

COMMUTATIVE RINGS WITH INFINITELY MANY MAXIMAL SUBRINGS

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ABSTRACT. It is shown that $RgMax(R)$ is infinite for certain commutative rings, where $RgMax(R)$ denotes the set of all maximal subrings of a ring R . It is observed that whenever R is a ring and D is a UFD subring of R , then $|RgMax(R)| \geq |Irr(D) \cap U(R)|$, where $Irr(D)$ is the set of all non-associate irreducible elements of D and $U(R)$ is the set of all units of R . It is shown that every ring R is either Hilbert or $|RgMax(R)| \geq \aleph_0$. It is proved that if R is a zero dimensional (or semilocal) ring with $|RgMax(R)| < \aleph_0$, then R has nonzero characteristic, say n , and R is integral over \mathbb{Z}_n . In particular, it is shown that if R is an uncountable artinian ring, then $|RgMax(R)| \geq |R|$. It is observed that if R is a noetherian ring with $|R| > 2^{\aleph_0}$, then $|RgMax(R)| \geq 2^{\aleph_0}$. We determine exactly when a direct product of rings has only finitely many maximal subrings. In particular, it is proved that if a semisimple ring R has only finitely many maximal subrings, then every descending chain $\cdots \subset R_2 \subset R_1 \subset R_0 = R$ where each R_i is a maximal subring of R_{i-1} , $i \geq 1$, is finite and the last terms of all these chains (possibly with different lengths) are isomorphic to a fixed ring, say S , which is unique (up to isomorphism) with respect to the property that R is finitely generated as an S -module.

INTRODUCTION

All rings in this paper are commutative rings with $1 \neq 0$. All subrings, ring extensions, homomorphisms and modules are unital. A proper subring S of a ring R is called a maximal subring if S is maximal with respect to inclusion in the set of all proper subrings of R . A ring with maximal subrings is called a submaximal ring in [2], [4], and [7]. The main aim in [1-7] is to determine rings R (or find conditions on R) such that $RgMax(R) \neq \emptyset$. In this paper, we study rings R for which $RgMax(R)$ is infinite.

Now let us first review some results from the literature about finiteness conditions on the set of all subrings of a ring. In [23], Rosenfeld proved that a (possibly noncommutative) ring with identity with only finitely many subrings (not necessarily unital) is finite. Bell and Gilmer have given elementary proofs of this result; see [8] and [14], respectively. Recently, Dobbs et. al, studied commutative unital rings with only finitely many unital subrings. They characterized such rings first in [11] for singly generated unital rings and later in [12] for general commutative rings. Korobkov characterized finite rings with exactly two maximal subrings, see [19]. In [6], it is observed that if $R = \prod_{i \in I} R_i$, where I is infinite and each R_i is a ring, then $|RgMax(R)| \geq 2^{|I|}$, and also if R is a noetherian integral domain with $|R| > 2^{\aleph_0}$, then $|Max(R)| \leq |RgMax(R)|$. Finally in [3], fields with only finitely many maximal subrings are completely characterized. In this article we are interested in showing that $RgMax(R)$ is infinite for certain commutative rings R . In [12], a ring is called to have the finite subring property (*FSP*) if it has only finitely many (unital) subrings. It is clear that when $RgMax(R)$ is infinite for a ring R , then R does not have *FSP*. In this paper we also study some connections between the existence of maximal subrings of a ring R (or the infinitude of $RgMax(R)$) and the existence of an infinite chain $\cdots \subseteq R_2 \subseteq R_1 \subseteq R_0 = R$, where each R_i is a maximal subring of R_{i-1} , for $i \geq 1$. In [3], it is shown that a field E has only finitely many maximal subrings if and only if E has no infinite chains in the previous form. See also [10], [16], [17], [22] and [3, Introduction] for more results about chain conditions on the set of subrings, intermediate rings and overrings. We refer the reader to [4, Introduction], for the notable role that $RgMax(E)$, for a field E , plays in algebraic geometry.

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Next, let us recall some standard definitions and notation from commutative ring theory which will be used throughout the paper (see [18]). An integral domain D is called G -domain if the quotient field of D is finitely generated as a D -algebra. A prime ideal P of a ring R is called G -ideal if R/P is a G -domain. A ring R is called Hilbert if every G -ideal of R is maximal. As usual, let $Char(R)$, $U(R)$, $N(R)$, $J(R)$, $Max(R)$, $Spec(R)$ and $Min(R)$, denote the characteristic, the set of all units, the nil radical ideal, the Jacobson radical ideal, the set of all maximal ideals, the set of all prime ideals and the set of all minimal prime ideals of a ring R , respectively. We also call a ring R , not necessarily noetherian, semilocal (resp. local) if $Max(R)$ is finite (resp. $|Max(R)| = 1$). For any ring R , let $Z = \mathbb{Z} \cdot 1_R = \{n \cdot 1_R \mid n \in \mathbb{Z}\}$, be the prime subring of R . We denote the finite field with p^n elements, where p is prime and $n \in \mathbb{N}$, by \mathbb{F}_{p^n} . If D is an integral domain, then we denote the set of all non-associate irreducible elements of D by $Irr(D)$. Also, we denote the set of all natural prime numbers by \mathbb{P} . Suppose that $D \subseteq R$ is an extension of domains. By Zorn's Lemma, there exists a maximal (with respect to inclusion) subset X of R which is algebraically independent over D . By maximality, R is algebraic over $D[X]$ (thus every integral domain is algebraic over a UFD; this can be seen by taking D to be the prime subring of R). If E and F are the quotient fields of D and R , respectively, then X can be shown to be a transcendence basis for F/E (that is, X is maximal with respect to the property of being algebraically independent over E). The transcendence degree of F over E is the cardinality of a transcendence basis for F/E (it can be shown that any two transcendence bases have the same cardinality). We denote the transcendence degree of F over E by $tr.deg(F/E)$.

Now, let us sketch a brief outline of this paper. In Section 1, we prove that whenever R is a ring and D is a UFD subring of R , then $|RgMax(R)| \geq |Irr(D) \cap U(R)|$. We show that if R is a ring and $x \in J(R)$ is not algebraic over the prime subring of R , then $RgMax(R)$ is infinite. We observe that every ring R is either Hilbert or $RgMax(R)$ is infinite. In particular, we show that if R is a reduced ring with $J(R) \neq 0$, then $RgMax(R)$ is infinite. We prove that if R is an integral domain with $|U(R)| \geq 2^{\aleph_0}$, then $|RgMax(R)| \geq |U(R)|$. We show that if R is an uncountable Dedekind domain with $|Max(R)| < |R|$, then $|RgMax(R)| \geq |R|$. We observe that if R is a reduced ring with $\max\{|R|, |Spec(R)|\} > 2^{\aleph_0}$ and $|Max(R)| \leq \aleph_0$, then $RgMax(R)$ is infinite. Finally in Section 1, we prove that the polynomials rings always have infinitely many maximal subrings.

In Section 2, we study the infinitude of $RgMax(R)$ for zero dimensional rings, semilocal rings, artinian rings and noetherian rings. We show that if a zero dimensional ring R has only finitely many maximal subrings, then R has nonzero characteristic, say n , and R is integral over \mathbb{Z}_n . We show that if R is a semilocal ring with $|RgMax(R)| < \aleph_0$, then R is a zero dimensional ring. Moreover, we observe that if R is an uncountable artinian ring, then $|RgMax(R)| \geq |R|$. We show that if R is noetherian ring with $|R| > 2^{\aleph_0}$, then $|RgMax(R)| \geq 2^{\aleph_0}$. We prove that if R is a ring with $|Max(R)| > 2^{\aleph_0}$, then $|Max(R)| \leq |RgMax(R)|$. Consequently, we show that if R is a ring with $|R/J(R)| > 2^{2^{\aleph_0}}$ or $|R| > \max\{2^{2^{\aleph_0}}, |U(R)|\}$ (resp. $|R| > \max\{2^{2^{\aleph_0}}, |N(R)|\}$), then $|RgMax(R)| \geq 2^{\aleph_0}$ (resp. $|RgMax(R)| \geq \aleph_0$). We show that if $|Spec(R)| > 2^{2^{\aleph_0}}$ for a ring R , then $|RgMax(R)| \geq \aleph_0$. We prove that if R is a reduced ring with $|R| > 2^{2^{\aleph_0}}$, then $|RgMax(R)| \geq \aleph_0$. Finally in Section 2, we show that if R is a finite dimensional noetherian ring, then either $|RgMax(R)| \geq \aleph_0$ or $|Spec(R)| \leq 2^{\aleph_0}$.

In Section 3, we study the structure of maximal subrings of finite direct product of rings and the finiteness of the set of all maximal subrings of direct products. We determine exactly the structure of maximal subrings of $K \times K$, where K is a field. In fact we show that if R is a maximal subring of $K \times K$, then either $R = S \times K$ or $K \times S$, where $S \in RgMax(K)$, or $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where $\sigma_i \in Aut(K)$ (the automorphism group of the field K) for $i = 1, 2$. In particular, $|RgMax(K \times K)| \geq 2|RgMax(K)| + |Aut(K)|$. Moreover, we show that if K is a field, then $K \times K$ has only finitely many maximal subrings if and only if K is a finite field. Consequently, we determine exactly when a direct product of rings has only finitely many maximal subrings. In particular, we characterize semilocal reduced rings with only finitely many maximal subrings. Moreover, we show that if E_1, \dots, E_n are fields, $n \in \mathbb{N}$, and $R = E_1 \times \dots \times E_n$ has only finitely many maximal subrings, then every descending chain $\dots \subset R_2 \subset R_1 \subset R_0 = R$ is finite, where each R_i is a maximal subring of R_{i-1} for every $i > 0$. The converse holds if for each $i \neq j$: If E_i and E_j are infinite, then $E_i \not\cong E_j$. Finally, we observe that although the lengths of the descending chains in the previous result for R need not be the same, the last terms in these chains are isomorphic to a fixed ring, say S ; furthermore, if R' is a non submaximal subring of R such that R is finitely generated as an R' -module, then $R' \cong S$ (in other words, up to isomorphism, S is the unique non submaximal subring of

R for which R is finitely generated as an S -module). Conversely, we prove that if R is a semilocal reduced ring which contains a non submaximal subring S over which R is finitely generated as an S -module, then R is a finite direct product of fields which have only finitely many maximal subbrings.

