

# RINGS WITH ARITHMETICAL CLOSURE PROPERTIES RELATIVE TO THE PRIME SPECTRUM AND ITS COMPLEMENT

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ABSTRACT. Let  $R$  be a commutative unital ring, and let  $\text{Spec}(R) := \mathcal{P}(R)$  be the collection of prime ideals of  $R$ . Further, let  $\mathcal{NP}(R)$  denote the collection of *proper* nonprime ideals of  $R$  and set  $\mathcal{NP}(R)^1 := \mathcal{NP}(R) \cup \{R\}$ . In this note, we investigate some closure properties relative to these collections of ideals with respect to ideal addition and multiplication. Among the main objects of our study are rings  $R$  for which  $\mathcal{P}(R) + \mathcal{NP}(R) \subseteq \mathcal{P}(R)$ ,  $\mathcal{NP}(R)^1 + \mathcal{NP}(R)^1 \subseteq \mathcal{NP}(R)^1$ , or  $\mathcal{P}(R) \cdot \mathcal{P}(R) \subseteq \mathcal{P}(R)$ . Directions for further research are indicated.

## 1. INTRODUCTION

Let  $R$  be a commutative unital ring. As stated in the abstract, the purpose of this note is to study various arithmetic closure properties relative to the set  $\text{Spec}(R)$  of prime ideals and its complement (within the set of all ideals of  $R$ ). We begin with two very simple examples of the sort of questions addressed in this paper.

**Proposition 1.** *Let  $R$  be a commutative ring with identity.*

- (1) *The sum of distinct prime ideals of  $R$  is nonprime (possibly  $R$  itself) if and only if  $R$  is zero-dimensional (that is, every prime ideal of  $R$  is maximal).*
- (2) *The sum of a prime ideal of  $R$  and a nonprime ideal of  $R$  is a nonprime ideal of  $R$  if and only if  $R$  is a field.*

*Proof.* Suppose  $R$  is a commutative ring with 1.

(1) If  $R$  is zero-dimensional, then distinct prime ideals  $P$  and  $Q$  of  $R$  are distinct maximal ideals of  $R$ . Thus  $P + Q = R$ , and  $P + Q$  is not prime. Conversely, suppose  $R$  is not zero-dimensional. Then there are prime ideals  $P$  and  $Q$  of  $R$  such that  $P \subsetneq Q$ . But  $P + Q = Q$  and hence  $P + Q$  is prime, a contradiction.

(2) If  $R$  is a field, then  $\{0\}$  is the only prime ideal of  $R$  and  $R$  is the only nonprime ideal of  $R$ . Moreover,  $\{0\} + R = R$ , which is not a prime ideal. Conversely, suppose that the sum of a prime ideal and a nonprime ideal is nonprime. We claim that  $\{0\}$  is a prime ideal of  $R$ . If not, then let  $P$  be any prime ideal of  $R$ . Then  $P + \{0\} = P$ , which is prime. This contradicts our assumption, and hence  $R$  is a domain. Suppose  $R$  is not a field, and let  $P$  be a nonzero prime ideal of  $R$ . Pick  $p \in P \setminus \{0\}$ . Then  $\langle p^2 \rangle$  is not a prime ideal, yet  $\langle p^2 \rangle + P = P$ , which is prime; this contradiction concludes the proof.  $\square$

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We pause to introduce some notation to which we shall refer throughout the paper for the sake of brevity. Let  $R$  be a commutative unital ring. Define  $\mathcal{NP}(R)$  to be the collection of *proper* nonprime ideals of  $R$  and set  $\mathcal{NP}(R)^1 := \mathcal{NP}(R) \cup \{R\}$ . Now suppose that  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  are sets of ideals of  $R$ . Write  $\mathcal{A} + \mathcal{B} \subseteq \mathcal{C}$  (respectively,  $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{C}$ ) if the sum of an ideal in  $\mathcal{A}$  and an ideal in  $\mathcal{B}$  results in an ideal in  $\mathcal{C}$  (respectively, if the product of an ideal in  $\mathcal{A}$  and an ideal in  $\mathcal{B}$  results in an ideal in  $\mathcal{C}$ ). If the sum (product) of *distinct* ideals of  $\mathcal{A}$  and  $\mathcal{B}$  results in an ideal of  $\mathcal{C}$ , then we denote this by  $\mathcal{A} +_d \mathcal{B} \subseteq \mathcal{C}$  (respectively,  $\mathcal{A} \cdot_d \mathcal{B} \subseteq \mathcal{C}$ ).

The paper is organized as follows. In the next section, we prove some results on rings  $R$  with various additive closure assumptions on  $\mathcal{P}(R)$  and  $\mathcal{NP}(R)$  ( $\mathcal{NP}(R)^1$ ), similar in spirit to Proposition 1 above (yet less trivial). The third section is devoted to studying rings  $R$  for which  $\mathcal{P}(R)$  is closed under ideal multiplication. Finally, we close the article with some open questions and directions for further research. **Throughout this paper, “ring” will always denote a commutative ring with identity  $1 \neq 0$ .** We list [3] and [4] as references for fundamental results in commutative ring theory.

## 2. SOME QUESTIONS INVOLVING ADDITIVE CLOSURE

**2.1. Rings  $R$  for which  $\mathcal{NP}(R) +_d \mathcal{NP}(R) \subseteq \mathcal{P}(R)$ .** Continuing in the spirit of Proposition 1, we consider a more difficult question. Before presenting the first result of this subsection, we recall that a ring  $V$  is a *discrete valuation ring* if  $V$  is a principal ideal domain with a unique nonzero prime ideal  $\mathfrak{m}$ . We now present our next proposition (compare to Proposition 1).

