals of D which contain I. Thus being a Dedekind domain is a sufficient condition for being ideal upper finite.

However, being Dedekind is not necessary, as we establish now. Let F be a finite field, and consider the ring $D := F[[X^2, X^3]]$ with quotient field K and maximal ideal J. By lemma 2, $D = (D, J)$ is a one-dimensional Noetherian local domain with unique proper valuation overring $F[[X]]$; moreover, $F[[X]]$ has center J on D. There is a natural injection $\eta \colon D/J \to F[[X]]/\langle X \rangle \cong F$. Since F is finite, we conclude that D/J is finite. Invoking Fact 3, we conclude that D is residually finite. But then for any nonzero ideal I of D, D/I is finite, thus has but finitely many ideals. Therefore, there are but finitely many ideals of D containing I, and D is ideal upper finite. Finally, observe that $X \in K$ is integral over D, but is not a member of D. We deduce that D is not integrally closed, and so by (2) of Fact 1, D is not Dedekind. □

Remark 1. Recall from Observation 2 that if a domain D is residually finite, then D is also upper ideal finite. However, the previous theorem shows that the converse fails. Indeed, let F be an infinite field. Then the polynomial ring $F[X]$ in X over F is Dedekind, thus ideal upper finite. But $F[X]/\langle X \rangle \cong F$ is infinite.

Corollary 1. *Let D be a one-dimensional Noetherian domain. Then the integral closure $\overline{D}$ of D is ideal upper finite.*

Now that we have seen that being Dedekind is a sufficient but not necessary condition for being ideal upper finite, we find some necessary conditions for a domain to be ideal upper finite.

Proposition 1. *Let D be an ideal upper finite domain which is not a field. Then D is a one-dimensional Noetherian domain.*

Proof. Let D be ideal upper finite. Then there can be no infinite, properly ascending chain of ideals of D, lest every ideal of the chain be contained in infinitely many ideals of D. Hence D is Noetherian. Now let P be a nonzero prime ideal of D. Then D/P is a domain with but finitely many ideals. It follows that D/P is a field, and hence P is a maximal ideal. □

Theorem 1 shows that being Dedekind is not necessary for a domain D to be ideal upper finite. However, we show that if D is sufficiently large, then being Dedekind becomes necessary. Toward this end, it will aid us tremendously to have a description of the commutative rings R with but finitely many ideals. This question was addressed rather recently in the thesis [7]. We present a different proof using a lemma from the literature, which we present now.

Lemma 3 ([6], Corollary 5(b))**.** *Let R be an infinite Noetherian ring. Then $|R/\text{Nil}(R)| = |R|$, that is, R and its reduction $R/\text{Nil}(R)$ have the same cardinality.*

Proposition 2 ([7], Theorem 3.3)**.** *A ring R has but finitely many ideals if and only if $R \cong S \times T$ for some finite ring S and Artinian principal ideal ring T.*[2]

Proof. Consider first a ring of the form $S \times T$ where S is a finite ring and T is an Artinian principal ideal ring. It is clear that S has but finitely many ideals. Now, (we can clearly assume T is nontrivial) T is a finite product of Artinian local principal ideal rings, hence by Fact 2, T is a finite product of chained rings, each with but finitely many ideals. It follows that T has finitely many ideals and, therefore, so does $S \times T$.

Conversely, let R be a ring with but finitely many ideals. If R is finite, we are done. Thus assume that R is infinite. Clearly R is Artinian, and hence $R = R_1 \times \cdots \times R_n$ for some Artinian local rings $R_1, \ldots, R_n$. It suffices to show that every infinite R_i is a principal ideal ring. We are done if we can show that an infinite local ring with but finitely many ideals is a principal ideal ring (PIR). Toward this end, let (S, J) be an infinite local ring with but finitely many ideals. We shall prove that S is a PIR. Since S has but finitely many ideals, S is Artinian and Noetherian. Hence J is the unique prime ideal of S, and so $\text{Nil}(S) = J$. We deduce from Lemma 3 that

$$|S| = |S/J|. \tag{3.1}$$

Since S is an Artinian ring, S possesses a minimal (nonzero) ideal I. We claim that

$$I \text{ is the unique minimal ideal of } S. \tag{3.2}$$

[2]Here, S or T could be trivial.