1. INTEGRAL DOMAINS WITH INFINITELY MANY MAXIMAL SUBBRINGS

The main aim in this section is to prove that for any ring R , either R has infinitely many maximal subbrings or R is a Hilbert ring. We need some observations about transferring existence of maximal subbrings in the ring extensions. In [1, Theorem 2.5], it is proved that a ring R is submaximal if and only if there exist a proper subring S of R and $x \in R \setminus S$ such that $S[x] = R$; moreover in this case, in fact R has a maximal subring T such that $S \subseteq T$ and $x \notin T$. The next result is in [5, Proposition 1.1] and for a more general result, see [6, Proposition 2.1]. We give its proof for the sake of the reader.

Proposition 1.1. *Let $F \subseteq K$ be an algebraic field extension and S be a maximal subring of F which is not a field. Then for any non-unit $x \in S \setminus \{0\}$, there exists a maximal subring R_x of K such that x is not a unit in R_x , $R_x \cap F = S$, $R_x[x^{-1}] = K$, and R_x is not a field, but it contains the integral closure of S in K .*

Proof. First note that $S[x^{-1}] = F$, since S is a maximal subring of F and $x^{-1} \in F \setminus S$. Let R be the integral closure of S in K . We claim $x^{-1} \notin R$ and $R[x^{-1}] = K$. Since S is not a field and is a maximal subring of F we infer that S is integrally closed in F and therefore $x^{-1} \notin R$ (note, $x^{-1} \in F$ and $S[x^{-1}] = F$). It remains to show that $R[x^{-1}] = K$. Clearly $R[x^{-1}] \subseteq K$; on the other hand assume that $u \in K$. Since K is algebraic over F , it follows that there exist a natural number n and a_0, a_1, \dots, a_{n-1} in F such that $u^n + a_{n-1}u^{n-1} + \dots + a_1u + a_0 = 0$. Since $F = S[x^{-1}]$ and $x \in S$, we conclude that there exists a non-negative integer m such that $a_i = x^{-m}s_i$ ($0 \leq i \leq n-1$), for some $s_i \in S$. Hence $u^n + x^{-m}s_{n-1}u^{n-1} + \dots + x^{-m}s_1u + x^{-m}s_0 = 0$. Multiplying the latter equation by x^{mn} , we obtain:

$$(x^m u)^n + s_{n-1}(x^m u)^{n-1} + \dots + s_1 x^{m(n-2)}(x^m u) + s_0 x^{m(n-1)} = 0.$$

Thus $x^m u$ is integral over S . Hence $x^m u \in R$, and therefore $u \in R[x^{-1}]$. Thus by the comments from the first paragraph of this section, we infer that K has a maximal subring R_x which contains R but not x^{-1} . Finally note that $S \subseteq R_x \cap F \subsetneq F$, since $x^{-1} \in F \setminus R_x$. Therefore $S = R_x \cap F$, by the maximality of S . \square

The following result which is the converse of [6, Proposition 2.1] is needed.

Proposition 1.2. [7, Theorem 2.19]. *Let $R \subseteq T$ be rings. If there exists a maximal subring V of T such that V is integrally closed in T and $U(R) \not\subseteq V$, then R is submaximal.*

Proof. We first claim:

$$(1) \quad \text{If } x \in U(R) \setminus V, \text{ then } x^{-1} \in V.$$

To prove this, assume $x \in U(R) \setminus V$. Then

$$(2) \quad x^{-1} \in R \subseteq V[x] \text{ (since } V \text{ is a maximal subring of } T \text{ and } x \in T \setminus V).$$

Thus $x^{-1} = v_0 + v_1x + \dots + v_nx^n$ for some $v_0, v_1, \dots, v_n \in V$. Multiplying the previous equation by x^{-n} , we see that x^{-1} is integral over V . But since $x^{-1} \in T$ and V is integrally closed in T , we conclude that $x^{-1} \in V$, and (1) above is established. We now complete the proof.

Since $U(R) \not\subseteq V$, clearly also $R \not\subseteq V$, whence $R \cap V$ is a proper subring of R . Choose any $x \in U(R) \setminus V$. We will prove that $R = (R \cap V)[x]$. The comments from the first paragraph of this section then yield that R is submaximal. Since $x \in R$, clearly $(R \cap V)[x] \subseteq R$. Let $r \in R$ be arbitrary. We will prove that $r \in (R \cap V)[x]$. Recall from (2) above that $R \subseteq T = V[x]$. Hence $r = b_0 + b_1x + \dots + b_mx^m$ for some $b_0, b_1, \dots, b_m \in V$. Multiplying this equation through by x^{-m} , we get $rx^{-m} = b_0x^{-m} + b_1x^{1-m} + \dots + b_{m-1}x^{-1} + b_m$. Since $r \in R$ and $x^{-1} \in R$, clearly $rx^{-m} \in R$. Recall from (1) above that $x^{-1} \in V$. This fact along with the previous equation implies that $rx^{-m} \in V$. But then $rx^{-m} \in R \cap V$, and we deduce that $r \in (R \cap V)[x]$, as required. \square

Next, we prove the following main result which is also needed in the sequel.

Theorem 1.3. *Let R be a ring and D be a subring of R which is a UFD. Then $|RgMax(R)| \geq |Irr(D) \cap U(R)|$.*

Proof. Let $D \subseteq R$ be an extension of rings with D a UFD. First, we claim that we can assume R is a domain without loss of generality. Indeed, suppose we have proven the theorem for domains. We will show that the theorem holds for R . Toward this end, let \mathcal{I} denote the collection of ideals of R which intersect D trivially. Then \mathcal{I} has maximal elements by Zorn's Lemma, and it is easy to show that any such maximal element is a prime ideal of R ; let Q be such a prime ideal. Since $D \cap Q = \{0\}$, D embeds naturally into R/Q , whence R/Q contains a copy of D . Thus

$$|RgMax(R)| \geq |RgMax(R/Q)| \geq |Irr(D) \cap U(R/Q)| \geq |Irr(D) \cap U(R)|.$$

The final inequality holds because every unit of R remains a unit module Q (and recall that $D \cap Q = \{0\}$). Thus we may assume that R is an integral domain. Next we claim that we may suppose that R is algebraic over D . For otherwise, let X be a transcendence basis for R over D . Then R is algebraic over $D[X]$. Now note that $D[X]$ is a UFD and clearly $Irr(D) \subseteq Irr(D[X])$. Thus we can replace D by $D[X]$ in this case. Hence assume that R is an integral domain which is algebraic over D . Now suppose that K and E are the quotient fields of D and R , respectively. Thus E/K is an algebraic extension, since R is algebraic over D . Now, note that for any $p \in Irr(D) \cap U(R)$, the field K has a maximal subring V_p such that $\frac{1}{p} \notin V_p$ and $Irr(D) \cap U(V_p) = Irr(D) \setminus \{p\}$ (note that $D_{(p)}$ is a local principal ideal domain, whence a DVR. It is well-known that the only proper overring of a DVR is its quotient field. Thus we can put $V_p = D_{(p)}$). Hence E has a maximal subring W_p such that $W_p \cap K = V_p$, by Proposition 1.1. Therefore $\frac{1}{p} \notin W_p$. Thus we have $U(R) \not\subseteq W_p$ which by the proof of the previous proposition implies that $(R \cap W_p)[\frac{1}{p}] = R$. Hence by the comment preceding Proposition 1.1, we infer that R has a maximal subring R_p such that $R \cap W_p \subseteq R_p$ and $\frac{1}{p} \notin R_p$. Since $(Irr(D) \cap U(R)) \setminus \{p\} \subseteq U(R_p)$ and $p \notin U(R_p)$, we conclude that $R_p \neq R_q$ for $p \neq q$ in $Irr(D) \cap U(R)$. Hence we are done. \square

Corollary 1.4. *If R is a ring with zero characteristic, then $|\mathbb{P} \cap U(R)| \leq |RgMax(R)|$. Moreover, if R has only finitely many maximal subrings, then $\mathcal{C} := \{Char(\frac{R}{M}) \mid M \in Max(R)\} \subseteq \mathbb{P}$ and $\mathbb{P} \setminus \mathcal{C}$ is finite.*

Proof. Let R be a ring of characteristic 0. Then $D := \mathbb{Z}$ is (up to isomorphism) the prime subring of R , and we may take $Irr(D)$ to be the set \mathbb{P} of prime numbers. It now follows immediately from the previous theorem that $(*) |\mathbb{P} \cap U(R)| \leq |RgMax(R)|$.

Suppose now that R has but finitely many maximal subrings. We claim that $0 \notin \mathcal{C}$. Indeed, suppose by way of contradiction that R/M has characteristic 0 for some $M \in Max(R)$. Since R has but finitely many maximal subrings, clearly this property is inherited by R/M . But then by $(*)$ we conclude that $|\mathbb{P} \cap U(R/M)|$ is finite. Since R/M is a field of characteristic 0, this is clearly impossible, and we have reached a contradiction. Lastly, we will show that $\mathbb{P} \setminus \mathcal{C}$ is finite. Since R has but finitely many maximal subrings, it follows from $(*)$ that $\mathbb{P} \cap U(R)$ is finite. Thus it suffices to show that $\mathbb{P} \setminus \mathcal{C} \subseteq \mathbb{P} \cap U(R)$. Let $p \in \mathbb{P} \setminus \mathcal{C}$, and let $M \in Max(R)$ be arbitrary. Since $p \neq Char(R/M)$, it follows that $p \notin M$. Since M was arbitrary, we see that p is a unit of R , and the proof is complete. \square

The next result shows that most fields have infinitely many maximal subrings.

