THE AXIOMATIZABILITY OF THE CLASS OF ROOT CLOSED MONOIDS

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Abstract. We prove that the theory of root closed monoids is axiomatizable, but not finitely axiomatizable. Some directions for further research are presented.

All monoids in this paper are assumed cancellative and commutative.

1. Main theorem

We begin this note by recalling that an (commutative) integral domain D is *integrally closed* provided every member α of the quotient field K of D which is a root of some monic $f(x) \in D[x]$ is actually in D. Integrally closed domains are ubiquitous in commutative ring theory. Indeed, most of the domains studied in multiplicative ideal theory are integrally closed. This massive class properly contains the classes of Prüfer domains and GCD domains, for example.

The notion of "integrally closed" has a natural analog in the realm of cancellative, commutative monoids. To wit, let S and T be monoids with $S \subseteq T$. Then the *root closure of S in T* is the monoid $S_r \subseteq T$ defined by $S_r := \{t \in T : t^n \in S$ for some $n > 0\}$. Say that S is *root closed in T* if and only if $S_r = S$. Now suppose that $T = \mathcal{Q}(S)$, the group of fractions of S defined by $\mathcal{Q}(S) := \{\frac{x}{y} : x, y \in S\}$ (with the canonical multiplication inherited from S). Then the *root closure $\overline{S}$ of S* is the root closure of S in $\mathcal{Q}(S)$. We say that S is *root closed* provided $\overline{S} = S$.

Examples of root closed monoids abound in semigroup theory as well. Recall that a monoid M is a *valuation monoid* if for any $x, y \in M$, either $x|y$ or $y|x$. It is a simple matter to check that every valuation monoid is root closed. More generally, a monoid M is a *GCD monoid* provided the intersection of any two principal ideals of M is again principal (this is equivalent to every pair of elements of M, not both zero, having a GCD in the usual ring-theoretic sense). It is an exercise to check that every GCD monoid is root closed (see pp. 76–77 of [2]).

We now turn our attention toward answering the following question:

Question 1. *Is the theory of root closed monoids finitely axiomatizable?*

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Toward this end, we shall require additional terminology. Let M be a monoid and p be a prime number. Say that M is *p-root closed* provided for all $g \in \mathcal{Q}(M)$: if $g^p \in M$, then $g \in M$. It is easy to see that

(1.1) M is root closed if and only if M is p-root closed for every prime p.

We will need the following lemma in the proof of our main result.

Lemma 1. *Let p be a prime. There exists a monoid M which is q-root closed for all primes $q \neq p$, but M is not p-root closed.*

Proof. We define an additive monoid M by $M := (\mathbb{N}^* \times \mathbb{N}) \cup \mathbb{N}(0, p)$. Observe that $\{(1,0),(1,1)\} \subseteq M$. Therefore, $\{(1,0),(0,1)\} \subseteq \mathcal{Q}(M)$, and $\mathcal{Q}(M) = \mathbb{Z} \times \mathbb{Z}$ follows. To see that M is not p-root closed, note that $p(0,1) = (0,p) \in M$, yet $(0,1) \notin M$. It is a triviality to prove that M is q-root closed for all primes $q \neq p$. □

We are now ready to answer Question 1 in the negative.

Theorem 1. *The theory of root closed monoids is axiomatizable, but not finitely axiomatizable*[1].

Proof. Let φ_M be the conjunction of the cancellative, commutative monoid axioms. Now let $\{p_i : i \in \mathbb{N}\}$ be an enumeration of the primes. For each $i \in \mathbb{N}$, we set $\varphi_i := \forall x \forall y (\exists z_1 (x^{p_i} = z_1 y^{p_i}) \Rightarrow \exists z_2 (x = z_2 y))$. Then the theory T of root closed monoids is axiomatized by $\sum := \{\varphi_M\} \cup \{\varphi_i : i \in \mathbb{N}\}$. We claim that T is not finitely axiomatizable. If so, then by The Compactness Theorem, there exists a finite $\sum_0 \subseteq \sum$ such that $T = Cn(\sum_0)$.[2] Let $n \in \mathbb{N}$ be such that $\sum_0 \subseteq \{\varphi_M\} \cup \{\varphi_i : 0 \leq i \leq n\}$. By Lemma 1, there exists a monoid M which is p_i-root closed for all i, $0 \leq i \leq n$, but is not p_{n+1}-root closed. However, such an M is a model of $\sum_0$, and hence also of T. This is a contradiction, and the proof is complete. □

2. Directions for further research

In this final section, we present two open problems. Before stating the first problem, we recall that a monoid M is *n-generated* provided M can be generated by a subset of size at most n.