Suppose that there is another minimal ideal I' of S. Then as S-modules, we have $I \oplus I' \cong S/J \times S/J$. The S-submodules of $S/J \times S/J$ coincide with the S sub-ideals of the ideal $I \oplus I'$ (note that the sum is necessarily direct by minimality of I and I'). Now, the S-submodules of $S/J \times S/J$ coincide with the S/J-subspaces of the vector space $S/J \times S/J$. By (3.1), $F := S/J$ is infinite. Moreover, $F \times F$ has infinitely many subspaces: for every $a \in F$, let $\ell_a := \{x(1, a) \colon x \in F\}$. One checks that $\ell_a \neq \ell_b$ for $a \neq b$. But from our work above, this means that there are infinitely many ideals of S contained in the ideal $I \oplus I'$. This countradicts our assumption that S has but finitely many ideals, and (3.2) is established.

Let $I := I_1$ be the unique minimal ideal of S. Because every nonzero ideal of S contains a minimal ideal (S being Artinian), we see that

(3.3) $$I_1 \subseteq X \text{ for every nonzero ideal } X \text{ of } S.$$

If $I_1 = S$, then S is a field, thus a PIR. Now assume that $I_1 \neq S$. Now, $|S| = |S/J| = |(S/I_1)/(J/I_1)|$. It follows that $(S/I_1, J/I_1)$ is an infinite local ring with but finitely many ideals. If $I_1 = J$, then S has exactly three ideals, $\{0\} \subsetneq I_1 = J \subsetneq S$. It follows that S is a chained Noetherian ring, hence a PIR. Now suppose that $I_1 \neq J$. Applying the above argument to the ring S/I_1, there is an ideal I_2 of S properly containing I_1 such that I_2 is contained in every ideal of S properly contained in I_1. If $I_2 = J$, then the ideals of S are given by $\{0\} \subsetneq I_1 \subsetneq I_2 = J \subsetneq S$. Again, S is a chained Noetherian ring, hence a PIR. This process must terminate after finitely many steps since S is Noetherian, showing that S is a PIR, as asserted. □

A natural question is whether one can find both necessary and sufficient conditions on a domain D in order for D to be ideal upper finite. We present such conditions below, and sketch a proof of the forward implication (for details, we refer the reader to Chapters 4 - 7 of [2]).

Proposition 3. *Let D be a domain which is not a field. Then D is ideal upper finite if and only if D is a one-dimensional Noetherian domain such that J is invertible for every maximal ideal J of D of infinite index.*

Sketch of Proof. Suppose D is ideal upper finite and not a field. By Proposition 1, D is one-dimensional and Noetherian. Next, let J be a maximal ideal of infinite index. We will show that J is invertible. Since D is Noetherian, it suffices to prove that JD_J is invertible in the local ring D_J. Toward this end, it suffices to prove that D_J is a Dedekind domain. Let X be an arbitrary nonzero ideal of D_J. By Fact 1(3), if suffices to prove that D_J/X is a principal ideal ring. Now, $X = ID_J$ for some nonzero ideal I of D contained in J. As D is ideal upper finite, D/I has but finitely many ideals. It follows that the quotient ring $D_J/ID_J \cong (D/I)_{(D\setminus J)/I}$ also has but finitely many ideals, since each ideal of $(D/I)_{D\setminus J)/I}$ is an extension of an ideal of D/I. Proposition 2 along with the fact that D_J/ID_J is *local* implies that either D_J/ID_J is finite or D_J/ID_J is a principal ideal ring. Recall above that D/J is infinite. Since $D_J/JD_J \cong D/J$, we see that D_J/JD_J is infinite. As $ID_J \subseteq JD_J$, it follows that $|D_J/JD_J| \leq |D_J/ID_J|$; thus D_J/ID_J is also infinite. Hence, finally, D_J/ID_J is a principal ideal ring, and we have shown that D_J is Dedekind. □

With Proposition 2 in hand, we are almost ready to prove that "Dedekind" and "upper finite" are equivalent for domains D of cardinality larger than $|\mathbb{R}| = 2^{\aleph_0}$. We shall make use of the following lemma from the literature.

Lemma 4 ([5], Lemma 3(1))**.** *Let R be a ring and I be a finitely generated ideal of R. If R/I is finite, then R/I^n is finite for every positive integer n.*

We now come to the result promised above; we give two proofs.