**Proposition 2.** *Let  $R$  be a ring. Then  $\mathcal{NP}(R) +_d \mathcal{NP}(R) \subseteq \mathcal{P}(R)$  if and only if  $R$  is a field,  $R$  is the product of two fields, or  $R \cong V/\langle \mathfrak{m}^2 \rangle$  for some discrete valuation ring  $(V, \mathfrak{m})$ .*

*Proof.* It is straightforward to check that the above rings satisfy the given condition. Conversely, let  $R$  be a ring and suppose that  $\mathcal{NP}(R) +_d \mathcal{NP}(R) \subseteq \mathcal{P}(R)$  (that is, the sum of two distinct, proper nonprime ideals of  $R$  is a prime ideal). For brevity, let us say that such a ring  $R$  has property  $(\dagger)$ . We consider two cases.

Case 1.  $R$  is an integral domain. We claim that  $R$  is a field in this case. Suppose not. Then there is some nonzero, nonunit  $d \in R$ . Observe that the ideals  $\langle d^2 \rangle$  and  $\langle d^3 \rangle$  are proper, distinct, nonprime ideals of  $R$ . Moreover,  $\langle d^2 \rangle + \langle d^3 \rangle = \langle d^2 \rangle$ , which is not prime. This contradicts property  $(\dagger)$ , and we are done with this case.

Case 2.  $R$  is not an integral domain. We claim that

$$(2.1) \quad \text{every proper, nonzero ideal of } R \text{ is prime.}$$

Suppose not, and let  $I$  be a proper, nonzero ideal of  $R$  which is nonprime. Since  $R$  is not a domain,  $\{0\}$  is a (proper) nonprime ideal of  $R$ . Since  $R$  has property  $(\dagger)$ ,  $\{0\} + I = I$  is prime, a contradiction. This proves (2.1).

Next, let  $I$  be an arbitrary proper, nonzero ideal of  $R$ . Then  $I$  is prime, and thus  $R/I$  is an integral domain. It is clear that  $R/I$  inherits property  $(\dagger)$  from  $R$ . By our work in Case 1 above,  $R/I$  is a field. But then  $I$  is a maximal ideal of  $R$ . We have shown that

(2.2) every proper, nonzero ideal of  $R$  is maximal.

Such an ideal  $I$  cannot properly contain a nonzero ideal  $J$  of  $R$ , since this would contradict the maximality of  $J$ . It follows that every proper, nonzero ideal of  $R$  contains only the zero ideal as a proper subideal. Therefore  $R$  is an Artinian ring. Thus

$$(2.3) \quad R = \prod_{i=1}^n R_i \text{ for some Artinian local rings } R_1, \dots, R_n.$$

We claim that  $n \leq 2$ . Suppose that  $n \geq 3$ . Then  $I := \{0\} \times \{0\} \times R_3 \times \cdots \times R_n$  is a proper, nonzero ideal of  $R$  which is not prime. This contradicts (2.1) above, and hence  $n \leq 2$ , as desired. We now treat the cases  $n = 1$  and  $n = 2$ .

Suppose first that  $n = 1$ . Then  $R = R_1$ , and hence  $R$  is an Artinian local ring. It follows from (2.1) that  $P$  is the only proper nonzero ideal of  $R$ . But then  $R$  is a local Artinian principal ideal ring, thus the homomorphic image of a discrete valuation ring  $(V, \mathfrak{m})$  (this is a consequence of Cohen's structure theorems for complete local rings; see [5], Theorem 3.3). Because  $R$  has but three ideals, it follows that  $R \cong V/\langle \mathfrak{m}^2 \rangle$ .

Now suppose that  $n = 2$ . Then  $R = R_1 \times R_2$ . We will show that  $R_1$  and  $R_2$  are fields. Let's suppose that  $R_1$  is not a field. Note that  $R$  is not an integral domain, and thus the zero ideal  $\{0\}$  of  $R$  is nonprime and proper. Also, the ideal  $I := \{0\} \times R_2$  is nonzero and proper, and since  $R/I \cong R_1$  is not a domain (since  $R_1$  is Artinian and not a field),  $I$  is not a prime ideal of  $R$ , and we have a contradiction to (2.1). Since  $R$  has property  $(\dagger)$ ,  $\{0\} + I = I$  is prime, and we have a contradiction. A symmetric argument shows that  $R_2$  is also a field, and this completes the proof.  $\square$

**2.2. Rings  $R$  for which  $\mathcal{NP}(R) + \mathcal{P}(R) \subseteq \mathcal{P}(R)$ .** It turns out that this class properly contains the class studied in the previous subsection. As such, it may not be a surprise that the proof of our next proposition is a bit more intricate.

**Proposition 3.** *Let  $R$  be a ring. Then  $\mathcal{NP}(R) + \mathcal{P}(R) \subseteq \mathcal{P}(R)$  if and only if  $R$  is a zero-dimensional local ring or  $R$  is a product of two fields.*

*Proof.* For brevity, let us say that a ring satisfying the above containment has property  $(\circ)$ . It is easy to see that a zero-dimensional local ring and the product of two fields both have property  $(\circ)$ .

To prove the converse, we establish two preliminary results.

(2.4) If  $S$  is a domain in which every proper ideal of  $S$  is prime, then  $S$  is a field.

Suppose that  $S$  is not a field. Then there is some nonzero, nonunit  $d \in S$ . Therefore,  $\langle d^2 \rangle$  is a proper ideal of  $S$ , hence is prime. Thus  $d \in \langle d^2 \rangle$ , and because  $S$  is a domain,  $d$  is a unit. This contradiction verifies (2.4).