Corollary 1.5. *Let $K \subseteq E$ be a field extension and F be the prime subfield of E . Then the following statements hold:*

- (1) *If E has zero characteristic, then $RgMax(E)$ is infinite.*
- (2) *$|RgMax(E)| \geq tr.deg(E/K)$. In particular, if E is uncountable, then $|RgMax(E)| \geq |E|$.*
- (3) *If $tr.deg(E/F) \neq 0$, then $RgMax(E)$ is infinite.*

In particular, if $RgMax(E)$ is finite, then E is algebraic over \mathbb{F}_p , for some prime number p .

Proof. Clearly (1) holds by the previous corollary. For item (2), let X be a transcendence basis for E/K . Clearly $K[X]$ is a UFD and therefore by Theorem 1.3, we have $|X| \leq |RgMax(E)|$. Hence the first part of (2) is true. For the final part of (2), note that $tr.deg(E/F) = |E|$. To prove (3), let $x \in E$ is not algebraic over F . Clearly $F[x]$ is a UFD and $Irr(F[x])$ is an infinite subset of E . Hence we are done by Theorem 1.3. The final assertion is now evident by (1) and (3). \square

We remind the reader that the structure of absolutely algebraic fields with only finitely many maximal subrings is completely determined in [3]. Now we have the following result.

Proposition 1.6. *Let R be a ring, and suppose that $x \in J(R)$ is not algebraic over the prime subring Z of R . Then $RgMax(R)$ is infinite.*

Proof. We prove the proposition in two cases. First assume that the characteristic of R is zero or a prime number p . Hence clearly $Z[x]$ is a UFD subring of R . Now, we have $1 - xZ[x] \subseteq U(R)$ and since $1 - xZ[x]$ contains infinity many non-associate irreducible elements of $Z[x]$ (note, if $\text{Char}(R) = p$, it is clear that $1 - x\mathbb{Z}_p[x]$ contains infinitely many non-associate irreducible elements; if $\text{Char}(R) = 0$, then note that for each prime number p , the polynomial $1 + x + \cdots + x^{p-1} \in 1 - x\mathbb{Z}[x]$ is irreducible), we infer that $\text{RgMax}(R)$ is infinite, by Theorem 1.3. Hence we are done in this case. Now assume that the characteristic of R is nonzero, say n , which is not prime. Thus $Z = \mathbb{Z}_n$ is (up to isomorphism) the prime subring of R . Suppose now that p is a prime divisor of n . Hence $P = \frac{p\mathbb{Z}}{n\mathbb{Z}}[x]$ is a minimal prime ideal of $\mathbb{Z}_n[x]$ (note, $\dim \mathbb{Z}_n[x] = 1$ and P is a non maximal prime ideal of $\mathbb{Z}_n[x]$). Hence by [18, Ex.1, P. 41], we conclude that there exists a (minimal) prime ideal Q of R such that $Q \cap \mathbb{Z}_n[x] = P$. Thus

$$(3) \quad \mathbb{Z}_p[x] \cong \frac{\mathbb{Z}_n[x]}{P} \cong \frac{\mathbb{Z}_n[x] + Q}{Q} \subseteq \frac{R}{Q},$$

hence R/Q is an integral domain which satisfies the assumption of the proposition and therefore by the first part of the proof we infer that $\text{RgMax}(R/Q)$ is infinite. Hence $\text{RgMax}(R)$ is infinite, as required. \square

We need the following lemma for the next observations.

Lemma 1.7. *Let R be a ring with zero characteristic. If $\text{Char}(R/J(R)) \neq 0$, then $|\text{RgMax}(R)| \geq \aleph_0$.*

Proof. Assume that $n = \text{Char}(R/J(R))$. Hence $1 - n\mathbb{Z} \subseteq U(R)$. Thus we are done by Corollary 1.4. \square

The next proposition will get us closer to our main result that we promised in the beginning of this section.

Proposition 1.8. *Let R be an integral domain with $J(R) \neq \{0\}$ (that is, R is not Jacobson semisimple). Then $\text{RgMax}(R)$ is infinite.*

Proof. If R has nonzero characteristic or $\text{Char}(R/J(R)) = 0$, then one can easily see that every nonzero element of $J(R)$ is not algebraic over the prime subring of R (note, if $0 \neq x \in J(R)$ and $a_n x^n + \cdots + a_1 x + a_0 = 0$, where $n \in \mathbb{N}$, $a_i \in Z$ and $a_0 \neq 0$, then we infer that $a_0 \in J(R)$ which is absurd). Hence we are done by Proposition 1.6. Thus assume that $\text{Char}(R) = 0$ but $\text{Char}(R/J(R)) \neq 0$. Hence we are done by Lemma 1.7. \square

We now establish several corollaries.

Corollary 1.9. *Let R be a ring. Then either R is a Hilbert ring or $\text{RgMax}(R)$ is infinite.*

Proof. If R is not Hilbert, then by [18, (c) of Ex. 9, P. 20], there exists a prime ideal P of R which is not an intersection of maximal ideals of R , that is $J(R/P) \neq 0$. Hence by the previous proposition $\text{RgMax}(R/P)$ and therefore $\text{RgMax}(R)$ is infinite. \square

Corollary 1.10. *Let R be a reduced ring with $J(R) \neq 0$. Then $\text{RgMax}(R)$ is infinite.*

Proof. If $\text{RgMax}(R)$ is finite, then by the previous corollary we infer that R must be a Hilbert ring. Thus by [18, (c) of Ex. 9, P. 20], we conclude that every prime ideal of R is an intersection of maximal ideals. Therefore $N(R)$ is an intersection of maximal ideals, which immediately implies that $J(R) = N(R) = 0$, which is absurd. Hence $\text{RgMax}(R)$ is infinite, and the proof is complete. \square

For the next corollary we need some observations. First note that one can easily see that by [18, Ex. 9, P. 20], the quotient ring of a Hilbert ring is Hilbert too. Also, by [18, Theorem 31], a ring R is Hilbert if and only if the polynomial ring $R[x]$ is a Hilbert ring. Thus we conclude that any finitely generated algebra over a Hilbert ring is Hilbert. In particular, if R is a Hilbert ring and R is a maximal subring of a ring T , then T is Hilbert (note, $T = R[t]$ for each $t \in T \setminus R$. Also we refer the reader to see [4, Proposition 2.18] for more results).

Corollary 1.11. *Let R be a non Hilbert ring. Then there exists an infinite chain $\cdots \subset R_2 \subset R_1 \subset R_0 = R$, where each R_i is a maximal subring of R_{i-1} .*

Proof. Since R is not Hilbert, we infer that R has a maximal subring R_1 , by Corollary 1.9. Now note that since R is not Hilbert, we conclude by the preceding comments that R_1 is not Hilbert too. Hence by induction we are done. \square

Corollary 1.12. *Let E be a field which either is not algebraic over its prime subfield or has zero characteristic. Then there exists an infinite chain $\cdots \subset R_2 \subset R_1 \subset R_0 = E$, where each R_i is a non-field G -domain maximal subring of R_{i-1} for every $i > 0$.*

Proof. Note that by Proposition 1.1 or the proof of Theorem 1.3, one can easily see that E has a maximal subring R_1 which is not a field, see also [5, Corollaries 1.2 and 1.3]. Now since there are no rings properly between R_1 and E , and E is the quotient field of R_1 , we infer that R_1 is a rank-one valuation domain (see [18, Ex. 29, P. 43]), whence R_1 is certainly not Hilbert. Thus by the previous corollary the infinite saturated chain exists. Finally, note that by [4, Remark 2.17], for each $i \in \mathbb{N}$, R_i is a non-field G -domain. Hence we are done. \square

The next proposition is need for subsequent observations.

Proposition 1.13. *Let R be an integral domain with quotient field K and F be the prime subfield of K . Then $|RgMax(R)| \geq tr.deg(F(U(R))/F)$. In particular,*

- (1) *If R is an integral domain with $|U(R)| > \aleph_0$, then $|RgMax(R)| \geq |U(R)|$.*
- (2) *If R is an uncountable integral domain with $J(R) \neq 0$, then $|RgMax(R)| \geq |R|$.*

Proof. Let $\alpha = tr.deg(F(U(R))/F)$. Note that $U(R)$ contains a transcendence basis X for $F(U(R))/F$ with $|X| = \alpha$. Thus by Theorem 1.3, we have $|X| \leq |Irr(Z[X]) \cap U(R)| \leq |RgMax(R)|$. Next, we prove items (1) and (2). For (1), note that since $|U(R)|$ is uncountable we infer that $tr.deg(F(U(R))/F) = |U(R)|$ (note, F is countable), hence we are done by the previous part. For (2), note that $|R| = |J(R)| \leq |U(R)|$, and therefore we are done by part (1). \square

Corollary 1.14. *Let R be an uncountable Dedekind domain. Suppose further that $|Max(R)| < |R|$. Then $|RgMax(R)| \geq |R|$.*