Theorem 2. *Let D be a domain of cardinality $\kappa > 2^{\aleph_0}$. Then D is a Dedekind domain if and only if D is ideal upper finite.*

Proof 1. Let D be as stated. If D is Dedekind, then Theorem 1 implies that D is ideal upper finite. Conversely, suppose that D is ideal upper finite. We shall prove that D is Dedekind. Utilizing (3) of Fact 1, let I be a nonzero ideal of D. We shall prove that D/I is a principal ideal ring. If $I = D$, this is clear, so suppose that I is a nonzero proper ideal of D. Because D is ideal upper finite, the ring D/I has but finitely many ideals. Proposition 2 shows us that $D/I = S \times T$ for some finite ring S and principal ideal ring T. It suffices to prove that S is trivial. Suppose not. Then we have surjections $D \to D/I \to S$; let K be the kernel of the composition. Then $D/K \cong S$. Because S is nontrivial, K is proper. Now, D is Noetherian since D is ideal upper finite. Thus by Krull's Intersection Theorem, we conclude that $\bigcap_{n=1}^{\infty} K^n = \{0\}$. The natural map $D \to \prod_{n=1}^{\infty} D/K^n$ is therefore injective. By Lemma 4, D/K^n is finite for every positive integer n. it follows that

$$|D| \leq \left| \prod_{n=1}^{\infty} D/K^n \right| \leq \aleph_0^{\aleph_0} = 2^{\aleph_0},$$

Contradicting that $|D| > 2^{\aleph_0}$. Hence S is indeed trivial, and D/I is a principal ideal ring. We have proven that D is Dedekind. □

Proof 2. Let D be as stated, and suppose that D is ideal upper finite. Proposition 1 yields that D is Noetherian. Let J be an arbitrary maximal ideal of D. By the argument given above invoking Krull's Intersection Theorem, D/J cannot be finite, lest $|D| \leq 2^{\aleph_0}$; thus D/J is infinite. Now, J/J^2 is naturally a D/J-vector space of dimension at least one since $J^2 \neq J$ (by Krull's Intersection Theorem). If J/J^2 had dimension greater than one, then J/J^2 would have infinitely many subspaces as D/J is infinite. But then there would be infinitely many ideals of D containing the nonzero ideal J^2, a contradiction to upper finiteness. Hence J/J^2 is a one-dimensional D/J-vector space, and so there are no ideals of D properly between J^2 and J. Fact 1(5) implies that D is Dedekind. □

Remark 2. Our work above yields the following curious result: we showed in Theorem 1 that if F is a finite field, then $F[[X^2, X^3]]$ is ideal upper finite. However, by the previous theorem, if F is large enough, then $F[[X^2, X^3]]$ is not ideal upper finite, as this domain is not Dedekind. But then the lattice of ideals of the ring $F[[X^2, X^3]]$ is *not* independent of the field F, as is the case with the power series ring $F[[X]]$.

We conclude this note by demonstrating that the cardinal $2^{\aleph_0}$ in Theorem 2 is best-possible in the sense that if κ is a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, then there is an ideal upper finite domain D of cardinality κ which is not Dedekind.

Theorem 3. *Let κ be a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Then there is an ideal upper finite domain D of cardinality κ which is not Dedekind.*

Proof. Let κ be a cardinal with $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Further, fix a finite field F. Let $F(X)$ denote the field of rational functions over F in the variable X and let $F((X))$ denote the field of formal Laurent series in X (that is, $F((X))$ is the field of fractions of the DVR $F[[X]]$). Then note that $F(X)$ is a field of size $\aleph_0$, $F((X))$ is a field of size $2^{\aleph_0}$, and $F(X)$ is a subfield of $F((X))$. If follows that there is a field K of cardinality κ which satisfies

(3.4) $$F(X) \subseteq K \subseteq F((X)).$$

Now, the power series ring $F[[X]]$ is a DVR on $F((X))$ which is not a field. It follows that

(3.5) $V := F[[X]] \cap K$ is a DVR on K with maximal ideal $XF[[X]] \cap K$.[3]

There is a natural injection $\eta\colon V/(XF[[X]] \cap K) \to F[[X]]/XF[[X]] \cong F$. It follows that

(3.6) $V/(XF[[X]] \cap K)$ is finite.

Next let $S := F[[X^2, X^3]]$ and set $D := S \cap V$. Then we see that D is a subring of V. Moreover,

(3.7) $$X \notin D, \text{ yet } X^2, X^3 \in D.$$

We now show that

(3.8) D is local with maximal ideal $DX^2 + DX^3$.