(2.5) If every proper, nonzero ideal of a ring  $T$  is prime, then  $T$  has at most two prime ideals.

To see this, suppose that  $I$  is a proper, nonzero ideal of  $T$ . Then by assumption,  $I$  is prime. Thus  $T/I$  is a domain with the property that every proper ideal of  $T/I$  is prime. By (2.4),  $T/I$  is a field. But then  $I$  is a maximal ideal of  $T$ . This shows that every proper, nonzero ideal of  $T$  is maximal, and hence (2.2) holds for  $T$ . We may invoke the proof of Proposition 2 to conclude that either  $T$  is an Artinian local ring or  $T \cong T_1 \times T_2$  for some Artinian local rings  $T_1$  and  $T_2$ . In any case,  $T$  has at most two prime ideals.

We are now ready to classify those rings which satisfy property  $(\circ)$ , that is, we determine the rings  $R$  satisfying  $\mathcal{NP}(R) + \mathcal{P}(R) \subseteq \mathcal{P}(R)$ . Let  $R$  be such a ring. We first claim that

(2.6) every prime ideal of  $R$  is maximal.

To see this, let  $J$  be a prime ideal of  $R$ . Then note that  $R/J := D$  also has property  $(\circ)$ . Now,  $\{0\}$  is a prime ideal of  $D$ . Because  $D$  has property  $(\circ)$ , it follows that  $D$  does not possess a proper, nonprime ideal. In other words, every proper ideal of  $D$  is prime. By (2.4),  $D$  is a field. It follows that  $J$  is maximal, completing the proof of (2.6).

Next, let  $\text{rad}(R)$  denote the Jacobson radical of  $R$ . Our next claim is that

(2.7) every proper, nonprime ideal of  $R$  is contained in  $\text{rad}(R)$ .

Indeed, let  $I$  be a proper, nonprime ideal of  $R$  and suppose by way of contradiction that  $I \not\subseteq \text{rad}(R)$ . Then there is some  $x \in I$  and maximal ideal  $M$  of  $R$  such that  $x \notin M$ . It follows that  $M + I = R$ . However, since  $M$  is maximal,  $M$  is also prime. Further,  $I$  is proper and nonprime. As  $R$  has property  $(\circ)$ ,  $M + I$  is prime, contradicting that  $M + I = R$ , and (2.7) is established.

Now, let  $S := R/\text{rad}(R)$ . Then it follows immediately from (2.7) that every proper, nonzero ideal of  $S$  is prime. Invoking (2.5), we conclude that  $S$  has at most two prime ideals. Observe that by (2.6),  $\text{rad}(R)$  coincides with  $\text{nil}(R)$ , the nilradical of  $R$ . But then for every prime ideal  $P$  of  $R$ ,  $\text{rad}(R) \subseteq P$ . We deduce that since  $S$  has at most two prime ideals, also

(2.8)  $R$  has at most two prime ideals.

If  $R$  has but one prime ideal, then  $R$  is a zero-dimensional local ring, and we are done. Thus suppose that  $R$  has two prime ideals  $P_1$  and  $P_2$ . We claim that

(2.9)  $P_1$  and  $P_2$  are principal.

It suffices to prove that  $P_1$  is principal. By (2.6),  $P_1$  and  $P_2$  are maximal, so  $P_1 \not\subseteq P_2$ . Let  $x \in P_1 \setminus P_2$ . We claim that  $\langle x \rangle$  is a prime ideal of  $R$ . Suppose not. Because  $R$  has property  $(\circ)$ ,  $P_2 + \langle x \rangle$  is a prime ideal of  $R$ , thus either  $P_2 + \langle x \rangle = P_2$  or  $P_2 + \langle x \rangle = P_1$ . The former is impossible, lest  $x \in P_2$ , and the latter is impossible since then  $P_2 \subsetneq P_1$ , contradicting the maximality of  $P_2$ . We have shown that  $\langle x \rangle$  is a prime ideal of  $R$ . Thus  $\langle x \rangle = P_1$  or  $\langle x \rangle = P_2$ . Since  $x \notin P_2$ , we deduce that  $\langle x \rangle = P_1$ , and  $P_1$  is principal, as claimed.

Applying Cohen's Theorem, we deduce that  $R$  is a principal ideal ring, thus Noetherian. By (2.6),  $R$  is zero-dimensional, and thus  $R$  is Artinian. Because  $R$  has exactly two prime ideals, it

follows from the structure theorem for Artinian rings that  $R = R_1 \times R_2$  for some Artinian local rings  $R_1$  and  $R_2$ . We will be done if we can show that  $R_1$  and  $R_2$  are fields. By symmetry, it suffices to prove that  $R_1$  is a field. Suppose not. Let  $M_2$  be the maximal (prime) ideal of  $R_2$  and  $M_1$  be the maximal (prime) ideal of  $R_1$ . Observe that  $R_1 + M_2$  is a maximal, thus prime, ideal of  $R$  and  $\{0\} + R_2$  is a nonprime ideal of  $R$  since  $R/(\{0\} + R_2) \cong R_1$  is not a field, hence also not a domain (since  $R_1$  is Artinian). However,  $R$  has property  $(\circ)$ , and thus  $(R_1 + M_2) + (\{0\} + R_2) = R$  is a prime ideal of  $R$ , which is absurd. This concludes the proof.  $\square$

**2.3. Rings  $R$  for which  $\mathcal{NP}(R)^1 + \mathcal{NP}(R)^1 \subseteq \mathcal{NP}(R)^1$ .** We conclude this section by studying rings  $R$  for which the sum of two nonprime ideals is again nonprime (we remind the reader that  $R$  is included in the collection  $\mathcal{NP}(R)^1$ ). This class of rings is quite broad. Indeed, this class contains all valuation domains and, more generally, all chained rings. We do not have a complete characterization of this class of rings, but we describe them completely within the class of Noetherian domains. First, recall that an integral domain  $D$  is a *unique factorization domain* (UFD) if every nonzero nonunit of  $D$  can be written uniquely (up to order and associates) as a finite product of prime elements of  $D$  (recall that an element  $p \in D$  is *prime* if  $p$  is a nonzero nonunit and whenever  $x, y \in D$  with  $p|xy$ , then either  $p|x$  or  $p|y$ ).