Proof. First note that if R is a field then we are done by (1) of the previous proposition or (2) of Corollary 1.5. Hence assume that R is an uncountable Dedekind domain which is not a field and that $|Max(R)| < |R|$. We first claim that $|U(R)| = |R|$. Suppose by way of contradiction that $|U(R)| < |R|$. Let $\mathcal{P} = \{(x_i) : i < \kappa\}$ be an enumeration of the nonzero principal ideals of R . We claim that $\kappa = |R|$. Suppose not, and set $X := \{\alpha x_i : \alpha \in U(R), i < \kappa\}$. Then clearly $|X| \leq |U(R)| \times \kappa < |R|$. Choose any nonzero $r \in R \setminus X$. Then $(r) \neq (x_i)$ for any i , and this is a contradiction since, of course, $(r) \in \mathcal{P}$. Thus $|R| = \kappa$ (that is, there are $|R|$ -many nonzero principal ideals of R). But since R is Dedekind, every proper nonzero ideal of R is a finite product of maximal ideals. As $|Max(R)| < |R|$ and R is uncountable, we deduce that R has fewer than $|R|$ ideals, and this is a contradiction. We conclude that $|U(R)| = |R|$. The result now follows from (1) of Proposition 1.13. \square

Proposition 1.15. *Let R be a reduced ring with $|Max(R)| \leq \aleph_0$. If $\max\{|R|, |Spec(R)|\} > 2^{\aleph_0}$, then $|RgMax(R)| \geq \aleph_0$.*

Proof. Assume that R has only finitely many maximal subrings and seek a contradiction. Since $RgMax(R)$ is finite, then we infer that $J(R) = 0$ by Corollary 1.10. We also note that for each maximal ideal M of R , the field $\frac{R}{M}$ is countable by Corollary 1.5. But we have the natural ring embedding $R \hookrightarrow \prod_{M \in Max(R)} \frac{R}{M}$, hence $|R| \leq 2^{\aleph_0}$. Therefore by our assumption we have $|Spec(R)| > 2^{\aleph_0}$. Since R has only finitely many maximal subrings, we deduce that R is a Hilbert ring by Corollary 1.9. Hence every prime ideal is an intersection of maximal ideals. In particular $|Spec(R)| \leq 2^{\aleph_0}$, which is a contradiction and we are done. \square

Finally, we conclude this section with the following result.

Theorem 1.16. *Let K be a field. Then $|RgMax(K[x])| \geq \aleph_0 |K|$. In particular, for any ring R , the ring $R[x]$ has infinitely many maximal subrings.*

Proof. We have two cases, either K is finite or not.

- (1) Assume that K is a finite field. Hence for each natural number $n \geq 2$, it is well-known that there exists an irreducible polynomial $q_n(x) \in K[x]$ of degree n . Since $K[x]/(q_n(x))$ is a vector space over K of dimension $n \geq 2$, we infer that the ring $K[x]/(q_n(x))$ has a maximal subring. Thus $K[x]$ has a maximal subring S_n which contains $(q_n(x))$. Next note that since $(q_n(x)) + (q_m(x)) = K[x]$ for $n \neq m$, we conclude that $S_n \neq S_m$. This proves that $|RgMax(K[x])| \geq \aleph_0$.
- (2) Now assume that K is infinite. Now note that for each $a \in K$, the ring $\frac{K[x]}{((x-a)^2)}$ is a vector space over K of dimension 2. Hence we infer that $\frac{K[x]}{((x-a)^2)}$ has a maximal subring (note, $\frac{K[x]}{((x-a)^2)}$ as a K -algebra is finitely generated and therefore by the comment from the first paragraph of this section we conclude that $\frac{K[x]}{((x-a)^2)}$ is submaximal). That is, $K[x]$ has a maximal subring S_a which contains $((x-a)^2)$. Now note that, whenever $a, b \in K$ and $a \neq b$, then we have $S_a \neq S_b$, since $((x-a)^2) + ((x-b)^2) = K[x]$. Hence $|RgMax(K[x])| \geq |K|$ and we are done.

The final part is evident. □

2. MORE RINGS WITH INFINITELY MANY MAXIMAL SUBRINGS

In this section we investigate the infinitude of the set of maximal subrings for various classes of rings including zero dimensional rings, semilocal rings, artinian rings and noetherian rings. First we have the following result which generalizes Corollaries 1.5 and 1.12.

Proposition 2.1. *Let R be a zero dimensional ring which satisfies at least one of the following conditions:*

- (1) R has zero characteristic.
- (2) R has nonzero characteristic and is not integral over its prime subring.

Then $|RgMax(R)| \geq \aleph_0$. Moreover, there exists an infinite chain $\cdots \subset R_2 \subset R_1 \subset R_0 = R$, where each R_i is a non-zero dimensional maximal subring of R_{i-1} , $i \geq 1$.

Proof. Assume that (1) holds. Then there exists a prime (maximal) ideal M of R such that $M \cap \mathbb{Z} = 0$. Hence R/M is a field with zero characteristic and therefore we are done by Corollary 1.5. Next assume that (2) holds. Then we recall that by [15, Theorem 1.3], R must have a prime (maximal) ideal M , such that $\frac{R}{M}$ is not algebraic over its prime subfield, hence $|RgMax(R)| \geq \aleph_0$, by Corollary 1.5. For the final part note that by Corollary 1.12 for the field R/M , there exists an infinite chain $\cdots \subset R_2/M \subset R_1/M \subset R_0/M$, such that each R_i/M is a non-field maximal subring of R_{i-1}/M , for $i \geq 1$. Thus we conclude that each R_i is a non-zero dimensional maximal subring of R_{i-1} (note, clearly M is a prime ideal of R_i), for $i \geq 1$. □

Corollary 2.2. *Let R be a semilocal ring. Then either $|RgMax(R)| \geq \aleph_0$ or R is a zero dimensional ring with nonzero characteristic which is integral over its prime subring. In particular, every semilocal ring with zero characteristic has infinitely many maximal subrings.*

Proof. If $RgMax(R)$ is finite, then by Corollary 1.9, we infer that R is Hilbert. Hence R must be zero dimensional, since R is semilocal. Thus we are done by the previous proposition. □

In the next proposition we study the infinitude of the set of all maximal subrings in localizations.

Proposition 2.3. *Let R be a ring. Then the following statements hold:*

- (1) *If R is an integral domain with quotient field $E \neq R$. Then $|RgMax(E)| \geq \aleph_0$.*
- (2) *If P is a non-maximal prime ideal of R , then $|RgMax(R_P)| \geq \aleph_0$.*
- (3) *If P is a prime ideal of R , then either $|RgMax(R_P)| \geq \aleph_0$ or $P \in Min(R) \cap Max(R)$.*

Proof. For item (1), note that if R has zero characteristic, then E has zero characteristic and therefore we are done by item (1) of Corollary 1.5. Next, assume that R has nonzero characteristic. Since R is not a field, we infer that E is not algebraic over its prime subfield. Hence we are done by item (3) of Corollary 1.5. For (2), first note that by [18, Ex.1, P. 24], we infer that $\frac{R_P}{P_P} \cong K$, as a ring, where K is the quotient field of the non-field integral domain $\frac{R}{P}$. Hence by item (1) we infer that $\frac{R_P}{P_P}$ and therefore R_P has infinitely many maximal subrings. Finally for item (3), note that if R_P has only finitely many maximal subrings, then by item (2), P is a maximal ideal of R . Now, by Corollary 2.2, we infer that R_P is a zero dimensional ring. Hence P is a minimal prime ideal and hence we are done. □

The proof of the following useful lemma can be found in [5, the proofs of Propositions 1.4 and 2.4]. We give its proof for completeness sake.

Lemma 2.4. *Let R be an uncountable artinian ring. Then R has a maximal ideal M such that $|R/M| = |R|$.*

Proof. First we prove the lemma for local artinian rings. Hence assume that (R, M) is an uncountable local artinian ring and $K = R/M$. We claim that $|K| = |R|$. Since R is artinian we have $M^n = 0$, for some integer n . Now by considering the chain $(0) = M^n \subseteq M^{n-1} \subseteq \cdots \subseteq M^2 \subseteq M$, and the fact that each $V_i = \frac{M^{i-1}}{M^i}$ ($1 \leq i \leq n$) is a finite dimensional vector space over the field K , we infer that for each i , $|V_i| \leq \aleph_0 \times |K|$. Thus we conclude that $|R| \leq |K| \leq |R|$, whence $|K| = |R|$.

Finally, assume that R is an arbitrary uncountable artinian ring. Thus $R \cong \prod_{i=1}^n R_i$, where each R_i is a local artinian ring with a maximal ideal M_i . Since R is infinite we conclude that there exist j , $1 \leq j \leq n$, such that $|R_j| = |R|$. Therefore by the first part of the proof we have $|R_j/M_j| = |R_j| = |R|$. Note that $M = R_1 \times \cdots \times R_{j-1} \times M_j \times R_{j+1} \times \cdots \times R_n$ is a maximal ideal of R and $|R/M| = |R_j/M_j| = |R|$. This concludes the proof. □

Now we are ready to determine properties of artinian rings with only finitely many maximal subrings.

Theorem 2.5. *Let R be an artinian ring. Then the following statements hold:*

- (1) *If R is uncountable, then $|RgMax(R)| \geq |R|$. Further, there exists an infinite chain $\cdots \subset R_2 \subset R_1 \subset R_0 = R$, where each R_i is a non-zero dimensional (and hence non-artinian) maximal subring of R_{i-1} , $i \geq 1$.*
- (2) *If $RgMax(R)$ is finite, then R is countable with nonzero characteristic and R is integral over its prime subring.*

Proof. For (1), first note that by Lemma 2.4, R has a maximal ideal M such that $|\frac{R}{M}| = |R|$. Hence by Corollary 1.5, we have $|RgMax(R)| \geq |RgMax(R/M)| \geq |R/M| = |R|$. Next, by Corollary 1.12, we infer that there exists an infinite chain $\cdots \subset R_2/M \subset R_1/M \subset R_0/M = R/M$ where each R_i/M is a maximal subring of R_{i-1}/M and each R_i/M is a non-field G -domain, for $i \geq 1$. Thus $\cdots \subset R_2 \subset R_1 \subset R_0 = R$ is an infinite chain in which each R_i is a non-zero dimensional maximal subring of R_{i-1} , for $i \geq 1$. Hence we are done. Item (2) is now clear by the first part and Corollary 2.2. \square

By combining the previous theorem and Corollary 2.2, the following is immediate.