To see this, observe that for any $f(X), g(X) \in D$, we have that $f(X)X^2 + g(X)X^3$ is not a unit of D, lest X be a unit of $F[[X]]$, which it isn't. It follows that $\langle X^2, X^3 \rangle$ is a proper ideal of D. We now show that every nonunit of D is a member $DX^2 + DX^3$. Let $f(X) \in D$ be a nonzero nonunit. We claim that $f(X)$ is not a unit of S. If it is, then $f(X)$ is also a unit of $F[[X]]$. Further, $f(X) \in D = S \cap V$, so $f(X) \in V = F[[X]] \cap K \subseteq K$. Therefore, $f(X)$ is a unit of $F[[X]]$ and a unit of K (since K is a field), thus a unit of V. But now $f(X)$ is a unit of S and a unit of V, thus a unit of D, a contradiction. Now, because $f(X)$ is a nonzero nonunit of S, we see that $f(X) := a_0 + a_1X + a_2X^2 + \cdots$ for some $a_i \in F$ such that $a_0 = a_1 = 0$. Therefore, $f(X) = X^2(a_2 + a_4X^2 + a_5X^3 + a_6X^6 + \cdots) + X^3(a_3)$. It is clear that both $a_2 + a_4X^2 + \cdots$ and a_3 are members of S. It remains to show that they are members of V (and thus members of D, by definition). Toward this end, $f(X) \in D = S \cap V$, so $f(X) \in V$. Moreover, K contains X^2, X^3 and

[3] Observe that $X \in V$ is not invertible in V, and so V is not a field.

F. Because $f(X) \in V \subseteq K$, it is now clear that $a_3 \in F[[X]] \cap K$ and $a_2 + a_4X^2 + a_5X^3 + a_6X^6 + \cdots \in F[[X]] \cap K = V$. This verifies (3.8).

Note trivially that every member of $F[[X]]$ has the form $f + aX$ for some $f \in S$ and $a \in F$. It follows easily that

(3.9) every member of V has the form $f + aX$ for some $f \in D$ and $a \in F$.

Finally, we are ready to complete our solution. Because F is finite, (3.9) shows that $|D| = |V| = |K| = \kappa$. We shall prove that D is not Dedekind yet ideal upper finite. That D is not Dedekind follows immediately from (3.7) above, since this shows that D is not integrally closed. Now let P be an arbitrary nonzero prime ideal of D and let W be a valuation overring of D with center P on D. Because W is integrally closed, it follows from (3.7) and (3.9) that $V \subseteq W$. Clearly $W \subseteq Q(D) \subseteq Q(V)$, where $Q(D)$ and $Q(V)$ are the quotient fields of D and V, respectively. Because V is a DVR, we conclude from Lemma 1 that $V = W$. But then $P = (XF[[X]] \cap K) \cap D$. This proves that D has but one nonzero prime ideal P; by (3.8), $P = DX^2 + X^3$. By Cohen's Theorem, D is Noetherian. To show that D is ideal upper finite, it suffices to prove that D is residually finite. By Fact 3, we simply must show that D/P is finite. Because V has center P on D, D/P embeds into $V/(XF[[X]] \cap K)$, which is finite by (3.6). This concludes the proof. □

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References

[1] K. Chew & S. Lawn, *Residually finite rings*, Canad. J. Math. **22** (1970), 92–101.
[2] R. Gilmer, *Multiplicative ideal theory. Corrected reprint of the 1972 edition.* Queen's Papers in Pure and Applied Mathematics, 90. Queen's University, Kingston, ON 1992.
[3] S. Lange, *Algebra. Revised third edition.* Graduate Texts in Mathematics, **211**. Springer-Verlag, New York, 2002.
[4] K.R. Mclean, *Commutative Artinian principal ideal rings*, Proc. London Math. Soc. **26** (1973), no. 3, 249–272.
[5] G. Oman, *Jónsson modules over Noetherian rings*, Comm. Alg. **38** (2010), no. 9, 3489–3498.
[6] G. Oman, *Small and large ideals of an associative ring*, J. Alg. Appl. **13** (2014), no. 5, 1350151, 20 pp.
[7] M.B.J. Zwann, *On commutative rings with only finitely many ideals*, Thesis (2014), Mathematisch Instituut, Universiteit Leiden. https://www.math.leidenuniv.nl/scripties/ZwaanBach.pdf

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