**Proposition 4.** *Let  $D$  be a Noetherian domain which is not a field. Then  $D$  is a discrete valuation ring (DVR) if and only if  $\mathcal{NP}(D)^1 + \mathcal{NP}(D)^1 \subseteq \mathcal{NP}(D)^1$ .*

*Proof.* Let  $D$  be an arbitrary Noetherian domain which is not a field. If  $D$  is a valuation ring, the result is clear since the sum of any two ideals in a chained ring is simply the max of the two (relative to set-theoretic containment). Suppose now that the collection of nonprime ideals of  $D$  is closed under addition. We shall prove that  $D$  is a DVR. Let  $P$  be a nonzero prime ideal of  $D$  (such a  $P$  exists since  $D$  is not a field). Since  $D$  is Noetherian, (as is well-known)  $D$  satisfies the descending chain condition on prime ideals (that is, if  $S$  is a nonempty set of prime ideals of  $D$ , then there is some  $Q \in S$  such that there is no  $M \in S$  such that  $M \subsetneq Q$ ). Therefore, there exists a nonzero prime ideal  $Q \subseteq P$  such that there is no prime ideal  $Q'$  such that  $\{0\} \subsetneq Q' \subsetneq Q$ . Since  $D$  is Noetherian,  $Q$  is finitely generated. Let  $n$  be the least positive integer such that  $Q$  can be generated by  $n$  elements. We claim that  $n = 1$ . Suppose by way of contradiction that  $n > 1$ , and let  $q_1, \dots, q_n$  generate  $Q$ . By minimality of  $Q$  and  $n$ , it follows that  $Q = \langle q_1 \rangle + \langle q_2, \dots, q_n \rangle$ , yet neither  $\langle q_1 \rangle$  nor  $\langle q_2, \dots, q_n \rangle$  are prime ideals, and this contradicts the assumption on  $D$ . We conclude that  $Q = \langle q \rangle$  for some  $q \in D$ . Since  $Q$  is a prime ideal, we deduce that  $q$  is a prime element of  $D$ . Since  $P$  was an arbitrary nonzero prime ideal of  $D$ , it follows that every nonzero prime ideal of  $D$  contains a prime element of  $D$ . As is well-known, this implies that  $D$  is a unique factorization domain (this is an exercise in many algebra texts; it can be found, for example, in Robert Gilmer's *Multiplicative Ideal Theory* ([3])). In particular,

(2.10) every nonzero nonunit of  $D$  can be written as a finite product of prime elements of  $D$ .

Next, we claim that

(2.11)  $D$  has a unique maximal ideal.

Suppose not, and let  $J_1$  and  $J_2$  be distinct maximal ideals of  $D$ . Then  $J_1 + J_2 = D$ , and thus

(2.12)  $j_1 + j_2 = 1$  for some  $j_1 \in J_1$  and  $j_2 \in J_2$ .

Since  $J_1$  and  $J_2$  are proper ideals, it follows that both  $j_1$  and  $j_2$  are nonzero nonunits. Now let  $\langle p \rangle$  be a minimal nonzero prime ideal of  $D$  (such exists by our work above). Then observe from (2.12) that  $\langle p \rangle = \langle j_1 p \rangle + \langle j_2 p \rangle$ . But then  $\langle j_1 p \rangle$  and  $\langle j_2 p \rangle$  are proper, nonzero subideals of  $\langle p \rangle$ ; by minimality of  $\langle p \rangle$ , we deduce that  $\langle j_1 p \rangle$  and  $\langle j_2 p \rangle$  are not prime ideals, and again, we have a contradiction to our hypothesis.

Now let  $J$  be the unique maximal ideal of  $D$ . We claim that

(2.13)  $J$  is a principal ideal.

Suppose not. By our work above,  $J$  contains a prime element  $p$  of  $D$ . Then  $\langle p \rangle$  is a prime ideal of  $D$  which is not maximal (since  $J$  is assumed to be nonprincipal). Thus  $D/\langle p \rangle$  is an integral domain which is not a field, and also inherits the same property as  $D$ , that is, the sum of two nonprime ideals of  $D/\langle p \rangle$  is again nonprime. By our work above,  $D/\langle p \rangle$  possesses a minimal principal nonzero prime ideal. It follows that there is some  $q \in D$ ,  $q \notin \langle p \rangle$  such that  $\langle p, q \rangle$  is a prime ideal of  $D$ . Observe that  $\langle p, q \rangle = \langle p^2, q \rangle + \langle q^2, p \rangle$ . We will have a contradiction if we can show that  $\langle p^2, q \rangle$  and  $\langle q^2, p \rangle$  are nonprime ideals. By symmetry, it suffices to show that  $\langle p^2, q \rangle$  is not prime. Suppose by way of contradiction that  $\langle p^2, q \rangle$  is prime. Then  $p \in \langle p^2, q \rangle$ . Thus

(2.14)  $p = p^2 x + qy$  for some  $x, y \in D$ .