Corollary 2.6. *Let R be a semilocal noetherian ring. Then either $|RgMax(R)| \geq \aleph_0$ or R is countable artinian ring.*

Next we are ready to present our first result about the infinitude of $RgMax(R)$ for a noetherian ring R .

Theorem 2.7. *Let R be a noetherian ring with $|R| > 2^{\aleph_0}$. Then $|RgMax(R)| \geq 2^{\aleph_0}$.*

Proof. We give the proof in four steps. **Step 1:** First assume that R is local with unique maximal ideal M . Now by Krull's Intersection Theorem we have $\bigcap_{n=1}^{\infty} M^n = 0$. Hence R is a subdirect product of the family $\{\frac{R}{M^n}\}_{n=1}^{\infty}$. Thus there exists a natural number n such that $\frac{R}{M^n}$ is uncountable, since $|R| > 2^{\aleph_0}$. Therefore we are done by (1) of Theorem 2.5, for the ring $\frac{R}{M^n}$. **Step 2:** Next assume that R is an integral domain. We show that in this case we have $|RgMax(R)| \geq 2^{\aleph_0}|Max(R)|$. For proof, note that for any maximal ideal M of R , we have $\bigcap_{n=1}^{\infty} M^n = 0$, by Krull's Intersection Theorem. Hence we infer that R is a subdirect product of the family $\{\frac{R}{M^n}\}_{n=1}^{\infty}$. Thus there exists a natural number n such that $|\frac{R}{M^n}| > 2^{\aleph_0}$. Thus by the previous step, $\frac{R}{M^n}$ has at least 2^{\aleph_0} maximal subrings. Hence R has at least 2^{\aleph_0} maximal subrings which contain M^n . Since for distinct maximal ideals M and N of R we have $M^r + N^s = R$ ($r, s \in \mathbb{N}$), we infer that if a maximal subring of R contains M^r for some r , then it can not contain N^s , for each natural number s . Hence $|RgMax(R)| \geq 2^{\aleph_0}|Max(R)|$. **Step 3:** Next, assume that R is reduced and $Min(R) = \{P_1, \dots, P_n\}$ (note, R is noetherian hence $Min(R)$ is finite). Since R is reduced we infer that R is a subdirect product of integral domains $\{\frac{R}{P_i}\}_{i=1}^n$. Thus there exists k , $1 \leq k \leq n$, such that $|\frac{R}{P_k}| = |R|$. Hence by the previous step we infer that $|RgMax(R)| \geq |RgMax(R/P_k)| \geq 2^{\aleph_0}$. **Step 4:** Finally, one can easily see that $|\frac{R}{N(R)}| > 2^{\aleph_0}$, see also [6, Lemma 2.8]. Therefore we are done by the previous step. \square

We remind the reader that if \mathcal{F} is the set of all fields, up to isomorphism, which are not submaximal, then $|\mathcal{F}| = 2^{\aleph_0}$, see [4, Corollary 1.15]. Also, in [1, Theorem 2.2], it is proved that for every ring R , the ring $R \times R$ is submaximal. In the following result we see that there are natural connections between the cardinality of $RgMax(R)$ and cardinalities of certain sets which are related to R .

Proposition 2.8. *Let R be a ring, then the following statements hold:*

- (1) *If $|Max(R)| > 2^{\aleph_0}$, then $|Max(R)| \leq |RgMax(R)|$.*
- (2) *Either $|RgMax(R)| \geq \aleph_0$ or $|Spec(R)| \leq 2^{2^{\aleph_0}}$.*
- (3) *If $|\frac{R}{J(R)}| > 2^{2^{\aleph_0}}$, then $|RgMax(R)| \geq 2^{\aleph_0}$.*
- (4) *If $|R| > \max\{2^{2^{\aleph_0}}, |U(R)|\}$, then $|RgMax(R)| \geq 2^{\aleph_0}$.*
- (5) *If $|R| > \max\{2^{2^{\aleph_0}}, |N(R)|\}$, then $|RgMax(R)| \geq \aleph_0$.*

Proof. (1) Let $X = Max(R)$ and $\mathcal{P} = \{X_i\}_{i \in I}$ be a partition of X with $|I| = |X_i| = |X|$ ($i \in I$). Now for each $i \in I$, we have $|X_i| > |\mathcal{F}| = 2^{\aleph_0}$. Hence by the paragraph preceding Proposition 2.8, we infer that there exists $M_i \in X_i$ such that $\frac{R}{M_i}$ is submaximal or there exist two distinct maximal ideals M_i and N_i in X_i such that $\frac{R}{M_i} \cong \frac{R}{N_i}$. Therefore by the paragraph preceding Proposition 2.8, R has a maximal subring which contains $M_i \cap N_i$. Now for each $i \in I$, let S_i be a maximal subring of R which either contains M_i or $M_i \cap N_i$. Since for $i \neq j$ in I , M_i (or $M_i \cap N_i$) and M_j (or $M_j \cap N_j$) are comaximal ideals, we infer that $S_i \neq S_j$. Thus $|RgMax(R)| \geq |I|$ and hence we are done.

(2) If $RgMax(R)$ is finite, then by Corollary 1.9, we infer that R is Hilbert ring. Thus every prime ideal of R is an intersection of maximal ideals (see the proof of Corollary 1.10). Hence $|Spec(R)| \leq 2^{|Max(R)|}$. Next, note that by part (1), we have $|Max(R)| \leq 2^{\aleph_0}$ and hence we are done.

(3) If R has a maximal ideal M such that $|\frac{R}{M}| \geq 2^{\aleph_0}$, then we are done by Corollary 1.5. Thus assume that $|\frac{R}{M}| < 2^{\aleph_0}$ for every $M \in Max(R)$. If $|Max(R)| > 2^{\aleph_0}$, then we are done by (1). Thus we may assume that $|Max(R)| \leq 2^{\aleph_0}$. Now, $\frac{R}{J(R)}$ embeds canonically into $\prod_{M \in Max(R)} R/M$. Thus

$$\left| \frac{R}{J(R)} \right| \leq \left| \prod_{M \in Max(R)} \frac{R}{M} \right| \leq (2^{\aleph_0})^{2^{\aleph_0}} = 2^{2^{\aleph_0}},$$

contradicting our assumption.

(4) By the previous part we may assume that $|\frac{R}{J(R)}| \leq 2^{2^{\aleph_0}}$. Thus $|R| = |J(R)|$ and therefore $|R| = |U(R)|$ (note, for each $x \in J(R)$ we have $1 - x \in U(R)$), which is a contradiction.

(5) If $RgMax(R)$ is finite, then by Corollary 1.9, we infer that R is Hilbert ring. Hence similar to the proof of Corollary 1.10 we conclude that $J(R) = N(R)$, which is impossible by part (3). \square

Corollary 2.9. *Let R be a reduced ring with $|R| > 2^{2^{\aleph_0}}$. Then $|RgMax(R)| \geq \aleph_0$. Moreover, there exists an infinite chain $\cdots \subset R_2 \subset R_1 \subset R_0 = R$, where each R_i is a maximal subring of R_{i-1} .*

Proof. The first part is an immediate consequence of (5) of the previous proposition. Now note that if S is a maximal subring of R , then clearly S is reduced and $|S| = |R| > 2^{2^{\aleph_0}}$. Hence the infinite chain exists by using induction and the first part of the corollary. \square

Corollary 2.10. *Let R be an integral domain with $|R| > 2^{2^{\aleph_0}}$. Then $|RgMax(R)| \geq 2^{\aleph_0}$.*

Proof. If $|R| > \max\{2^{2^{\aleph_0}}, |U(R)|\}$, then we are done by (4) of Proposition 2.8. Hence assume that $|R| = |U(R)|$, therefore we are done by (1) of Proposition 1.13. \square

Let R be a ring, then we denote by $Ht_n(R)$ (resp. $CoHt_n(R)$) the set of all prime ideals of R of height (resp. coheight) n .

Lemma 2.11. *Let R be a noetherian Hilbert ring. Then any prime ideal in $CoHt_n(R)$, $n \geq 0$, is a countable intersection of maximal ideals.*

Proof. First note that if R is a noetherian integral domain, then for any infinite set $\{P_i\}_{i=1}^{\infty} \subseteq Ht_1(R)$, we have $I = \bigcap_{i=1}^{\infty} P_i = 0$. For proof note that, if $I \neq 0$, then $Min(I)$ is infinite which is impossible, since R is noetherian. Now we prove the theorem by induction on $n \geq 1$. Hence assume that $P \in CoHt_1(R)$. Since R/P is an integral domain, we infer that P is a countable intersection of maximal ideals, by the first comment of the proof. Now suppose that the theorem holds for each $P \in CoHt_k(R)$, where $k \leq n - 1$. Assume that $P \in CoHt_n(R)$. Since R/P is an integral domain, by the first comment of the proof we infer that there exists a countable set $\{Q_i\}_{i=1}^{\infty}$ of prime ideals of R such that $P = \bigcap_{i=1}^{\infty} Q_i$ and $ht_{R/P}(Q_i/P) = 1$ (note, $J(R/P) = 0$), see also [18, Theorem 147]. Thus for each $i \geq 1$ we have $Q_i \in CoHt_{k_i}(R)$, for some $k_i \leq n - 1$ (note, $P \in CoHt_n(R)$). Now by induction hypothesis each Q_i is a countable intersection of maximal ideals. Thus P is a countable intersection of maximal ideals too. Hence we are done. \square

Corollary 2.12. *Let R be a finite dimensional noetherian ring. Then either $RgMax(R)$ is infinite or $|Spec(R)| \leq 2^{\aleph_0}$.*

Proof. Assume that $RgMax(R)$ is finite. Hence by (1) of Proposition 2.8, we infer that $|Max(R)| \leq 2^{\aleph_0}$. Next, note that by Corollary 1.9, R must be a Hilbert ring. Thus by the previous lemma, we infer that $|Spec(R)| \leq 2^{\aleph_0}$ and hence we are done. \square

We conclude this section with the following remark.