Problem 1. *For every positive integer n, does there exist a sentence σ_n such that an n-generated monoid M is root closed if and only if $M \models \sigma_n$?*

We prove that such a σ_n exists for $n \leq 2$, but leave the general problem open. First, we prove a simple lemma.

[1] in the language **L** consisting of equality, a binary function symbol $\cdot$, and a constant symbol **1**.

[2] $Cn(\sum_0)$ is the set of all sentences logically implied by $\sum_0$.

Lemma 2. *Let M be a root closed monoid for which $\mathcal{Q}(M)$ is cyclic. Then M is a valuation monoid.*

Proof. If $\mathcal{Q}(M)$ is a finite cyclic group, then M is a group, and the conclusion is patent. Thus assume that M is a nontrivial submonoid of the additive group $\mathbb{Z}$ of integers. If M contains both positive and negative integers, then M is a group ([1], Theorem 2.6), and we are done as above. So assume that $M \subseteq \mathbb{N}$. We have $\mathcal{Q}(M) = \langle g \rangle$ for some positive integer g. Now pick any $m \in M - \{0\}$. Then $mg \in M$. Since M is root closed, $g \in M$. It is now clear that $M \cong \mathbb{N}$, and hence M is a valuation monoid in this case as well. □

Corollary 1. *We may take $\sigma_1 := \forall x \forall y \exists z(xz = y \vee yz = x)$.*

It is easy to see that σ_1 defined above must be modified to characterize the 2-generated root closed monoids. To wit, simply consider the additive monoid $M := \mathbb{N} \times \mathbb{N}$. It is obvious that M is root closed, however, neither of $(0, 1)$ and $(1, 0)$ divides the other in M. On the other hand,

Lemma 3. *Every root closed 2-generated monoid is a GCD monoid.*

Proof. Let M be 2-generated. If $\mathcal{Q}(M)$ is cyclic, then by Lemma 2, M is a valuation monoid. Thus M is a GCD monoid as well. So assume that $\mathcal{Q}(M)$ is not cyclic. If $\mathcal{Q}(M)$ is finite, then M is a group and we are clearly done. We have two remaining cases to consider.

Case 1. $\mathcal{Q}(M) \cong \mathbb{Z} \times \mathbb{Z}$. Then we may assume that $M := \mathbb{N}\mathbf{a} + \mathbb{N}\mathbf{b}$ is a submonoid of $\mathbb{Z} \times \mathbb{Z}$. Since M does not embed into $\mathbb{Z}$, it follows that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent over $\mathbb{Q}$. But then $M \cong \mathbb{N} \times \mathbb{N}$, and M is a GCD monoid.

Case 2. $\mathcal{Q}(M) \cong \mathbb{Z} \times \mathbb{Z}/\langle m \rangle$ for some positive integer m. Again, we assume M is a submonoid of $\mathbb{Z} \times \mathbb{Z}/\langle m \rangle$, and let $\pi_1 : M \to \mathbb{Z}$ be projection onto the first coordinate. It is easy to see that if $\pi_1(M)$ contains both positive and negative integers, then M is a group, and we're done. So assume that $\pi_1(M) \subseteq \mathbb{N}$. For any $g \in \mathbb{Z}/\langle m \rangle$, $m(0, g) = (0, 0) \in M$. Since M is root closed, we see that

$$\{0\} \times \mathbb{Z}/\langle m \rangle \subseteq M. \tag{2.1}$$

Now pick $x \in \pi_1(M) - \{0\}$ and let $n > 0$ and $g \in \mathbb{Z}/\langle m \rangle$ be arbitrary. Then $(xn, \alpha) \in M$ for some $\alpha \in \mathbb{Z}/\langle m \rangle$. It follows at once from (2.1) that $x(n, g) = (xn, xg) \in M$. Since M is root closed, $(n, g) \in M$. We have shown that $M = \mathbb{N} \times \mathbb{Z}/\langle m \rangle$, and hence M is a valuation monoid. We deduce that M is a GCD monoid, and the proof is complete. □

Corollary 2. *We may take σ_2 to be the sentence which formalizes "M is a GCD monoid."*

We leave the general investigation of Problem 1 to the reader. However, we remark that there exist root closed 3-generated monoids which are not GCD monoids. For example, let M be the submonoid of $\mathbb{Z} \times \mathbb{Z}$ generated by $a := (-1, 2)$, $b := (0, 1)$, and $c := (1, 1)$. Then it is easy to see that M is root closed, but $(M + a) \cap (M + b)$ is not principal.

Finally, we conclude the paper with the following related problem:

Problem 2. *Does there exist a sentence σ such that for all finitely generated monoids M: M is root closed if and only if $M \models \sigma$?*

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References

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