But then clearly  $p|qy$ . Because  $q \notin \langle p \rangle$ ,  $p \nmid q$ . As  $p$  is prime,  $p \mid y$ . Thus  $y = pz$  for some  $z \in D$ . Therefore, (2.14) becomes  $p = p^2 x + pqz$ . Since  $D$  is a domain and  $p \neq 0$ , we may cancel  $p$  to get  $1 = px + qz$ . But  $\langle p \rangle$  and  $\langle q \rangle$  are proper ideals of  $D$ , and thus they are both contained in the unique maximal ideal  $J$ . But then  $1 \in J$ , and this is a contradiction. We have now verified (2.13).

Let  $J = \langle p \rangle$ , and let  $q$  be an arbitrary prime element of  $D$ . Then  $q \in J$ , and so  $q = pd$  for some  $d \in D$ . Thus  $q$  divides  $pd$  and since  $q$  is prime, either  $q|p$  or  $q|d$ . The latter is impossible since  $p$  is not a unit, and so  $q|p$ . Thus  $p = qt$  for some  $t \in D$ . Because  $q = pd$ , we deduce that  $d$  is a unit. We have shown that the only prime elements of  $D$  are of the form  $up$  for some unit  $u \in D$ . Because  $D$  is a UFD, we conclude that every nonzero nonunit of  $D$  has the form  $up^n$  for some unit  $u$  of  $D$  and some positive integer  $n$ . It is now clear that  $D$  is a principal ideal domain with unique nonzero prime ideal  $J$ , hence a DVR.  $\square$

**Remark 1.** Note that the definition we gave of “prime element” preceding Proposition 4 can be applied in the more general setting of commutative rings. Moreover, if one assumes in Proposition 4 only that  $D$  is a Noetherian ring for which every prime element is regular (that is, not a zero divisor),

then the same conclusion goes through. However, since every chained ring<sup>1</sup> has the property that the sum of two nonprime ideals is nonprime, and because such rings (Noetherian, even) exist which are not domains, we cannot completely dispense with the assumption that every prime element is regular.

### 3. WHEN IS $\text{Spec}(R)$ MULTIPLICATIVELY CLOSED?

There are several analogous questions one can study relative to multiplicative closure properties of  $\mathcal{P}(R)$  and  $\mathcal{NP}(R)$ . Yet again, some are very easy to settle. We present two such examples below.

**Example 1.** *Let  $R$  be a ring. Then  $(\mathcal{NP}(R) \cdot \mathcal{NP}(R)) \cap \mathcal{P}(R)$  is empty.<sup>2</sup>*

*Proof.* Suppose by way of contradiction that there exists a ring  $R$  and proper nonprime ideals  $I$  and  $J$  of  $R$  such that  $IJ$  is prime. Then either  $I \subseteq IJ$  or  $J \subseteq IJ$ . Because  $IJ \subseteq I \cap J$ , it follows that  $IJ = I$  or  $IJ = J$ . But then either  $I$  or  $J$  is prime, a contradiction.  $\square$

**Example 2.** *Let  $R$  be a ring. Then  $\mathcal{NP}(R) \cdot \mathcal{P}(R) \subseteq \mathcal{P}(R)$  iff  $R$  is a field.*

*Proof.* If  $R$  is a field, then  $R$  has no proper nonprime ideals, and so the containment vacuously holds. Conversely, suppose that  $R$  is a ring which satisfies the above containment. We claim that every proper ideal of  $R$  is prime. Suppose not, and let  $I$  be a proper nonprime ideal of  $R$ . Further, let  $J$  be a maximal ideal of  $R$ . Then  $J$  is prime, and thus so is  $IJ$ . It follows that either  $I \subseteq IJ$  or  $J \subseteq IJ$ . If the former holds, then  $I \subseteq IJ \subseteq I$ , and  $IJ = I$ . But  $IJ$  is prime yet  $I$  isn't, and we have a contradiction. It follows that  $J \subseteq IJ \subseteq J$ , and thus  $IJ = J$ . Hence  $J = IJ \subseteq I$ , and so  $J \subseteq I$ . The maximality of  $J$  implies  $I = J$ . But then again,  $I$  is prime, a contradiction. Thus every proper ideal of  $R$  is prime. Since  $\{0\}$  is prime,  $R$  is a domain. To show that  $R$  is a field, it suffices to argue that  $R$  has no nonzero nonunits. Suppose  $r \in R$  is such an element. Then  $\langle r^2 \rangle$  is a proper ideal of  $R$  which is not prime, a contradiction. The proof is complete.  $\square$

In this section, we study the question of when  $\mathcal{P}(R) \cdot \mathcal{P}(R) \subseteq \mathcal{P}(R)$ , that is, of when the prime spectrum of a ring  $R$  is closed under ideal multiplication. For brevity, let us agree to call a ring for which  $\text{Spec}(R)$  is closed under multiplication a *PC ring*. First, we present two examples.

**Example 3.** *Let  $F$  be a field. Then  $\text{Spec}(R)$  consists solely of the zero ideal. Since the product of the zero ideal with itself is the zero ideal, we see that  $\text{Spec}(R)$  is closed under multiplication, and so  $R$  is a PC ring.*

**Example 4.** *Let  $F$  be a field, and consider the monoid ring  $D := F[X^r : r \in [0, \infty)]$  over the real interval  $[0, \infty)$  with coefficients in  $F$  (the reader unfamiliar with monoid rings may view elements of  $D$  as polynomials with coefficients in  $F$  except with positive integer powers of  $X$  replaced by positive real powers of  $X$ ). With this notation, the ideal  $J$  of  $D$  defined by  $J := \langle X^r : r > 0 \rangle$  is a maximal ideal of  $D$ . It is not hard to check that the local ring  $D_J$  has a unique nonzero prime ideal (namely,  $J_J$ ) and that, as a result of the Archimedean property of the usual order  $<$  on  $R$ ,  $J_J$  is idempotent. It follows that  $D_J$  is a PC domain.*

<sup>1</sup>That is, the set of ideals of  $R$  is linearly order by set inclusion.