Remark 2.13. We remind the reader that most results in Sections 1 and 2 can be generalized as follows. In fact in most of them the conclusions are valid for any R -algebra. To see this we prove for example Corollary 1.10 for any R -algebra. Hence assume that R is a reduced ring with $J(R) \neq 0$ and T be an R -algebra. We claim that $|RgMax(T)| \geq \aleph_0$. To see this first note that $T/N(T)$ contains a copy of R , thus we may assume that T is a reduced ring. Now if $J(T) \neq 0$, then we are done by Corollary 1.10. Hence assume that $J(T) = 0$, we show that in this case there exists a maximal ideal M of T such T/M is

not an absolutely algebraic field and therefore we are done by Corollary 1.5. For otherwise, we infer that $R \cap M$ is a maximal ideal of R for each maximal ideal M of T (note, $R/(R \cap M) \cong (R + M)/M \subseteq T/M$). Hence $J(R) \subseteq J(T) \cap R = 0$, which is absurd.

3. DIRECT PRODUCTS AND SEMILOCAL REDUCED RINGS

Finally in this section we determine exactly when a direct product of rings has only finitely many maximal subrings. We first have a closer look to the structure of maximal subrings of a finite direct product of rings. In particular, the structure of maximal subrings of $K_1 \times K_2$, where K_1 and K_2 are fields, are completely determined by $Isom(K_1, K_2)$ (i.e., the set of all field isomorphisms from K_1 into K_2), $RgMax(K_i)$ and $Aut(K_i)$, $i = 1, 2$. Finally, the structure of semilocal reduced rings with only finitely many maximal subrings are characterized.

We need the following result, whose proof could be found in [13], [21] and [9, Theorems 2.3 and 2.4]. We give its proof for the sake of completeness. Before presenting it, let us recall some observations. Let S be a subring of a ring R . Then $(S : R) := \{x \in R \mid Rx \subseteq S\}$ is the largest ideal of R which is contained in S . Also one can easily prove that if I is a common ideal between S and R , then the extension $S \subseteq R$ is integral if and only if $S/I \subseteq R/I$ is integral. Further, if S is a maximal subring of R , then one can easily see that either R is integral over S or S is integrally closed in R . Finally, note that if S is a maximal subring of R , then R is algebraic over S (note, if $x \in R \setminus S$ then either $x^2 \in S$ or $x \in S[x^2]$).

Theorem 3.1. *Let S be a maximal subring of a ring R . Then the following statements are true,*

- (1) $(S : R) \in Spec(S)$.
- (2) $(S : R) \in Max(S)$ if and only if R is integral over S .
- (3) If S is integrally closed in R , then $(S : R) \in Spec(R)$.

Proof. First note that (1) is an immediate consequence of (2) and (3), but we give a direct proof for it. Let $ab \in P := (S : R)$, where $a, b \in S$, but $a \notin P$. Then $Ra + S = R$, by the maximality of S . Multiplying the latter equality by b , we have $Rab + Sb = Rb$. Since $Rab \subseteq P \subseteq S$, we infer that $Rb \subseteq S$, i.e., $b \in P$ and we are done.

Next we prove (2). First assume that R is integral over S . We claim that $P = (S : R)$ is a maximal ideal of S . Indeed, suppose by way of contradiction that I be a proper ideal in S with $P \subsetneq I$. Now by the maximality of S we have either $S = S + IR$ or $R = S + IR$. But the former equality implies $I \subseteq P$ which is absurd, hence $R = S + IR$. The latter equality immediately implies that $I(R/S) = R/S$ as an S -module. Since for each $u \in R \setminus S$ we have $R = S[u]$ and u is integral over S , we infer that R is finitely generated as an S -module. Now by the so-called determinant trick (see [18, Theorem 76]) there exists an element $a \in I$ with $(1 + a)(R/S) = 0$, whence $(1 + a)R \subseteq S$, which means $1 + b \in P \subset I$, and hence $1 \in I$, a contradiction. Thus P is a maximal ideal of S . Conversely, assume that P is a maximal ideal of S . Hence the field S/P is a maximal subring of R/P , which by the preceding comments, immediately implies that R is integral over S .

Finally for (3), assume that S is a maximal subring of R which is integrally closed in R . We claim that P is a prime ideal of R . To see this, assume that $a, b \in R$ and $ab \in P$ but $a \notin P$. We prove that $b \in P$, that is $Rb \subseteq S$. Suppose that $b' \in Rb$, clearly $ab' \in P$. Since $a \notin P$, we infer that $R = S + Ra$, by the maximality of S . Thus $b' = s + r'a$ for some $s \in S$ and $r' \in R$. By multiplying the latter equation by b' , we get $b'^2 = sb' + r'ab'$. Now since $r'ab' \in P \subseteq S$, we conclude that b' is a root of the monic polynomial $x^2 - sx - r'ab' \in S[x]$, that is b' is integral over S and since S is integrally closed in R , we infer that $b' \in S$. Thus $Rb \subseteq S$, i.e., $b \in P$, as required. \square

In the following proposition which is a generalization of [5, Lemma 2.2, Corollary 2.3], we determine the structure of maximal subrings of a finite direct product of rings.

Proposition 3.2. *Let R_1, \dots, R_n be rings and $n \geq 2$ and S be a maximal subring of $R = \prod_{i=1}^n R_i$. Then S satisfies at least one of the following conditions:*

- (1) $S = R_1 \times \dots \times R_{i-1} \times S_i \times R_{i+1} \times \dots \times R_n$, where $S_i \in RgMax(R_i)$.
- (2) There exist $1 \leq i < j \leq n$ and $M_k \in Max(R_k)$ ($k = i, j$) such that $P = R_1 \times \dots \times R_{i-1} \times M_i \times R_{i+1} \times \dots \times R_{j-1} \times M_j \times R_{j+1} \times \dots \times R_n \subseteq S$ and $\frac{R_i}{M_i} \cong \frac{R_j}{M_j} \cong \frac{S}{P}$.

Moreover, if condition (2) holds, then R is integral over S , $P = (S : R)$ is a maximal ideal of S (also note that $P = M \cap N$, for some $M, N \in Max(R)$) and $S \cong S' \times \prod_{k \neq i, j} R_k$, where $S' \in RgMax(R_i \times R_j)$.

Proof. We prove the proposition by induction on n . It is clear that it suffices to prove the proposition only for $n = 2$ (then one can easily complete the proof). Hence assume that $n = 2$, $J_1 = R_1 \times \{0\}$ and $J_2 = \{0\} \times R_2$. Now, if S is a maximal subring of R which contains either J_1 or J_2 , then clearly (1) holds. Hence assume that S does not contain J_1 or J_2 . Thus $J'_1 = J_1 \cap S = S_1 \times \{0\}$ and $J'_2 = J_2 \cap S = \{0\} \times S_2$, where $S_i \subsetneq R_i$. Now we claim that $J'_1 + J'_2 = S_1 \times S_2$ is a proper ideal of S . For otherwise, we have $S = S_1 \times S_2 \subsetneq S_1 \times R_2 \neq R$, which is absurd, since S is a maximal subring of R . Hence $(1, 0), (0, 1) \notin S$ and therefore we infer that R is integral over S , since $(1, 0)^2 = (1, 0)$. Hence by the previous theorem $P = (S : R) = M_1 \times M_2$ is a maximal ideal of S , where each M_i is a proper ideal of R_i , for $i = 1, 2$ (note, $S_i \neq R_i$). Therefore the field $K = \frac{S}{P}$ is a maximal subring of $\frac{R}{P} = \frac{R_1 \times R_2}{M_1 \times M_2} \cong \frac{R_1}{M_1} \times \frac{R_2}{M_2}$. Now by [13, Lemma 1.2], we conclude that $\frac{R_i}{M_i} \cong K$, for $i = 1, 2$. Hence (2) and the final assertions of the proposition hold and we are done. \square

Remark 3.3. Let R_1, \dots, R_n be rings, $n \geq 2$ and $R = \prod_{i=1}^n R_i$. It is clear that if S is a subring of R which satisfies condition (1) of the previous proposition, then S is a maximal subring of R . But, if a subring S of R satisfies in condition (2) of the previous proposition, then it can not be a maximal subring of R . To see this, let K be any field and x be an indeterminate over K . Now, put $R = K(x) \times K(x)$ and $S = \{(t, t) \mid t \in K(x^2)\}$. Then one can easily see that S satisfies condition (2) of the previous proposition, but S is not a maximal subring of R , since S is properly contained in $T = \{(t, t) \mid t \in K(x)\}$ which is a proper subring of R .

Now we are ready to determine exactly the structure of maximal subrings of $K \times K$, where K is a field.

Theorem 3.4. *Let K be a field. Then R is a maximal subring of $K \times K$ if and only if R satisfies at least one of the following conditions:*

- (1) $R = S \times K$ or $R = K \times S$, for some $S \in \text{RgMax}(K)$.
- (2) $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where $\sigma_i \in \text{Aut}(K)$ for $i = 1, 2$.

In particular, $|\text{RgMax}(K \times K)| \geq 2|\text{RgMax}(K)| + |\text{Aut}(K)|$.

Proof. First assume that R satisfies at least one of conditions (1) or (2), we claim that R is a maximal subring of $K \times K$. If R satisfies condition (1), then clearly R is a maximal subring of $K \times K$. Hence assume that $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$, where $\sigma_i \in \text{Aut}(K)$. Clearly, R is a proper subring of $K \times K$ and $R \cong K$, i.e., R is a field. Thus for the maximality of R , it suffices to show that for each $(x, y) \in (K \times K) \setminus R$, we have $T := R[(x, y)] = K \times K$. Since $\sigma_i \in \text{Aut}(K)$, we infer that there exist $a, b \in K$ such that $\sigma_1(a) = x$ and $\sigma_2(b) = y$. Thus $(x, \sigma_2(a)), (\sigma_1(b), y) \in R$ and therefore $A := (0, y - \sigma_2(a)), B := (x - \sigma_1(b), 0) \in T$. Clearly, $x_0 := x - \sigma_1(b) \neq 0$ and $y_0 := y - \sigma_2(a) \neq 0$ (note, $(x, y) \notin R$), hence there exist $a_0, b_0 \in K$ such that $\sigma_1(a_0) = x_0$ and $\sigma_2(b_0) = y_0$. Thus $C := (x_0, \sigma_2(a_0)), D := (\sigma_1(b_0), y_0) \in R \setminus \{0\}$, and since R is a field we infer that $C^{-1}B = (1, 0) \in T$ and $AD^{-1} = (1, 0) \in T$. Now note that for each $(s, t) \in K \times K$, there exist $s_0, t_0 \in K$ such that $(s, \sigma_2(s_0)), (\sigma_1(t_0), t) \in R$. Therefore $(s, t) = (s, \sigma_2(s_0))(1, 0) + (\sigma_1(t_0), t)(0, 1) \in T$, i.e., $T = K \times K$. Hence R is a maximal subring of $K \times K$.

Conversely, assume that R is a maximal subring of $K \times K$ which does not satisfy condition (1). We prove that R satisfies condition (2). Since R does not satisfies condition (1), by the previous proposition we infer that $R \cong K$, and thus R is a field. Let $\sigma : K \rightarrow R$ be a ring isomorphism and $\pi_i : R \rightarrow K$ be the natural projection maps ($i = 1, 2$). Now put $\sigma_i = \pi_i \sigma$, for $i = 1, 2$. It is clear that $R = \{(\sigma_1(x), \sigma_2(x)) \mid x \in K\}$. We claim that $\sigma_i \in \text{Aut}(K)$, for $i = 1, 2$. The fact that R is a field implies that $(1, 0), (0, 1) \notin R$, and since R is a maximal subring of $K \times K$, we conclude that $K \times K = R[(0, 1)] = R + R(0, 1)$ (note, $(0, 1)^2 = (0, 1)$). Now if $y \in K$, then $(y, 0) \in R[(0, 1)] = R + R(0, 1)$, which immediately implies that there exists $x \in K$ such that $\sigma_1(x) = y$. Hence $\sigma_1 \in \text{Aut}(K)$ (note, since R is a field σ_1 is one-one) and similarly $\sigma_2 \in \text{Aut}(K)$. For the final part note that for each $\sigma \in \text{Aut}(K)$, $R_\sigma := \{(x, \sigma(x)) \mid x \in K\}$ is a maximal subring of $K \times K$, and clearly whenever $\sigma \neq \tau$ are in $\text{Aut}(K)$, then $R_\sigma \neq R_\tau$. Thus we are done. \square

The following result is now in order.

Corollary 3.5. *Let K be a field. Then $K \times K$ has only finitely many maximal subrings if and only if K is a finite field.*

Proof. It is clear that if K is a finite field, then $K \times K$ has only finitely many maximal subrings. Conversely, assume that $K \times K$ has only finitely many maximal subrings. Indeed, suppose by way of contradiction that K is infinite. Since $K \times K$ has only finitely many maximal subrings, we infer that K has only finitely many maximal subrings and $\text{Aut}(K)$ is finite, by the previous theorem. Therefore

by Corollary 1.5, we conclude that K is an algebraic extension of \mathbb{Z}_p , for some prime number p . Let $\sigma : K \rightarrow K$ be the Frobenius automorphism defined by $\sigma(x) = x^p$ for all $x \in K$. It is clear that $\sigma \in \text{Aut}(K)$ and σ has infinite order (note, if σ has order k , then $f(x) := x^{p^k} - x = 0$ for all $x \in K$. However, f has degree p^k , whence can have at most p^k roots in K , contradicting that K is infinite). Thus $\text{Aut}(K)$ is an infinite group which is impossible and we have reached a contradiction. Therefore K is finite and the proof is complete. \square

Now we have the following immediate corollaries.

Corollary 3.6. *Let K_1 and K_2 be fields. Then R is a maximal subring of $K_1 \times K_2$ if and only if R satisfies exactly one of the following conditions:*

- (1) $R = S_1 \times K_2$, for some $S_1 \in \text{RgMax}(K_1)$.
- (2) $R = K_1 \times S_2$, for some $S_2 \in \text{RgMax}(K_2)$.
- (3) $R = \{(\sigma_1(x), \tau(\sigma_2(x))) \mid x \in K_1\}$, where $\tau : K_1 \rightarrow K_2$ is a field isomorphism and $\sigma_i \in \text{Aut}(K_i)$, for $i = 1, 2$.

Corollary 3.7. *Let K_1 and K_2 be fields. Then $K_1 \times K_2$ has only finitely many maximal subrings if and only if exactly one of the following conditions holds:*

- (1) $K_1 \not\cong K_2$ and each K_i has only finitely many maximal subrings, for $i = 1, 2$.
- (2) $K_1 \cong K_2$ and K_1 is finite.

Remark 3.8. The only maximal subring of $\mathbb{R} \times \mathbb{R}$ which is a field is $\{(x, x) \mid x \in \mathbb{R}\}$. To see this, note that it is well-known that $\text{Aut}(\mathbb{R}) = \{i\}$, hence we are done by Theorem 3.4.

Now we are ready to determine exactly when a direct product of rings has only finitely many maximal subrings.

Theorem 3.9. *Let $\{R_i\}_{i \in I}$ be a family of rings and $R := \prod_{i \in I} R_i$. Consider the following conditions:*

- (1) $|I| < \infty$.
- (2) Each R_i has only finitely many maximal subrings.
- (3) For each $i \neq j$ in I , if M_k is a maximal ideal of R_k , $k = i, j$, and $R_i/M_i \cong R_j/M_j$, then R_i/M_i is finite.
- (3') For any $i \in I$, if M_i and N_i are distinct maximal ideals of R_i and $R_i/M_i \cong R_i/N_i$, then R_i/M_i is finite.
- (4) For each $i \neq j$ in I , the set $C_{ij} := \{(M, N) \in \text{Max}(R_i) \times \text{Max}(R_j) \mid R_i/M \cong R_j/N\}$ is finite.
- (4') For any $i \in I$, the set $C_i := \{(M, N) \in \text{Max}(R_i) \times \text{Max}(R_i) \mid R_i/M \cong R_i/N, M \neq N\}$ is finite.

If R has only finitely many maximal subrings, then all of the above conditions hold. Conversely, if conditions (1), (2), (3), and (4) hold, then R has only finitely many maximal subrings.

Proof. Suppose first that R has only finitely many subrings. To prove (1), we suppose by way of contradiction that I is infinite. Then by [6, Remark 3.18], we infer that $|\text{RgMax}(R)| \geq 2^{|I|}$, which is absurd. Hence I is finite and we may assume that $I = \{1, \dots, n\}$. As for (2), if some R_i has infinitely many subrings, then by Remark 3.3, the same is true of R , and this is a contradiction. We now prove (3). So suppose that $i \neq j$, M_i is a maximal ideal of R_i and M_j is a maximal ideal of R_j . Note first that $R_i/M_i \times R_j/M_j$ has only finitely maximal subrings, lest $R_i \times R_j$ has infinitely many maximal subrings, which would imply that R has infinitely many maximal subrings, a contradiction. The conclusion now follows from Corollary 3.7. Condition (3') is proved analogously via The Chinese Remainder Theorem. Now for (4). Suppose by way of contradiction that there exist $i \neq j$ such that C_{ij} is infinite. We now consider cases.

Case 1: There exists M such that $(M, N) \in C_{ij}$ for infinitely many N . Then we may assume $\{(M, N_1), (M, N_2), (M, N_3), \dots\} \subseteq C_{ij}$, where $N_a \neq N_b$ for $a \neq b$. By definition, we see that $R_j/N_1 \cong R_j/N_2 \cong R_j/N_3 \dots$. For every positive integer n , let $T_n := R_j/(N_{2n-1} \cap N_{2n})$. Then note that $T_n \cong R_j/N_{2n-1} \times R_j/N_{2n} \cong R_j/N_1 \times R_j/N_1$. By [1, Theorem 2.2], it follows that T_n is submaximal. Thus for every positive integer n , there exists a maximal subring S_n of R_j which contains $N_{2n-1} \cap N_{2n}$. We claim that for $m \neq n$, also $S_m \neq S_n$. If $S_m = S_n$, then S_m contains both $N_{2n-1} \cap N_{2n}$ and $N_{2m-1} \cap N_{2m}$. But since $N_{2n-1}, N_{2n}, N_{2m-1}$, and N_{2m} are *distinct* maximal ideals of R_j , it follows that $N_{2n-1} \cap N_{2n}$ and $N_{2m-1} \cap N_{2m}$ are coprime, whence $S_m = R_j$, a contradiction. But now R_j has infinitely many maximal subrings, contradicting (2).