<sup>2</sup>Here,  $\mathcal{NP}(R) \cdot \mathcal{NP}(R)$  denotes the set  $\{IJ : I, J \in \mathcal{NP}(R)\}$ .

We now delve into the elementary theory of PC rings. We begin with a lemma.

**Lemma 1.** *Let  $R$  be a ring, and suppose that  $\text{Spec}(R)$  is closed under multiplication. Then for any prime ideals  $P$  and  $Q$  of  $R$ , either  $PQ = P$  or  $PQ = Q$ .*

*Proof.* Let  $P$  and  $Q$  be arbitrary prime ideals of the PC ring  $R$ . Then  $PQ$  is prime. Trivially  $PQ \subseteq PQ$ , and since  $PQ$  is prime, we deduce that either  $P \subseteq PQ$  or  $Q \subseteq PQ$ . In the former case, we have  $P \subseteq PQ \subseteq P$ , and thus  $P = PQ$ . The latter case is analogous.  $\square$

We now establish the following proposition, which gives a collection of further necessary conditions on a ring  $R$  in order for  $R$  to be PC.

**Proposition 5.** *Let  $R$  be a ring for which  $\text{Spec}(R)$  is closed under multiplication. Then the following hold:*

- (1)  $\text{Spec}(R)$  is linearly ordered by  $\subseteq$ ,
- (2) if  $P$  and  $Q$  are prime ideals of  $R$  such that  $P \subseteq Q$ , then  $PQ = P$ ,
- (3)  $R$  is a local ring, and
- (4) all nonzero prime ideals of  $R$  are infinitely generated.

*Proof.* Suppose that  $R$  is a ring for which  $\text{Spec}(R)$  is closed under multiplication.

(1) Let  $P, Q \in \text{Spec}(R)$  be arbitrary. We may assume by Lemma 1 that  $PQ = P$ . But then  $P = PQ \subseteq Q$ , and  $P \subseteq Q$ .

(2) Assume  $P, Q \in \text{Spec}(R)$  and  $P \subseteq Q$ . Assume by way of contradiction that  $PQ \neq P$ . Applying Lemma 1,  $PQ = Q$ . But now we have  $PQ \subseteq P \subsetneq Q$ . We deduce that  $PQ \subsetneq Q$ , contradicting  $PQ = Q$ .

(3) Let  $J_1$  and  $J_2$  be maximal ideals of  $R$ . Then both  $J_1$  and  $J_2$  are prime. By (1), we may suppose that  $J_1 \subseteq J_2$ . By maximality of  $J_1$ , we conclude that  $J_1 = J_2$ .

(4) Let  $J = J(R)$  be the unique maximal ideal of  $R$ , and suppose by way of contradiction that  $P$  is a nonzero, finitely generated prime ideal of  $R$ . By (2),  $JP = P$ . Nakayama's Lemma implies that  $P = \{0\}$ , a contradiction.  $\square$

A natural question is whether one can give a list of necessary and sufficient conditions in order for  $\text{Spec}(R)$  to be closed under multiplication. Indeed, this is not difficult to do and is a simple corollary of Proposition 5.

**Corollary 1.** *Let  $R$  be a ring. Then  $\text{Spec}(R)$  is closed under multiplication if and only if  $\text{Spec}(R)$  is linearly ordered by set inclusion and every prime ideal of  $R$  is idempotent.*

*Proof.* If  $\text{Spec}(R)$  is closed under multiplication, then it is immediate from (1) and (2) of Proposition 5 that  $\text{Spec}(R)$  is linearly ordered and every prime ideal is idempotent. Conversely, suppose that  $\text{Spec}(R)$  is linearly ordered by inclusion and every prime ideal of  $R$  is idempotent. Let  $P$  and  $Q$  be arbitrary prime ideals of  $R$ . We may suppose without loss of generality that  $P \subseteq Q$ . Of course,  $PQ \subseteq P$ . Conversely, since  $P$  is idempotent and  $P \subseteq Q$ , we see that  $P = P^2 \subseteq PQ$ . Thus  $PQ = P$ , and thus  $PQ$  is a prime ideal.  $\square$



**Remark 2.** Let  $R$  be an arbitrary PC ring. Then observe that  $\text{Spec}(R)^1 := \text{Spec}(R) \cup \{R\}$  is a commutative monoid via ideal multiplication. Recall that *Green's relation*  $\leq$  is defined on a commutative monoid  $M$  by  $x \leq y$  if and only if  $y|x$ . It follows from our work above that the subset relation on  $\text{Spec}(R)^1$  is the same as Green's relation on  $\text{Spec}(R)^1$ . Thus Green's relation on  $\text{Spec}(R)^1$  is a total order and, moreover, completely determines the monoid multiplication: for any  $P, Q \in \text{Spec}(R)^1$ ,  $PQ = \min(P, Q)$ .

We shall soon present another corollary of Proposition 5. First, we recall that a domain  $D$  is an *almost Dedekind domain* if the localization  $D_J$  is either a field or a discrete valuation ring, for every maximal ideal  $J$  of  $D$ . It is well-known that there exist UFDs and almost Dedekind domains which are not Noetherian.

Our next result gives some equivalent conditions for a PC ring to be a field.

**Corollary 2.** *Suppose that  $R$  is a PC ring. Then the following are equivalent:*

- (1)  $R$  is a field,
- (2)  $R$  is Noetherian,
- (3)  $R$  is a unique factorization domain, and
- (4)  $R$  is an almost Dedekind domain.

*Proof.* Let  $R$  be a PC ring. It is clear that (1) implies (2)–(4); we show that each of (2)–(4) implies (1). Suppose that  $R$  is Noetherian, and let  $J$  be (by (3) of Proposition 5) the maximal ideal of  $R$ . Then  $J$  is a prime ideal. Invoking (4) of Proposition 5, it follows that  $J = \{0\}$ . Thus  $R$  is a field, as claimed. Next, assume that  $R$  is a UFD. To show that  $R$  is a field, it suffices to prove that  $R$  has no prime elements. Suppose by way of contradiction that  $p \in R$  is prime. Then  $\langle p \rangle$  is a nonzero finitely generated prime ideal, which contradicts (4) of Proposition 5. Finally, suppose that  $R$  is an almost Dedekind domain. Because  $R$  is local, we conclude that  $R_J = R$ , and thus  $R$  is a discrete valuation ring. Hence  $R$  is Noetherian, and by our work above,  $R$  is a field. The proof is complete.  $\square$

Next, recall that a domain  $D$  is a *Prüfer domain* if every nonzero, finitely generated ideal  $I$  of  $D$  is invertible (that is, there is some ideal  $J$  of  $D$  such that  $IJ$  is a nonzero principal ideal). There are some 40 well-known characterizations of these rings, which gives testament to their status in the field of multiplicative ideal theory (see [3] for further details). Equivalently,  $D$  is a Prüfer domain if  $D$  is *locally a valuation ring (domain)*, that is, a domain  $D$  for which the ideals of  $D_J$  are linearly ordered by set inclusion for every maximal ideal  $J$  of  $D$ . As every PC ring is local, it follows that a Prüfer PC domain is a valuation ring. Given this fact along with Example 4, it is reasonable to study PC rings in the context of valuation rings to get a foothold on their structure. We will soon prove that there exist PC valuation rings of arbitrary Krull dimension. First, we recall some standard results in the theory of ordered groups for the convenience of the reader; these results will play an important role shortly.

Let  $(G, +, \leq)$  be a totally ordered abelian group. The *nonnegative cone* of  $G$ , which we denote by  $G_+$ , is defined by  $G_+ := \{g \in G : g \geq 0\}$ . A nonempty subset  $F \subseteq G_+$  is called a *filter* provided  $0 \notin F$  and  $F$  is *closed upwards*, that is, if  $x \in F$  and  $x \leq y$ , then  $y \in F$ . If in addition,  $G_+ \setminus F$  is

closed under addition, then  $F$  is called a *prime filter*. Next, a subgroup  $C$  of  $G$  is called *convex* provided that whenever  $c_1 \leq c_2$  are members of  $C$  and  $g \in G$  satisfies  $c_1 \leq g \leq c_2$ , then also  $g \in C$ . It is well-known that the map  $C \mapsto G_+ \setminus C_+$  gives a one-to-one correspondence between the proper convex subgroups of  $G$  and the prime filters of  $G$  (see the discussion on pp. 196–198 of [3]). Now, let  $\kappa$  be a cardinal number and for each  $i < \kappa$ , suppose that  $(G_i, \leq_i)$  is a totally ordered abelian group. Then the group  $G := \bigoplus_{i < \kappa} G_i$  is also totally ordered as follows: for a nonzero  $(g_i) \in G$ , let  $j$  be least such that  $g_j \neq 0$ . We say that  $(g_i)$  is *positive* provided  $g_j >_j 0$ .<sup>3</sup> Letting  $P$  be the set of positive elements of  $G$ , one checks easily that  $P$  is closed under addition and that  $\{P, \{0\}, -P\}$  partitions  $G$ . Thus the order  $\leq$  on  $G$  defined by  $f \leq g$  if and only if  $f = g$  or  $g - f \in P$  gives a translation-invariant total order on  $G$ , called the *lexicographic order* (relative to the orders on each  $G_i$ ) on  $G$ . Finally, the *rank* of a totally ordered abelian group is the order type of the collection of proper convex subgroups, which is totally ordered by set inclusion. We shall make use of the following elementary result (whose proof is straightforward).

**Lemma 2** ([3], p. 221, Exercise 6). *Suppose that  $\kappa > 0$  is a cardinal and that for each  $i < \kappa$ ,  $(G_i, \leq_i)$  is a rank one totally ordered abelian group (in other words, each  $G_i$  is a nontrivial subgroup of  $(\mathbb{Q}, +)$ , up to isomorphism). Next, set  $G := \bigoplus_{i < \kappa} G_i$  and equip  $G$  with the lexicographic order defined above. Set  $C_0 := \{0\}$  and for  $0 < i < \kappa$ , set  $C_i := \{f \in G : f(j) = 0 \text{ for all } j < i\}$ . Then the  $C_i$ s are exactly the proper convex subgroups of  $G$ .*

We are now equipped to construct PC rings of arbitrary (classical) Krull dimension. Rudimentary knowledge of Krull dimension and valuation theory is assumed.

**Proposition 6.** *For every cardinal number  $\kappa$ , there exists a PC valuation ring  $V$  of Krull dimension  $\kappa$ .*

*Proof.* If  $\kappa = 0$ , simply take  $V$  to be any field. Now suppose that  $\kappa > 0$ , and consider the direct sum  $G := \bigoplus_{i < \kappa} \mathbb{Q}$  of  $\kappa$  many copies of  $\mathbb{Q}$  equipped with the lexicographic order. As  $(\mathbb{Q}, +, \leq)$  (here  $\leq$  denotes the usual order on  $\mathbb{Q}$ ) is Archimedean, it is clear that  $\{0\}$  is the only proper convex subgroup of  $\mathbb{Q}$ , and therefore  $(\mathbb{Q}, +, \leq)$  has rank one. By Lemma 2 and the paragraph preceding it, we conclude that the prime filters of  $G$  are precisely

$$(3.1) \quad F_0 := G_+ \setminus \{0\} \text{ and for } 0 < i < \kappa, F_i := \{f \in G_+ : f(j) \neq 0 \text{ for some } j < i\}.$$

Next, by the Jaffard-Ohm-Kaplansky Theorem, there is a field  $K$  and a valuation  $v$  on  $K$  with value group  $G$ ; let  $V$  be the associated valuation ring. As is well-known, the nonzero prime ideals of  $V$  are in one-to-one correspondence with the prime filters of  $G$  via the map  $P \mapsto v(P \setminus \{0\})$ . Recall above that there are exactly  $\kappa$  prime filters of  $G$ , thus also exactly  $\kappa$  many nonzero prime ideals of  $V$ . Because the prime ideals of  $V$  are linearly ordered by inclusion, it follows that  $V$  has Krull dimension  $\kappa$ . It remains to show that  $V$  is PC. Toward this end, by Corollary 1, it suffices to prove that every prime ideal of  $V$  is idempotent. This is immediate if  $P$  is the zero ideal, so suppose that  $P$  is a nonzero prime ideal of  $V$ . We simply must prove that  $P \subseteq P^2$  as the reverse containment is

<sup>3</sup>Formally, a sequence  $(g_i) \in G$  is a function  $f$  with domain  $\kappa$  such that  $f(i) \in G_i$  for all  $i < \kappa$ .

obvious. Let  $x \in P$  be arbitrary. If  $x = 0$ , then  $x \in P^2$  and we are done. Thus suppose that  $x \neq 0$ . It follows from our comments above that  $v(P \setminus \{0\}) = F_i$  for some  $i < \kappa$ . Hence  $v(x) \in F_i$ . Since  $G$  is torsion-free divisible, it is clear that each  $F_i$  is also divisible (that is, if  $f \in F_i$  and  $n$  is a positive integer, then  $\frac{f}{n} \in F_i$ ). Therefore,  $\frac{v(x)}{2} \in F_i$ . We deduce that there is some  $y \in P \setminus \{0\}$  such that  $v(y) = \frac{v(x)}{2}$ . But then  $v(x) = 2v(y) = v(y^2)$ ; thus  $x$  and  $y^2$  are associates in  $V$ . Because  $y \in P$ , of course  $y^2 \in P^2$ . As  $x$  and  $y^2$  are associates,  $x \in P^2$ , as claimed.  $\square$

In light of Example 4 and Proposition 6, another canonical question arises: is every PC ring a valuation ring? We conclude this section with an example showing that the answer is “no”.

**Example 5.** *There exists a PC integral domain which is not a valuation ring.*

*Proof.* Let  $K$  be a field, and let  $F := K(Y)$  be the field of rational functions in  $Y$  with coefficients in  $K$ . As in Example 4, consider the monoid ring  $T := F[X^r : r \in [0, \infty)]$  and let  $J := \langle X^r : r > 0 \rangle$ . Then  $J$  is an idempotent maximal ideal of  $T$ . As stated in Example 4,  $T_J$  is a local ring with unique nonzero prime ideal  $J_J$ ; it follows that  $T_J$  has Krull dimension one. We now obtain the pullback  $R := K + J$  defined by the following conductor square:

$$\begin{array}{ccc} R := K + J & \xrightarrow{i} & T \\ \downarrow & & \downarrow \\ K & \xrightarrow{i} & F \end{array}$$

We may now localize this square at  $J$  to obtain the conductor square

$$\begin{array}{ccc} R_J = K + J_J & \xrightarrow{i} & T_J \\ \downarrow & & \downarrow \\ K & \xrightarrow{i} & F. \end{array}$$

Since  $F$  is not an overring of  $K$ , the pullback  $R_J$  is not a valuation ring (see Theorem 4.7 of [1]). Because  $T_J$  is one-dimensional, it follows from [2], Section 2, Corollary 2 that  $\dim(R_J) = \dim(K) + \dim(T_J) = 0 + 1 = 1$ . As  $J$  is a height-one idempotent maximal ideal of  $T$ ,  $J_J$  is the unique nonzero prime ideal of  $R_J$ , and  $J_J$  is idempotent. We deduce that  $R_J$  is PC but not a valuation ring.  $\square$

#### 4. OPEN QUESTIONS

We conclude this note with the following questions which we feel are interesting:

**Question 1.** Investigate more deeply the structure of PC rings. In light of Proposition 6 and Example 5, how “close” is a PC domain to being a valuation domain?

**Question 2.** If  $D$  is a domain for which the set of nonprime ideals of  $D$  is closed under addition, is  $D$  necessarily a valuation ring? More generally, is a ring  $R$  for which the set of nonprime ideals of  $R$  is closed under addition necessarily chained?

Finally, we remark that given the results presented in this paper, yet another natural problem is to study the rings  $R$  for which  $\text{Spec}(R)$  is closed under addition. Recall that if  $R$  is a PC ring, then the prime ideals of  $R$  are linearly ordered by inclusion. Therefore,  $\text{Spec}(R)$  is closed under addition. On the other hand, if  $V$  is a DVR, then  $\text{Spec}(V)$  is closed under addition but not multiplication. Thus the class of rings  $R$  for which  $\text{Spec}(R)$  is closed under addition properly contains the class of rings  $R$  for which  $\text{Spec}(R)$  is closed under multiplication. Our final question is the following:

**Question 3.** What is the structure of rings  $R$  for which  $\text{Spec}(R)$  is closed under addition?

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