Case 2: There exists N such that $(M, N) \in C_{ij}$ for infinitely many M . This case is analogous to Case 1 and the argument is omitted.

Case 3: For every M there are only finitely many N for which $(M, N) \in C_{ij}$ and for every N , there are only finitely many M for which $(M, N) \in C_{ij}$. Let $(M_1, N_1) \in C_{ij}$ be arbitrary. Let $X := \{(M, N) \in C_{ij} | M = M_1\}$ and let $Y := \{(M, N) \in C_{ij} | N = N_1\}$. By our assumption, X and Y are finite, yet C_{ij} is infinite. Pick $(M_2, N_2) \in C_{ij} - (X \cup Y)$. Thus $M_2 \neq M_1$ and $N_2 \neq N_1$. Continuing recursively, we obtain members $(M_1, N_1), (M_2, N_2), (M_3, N_3), \dots$ of C_{ij} such that for $a \neq b$, we have $M_a \neq M_b$ and $N_a \neq N_b$. Analogous to Case 1, for every positive integer n , there exists a maximal subring S_n of $R_i \times R_j$ which contains $M_n \times N_n$. For $m \neq n$, it is easy to see that $M_m \times N_m$ and $M_n \times N_n$ are coprime in $R_i \times R_j$. Thus $S_m \neq S_n$ for $m \neq n$, and $R_i \times R_j$ has infinitely many maximal subrings, which is a contradiction. The proof of (4') proceeds analogously to the proof of (4) and is omitted.

We now suppose that conditions (1), (2), (3), and (4) hold, and we show that R has only finitely many maximal subrings. We may suppose that $I = \{1, \dots, n\}$. Suppose by way of contradiction that R has infinitely many maximal subrings. Since for each i , $1 \leq i \leq n$, R_i has only finitely many maximal subrings, we infer that R has only finitely many maximal subrings of the form (1) of Proposition 3.2. Thus R has infinitely many maximal subrings of the form (2) of Proposition 3.2. Thus there exist $i \neq j$ such that $R_i \times R_j$ has infinitely many maximal subrings of the form (2) of Proposition 3.2. Since C_{ij} is finite, we conclude that there exist maximal ideals M_k of R_i and N_k of R_j such that $R_i/M_k \cong R_j/N_k$, R_i/M_k is finite, and $R_i/M_k \times R_j/N_k$ contains infinitely many maximal subrings; this is absurd since $R_i/M_k \times R_j/N_k$ is finite. This completes the proof. \square

The following corollaries are immediate now.

Corollary 3.10. *Let R be a ring. Then $R \times R$ has only finitely many maximal subrings if and only if R is a semilocal ring with only finitely many maximal subrings and for each maximal ideal M of R , the field R/M is finite.*

Corollary 3.11. *Let R_1, \dots, R_n be rings such that for each i , $\text{Char}(R_i) = c_i > 0$. If for each $i \neq j$: $(c_i, c_j) = 1$, then $R = \prod_{i=1}^n R_i$ has only finitely many maximal subrings if and only if each R_i has only finitely many maximal subrings.*

Finally in this article we give some results about semilocal reduced rings with only finitely many maximal subrings. In the next corollary we see that these rings are semisimple.

Corollary 3.12. *Let R be a semilocal reduced ring. Then either $\text{RgMax}(R)$ is infinite or $R \cong E_1 \times \dots \times E_m$, where $m \in \mathbb{N}$ and each E_i is a field with only finitely many maximal subrings (hence is algebraic over \mathbb{F}_{p_i} for some $p_i \in \mathbb{P}$, by Corollary 1.5) and for each $i \neq j$: If E_i and E_j are infinite, then $E_i \not\cong E_j$. Consequently, if R is an uncountable semilocal reduced ring, then $|\text{RgMax}(R)| \geq \aleph_0$. In particular, a semilocal domain R has only finitely many maximal subrings if and only if R is a field with only finitely many maximal subrings.*

Proof. Assume that R has only finitely many maximal subrings. Thus by Corollary 1.10, we infer that $J(R) = 0$. Hence R is a finite direct product of fields. Now Theorem 3.9 completes the proof of the first part of the theorem. Also note that if R is semilocal reduced ring with only finitely many maximal subrings, then by the first part we infer that R is a finite direct product of countable fields. Hence R is countable and therefore the second part of the corollary is proved. The final part is evident. \square

Let R be a ring. The chain $\dots \subset R_2 \subset R_1 \subset R_0 = R$ of subrings of R is called a saturated descending chain of subrings of R (or a saturated chain of maximal subrings in R) whenever each R_i is a maximal subring of R_{i-1} , for each $i \geq 1$. Moreover, if $R_m \subset R_{m-1} \subset \dots \subset R_2 \subset R_1 \subset R_0 = R$ is a chain of maximal subrings and R_m is not submaximal, then the integer m is called the length of the chain. We cite the following result from [3].

Theorem 3.13. *Let E be a field. Then the following conditions are equivalent:*

- (1) $\text{RgMax}(E)$ is finite.
- (2) E has a non submaximal subfield F such that E/F is a finite field extension.
- (3) Every descending chain

$$\dots \subset R_2 \subset R_1 \subset R_0 = E$$

is finite, where each R_i is a maximal subring of R_{i-1} for $i \geq 1$.

Moreover, if one of the above equivalent conditions holds, then F is unique and contains all non submaximal subrings of E . Furthermore all saturated descending chains in (3) have the same length, $m = [E : F]$, $R_m = F$, and E has only finitely many saturated descending chains of the form which is presented in (3).

Now we are ready to generalize the above characterization to semilocal reduced rings. In the next example we show that condition (3) does not imply condition (1) for semisimple rings which are not fields.

Example 3.14. Let F be the algebraic closure of \mathbb{F}_p . Then $F \times F$ has infinitely many maximal subrings by Corollary 3.5, but every descending chain

$$\cdots \subset R_2 \subset R_1 \subset R_0 = F \times F$$

is finite, where each R_i is a maximal subring of R_{i-1} for $i \geq 1$. To see this first note that $R_1 \cong F$, by Proposition 3.2 or Theorem 3.4. Now by [4, Remark 1.13 or Remark 2.11], F and therefore R_1 is not submaximal. Hence the chain is finite and we are done.

We need the following useful proposition for the next observations.

Proposition 3.15. Let E_1, \dots, E_n be absolutely algebraic fields of prime characteristic (the characteristics are not assumed to all be equal), and let $R := E_1 \times \cdots \times E_n$. Then the following hold:

- (1) If S is a maximal subring of R , then either there exists i such that $S = E_1 \times \cdots \times E_{i-1} \times E'_i \times E_{i+1} \times \cdots \times E_n$, where E'_i is a maximal subring of E_i , or $S \cong E_1 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_n$. In any case, S is a direct product of k fields, where $n - 1 \leq k \leq n$. Further, if each E_i has only finitely many maximal subrings, then S is a product of k fields which each have only finitely many maximal subrings.
- (2) If $R_m \subsetneq R_{m-1} \subsetneq \cdots \subsetneq R_1 \subsetneq R$ is a descending chain of maximal subrings, then R is a finite extension of R_m .
- (3) If F is a field with $F \subseteq R$ and if R is a finite-dimensional vector space over F , then each E_i contains an isomorphic copy F' of F such that E_i is a finite field extension of F' .

Proof. Let E_1, \dots, E_n and R be as stated.

(1) Suppose that S is a maximal subring of R . If S has the form (1) of Proposition 3.2, then S has the first form. In this case, we claim that E'_i is a field. To see this, simply note that for some prime p , $\mathbb{F}_p \subseteq E'_i \subseteq E_i$ and E_i is algebraic over \mathbb{F}_p . Thus E'_i is a field. Otherwise (again, by Proposition 3.2) it follows that $S \cong S' \times \prod_{k \neq i, j} E_k$ where $S' \in \text{RgMax}(E_i \times E_j)$ ($i \neq j$) and S' has form (2) of Proposition 3.2. But in this case, $S' \cong E_i \cong E_j$. To complete the proof of (1), suppose that each E_i has only finitely many maximal subrings. If S has the second form, then trivially S is a product of $n - 1$ fields which each have only finitely many maximal subrings. If S has the first form, we simply must show that E'_i has only finitely many maximal subrings. If not, then by Theorem 3.13, E'_i has an infinite descending chain of maximal subrings, and thus the same is true of E_i . But then by Theorem 3.13 again, $\text{RgMax}(E_i)$ is infinite, a contradiction.

(2) It suffices to show that R is a finite extension of R_1 . Let F_i be the prime subfield of E_i and Z be the prime subring of R . Now it is clear that $Z \subseteq S := F_1 \times \cdots \times F_n$ is an integral extension (note, S is finite) and also $S \subseteq R$ is an integral extension. These immediately imply that $S \subseteq R$ and therefore $R_1 \subseteq R$ is an integral extension. Since R_1 is a maximal subring of R , it follows that R is a finite extension of R_1 and we are done.

(3) We prove (3) by induction on n . The claim is clearly true if $n = 1$. Now let $n > 0$ be arbitrary and suppose that (3) holds for n . Now assume that E_1, \dots, E_{n+1} are absolutely algebraic fields of prime characteristic p , and let $R := E_1 \times \cdots \times E_{n+1}$. Suppose further that F is a field and $F \subseteq R$ with R a finite-dimensional vector space over F . It follows that there exists a descending chain $F = R_m \subsetneq R_{m-1} \subsetneq R_{m-2} \cdots \subsetneq R_1 \subsetneq R_0 := R$ of maximal subrings of R ($m \geq 1$). Now let k be greatest such that R_k is a product of $n + 1$ fields. Then $k < m$ and (by (1)) R_{k+1} is a product of n fields, each of which is a summand of R_k . Moreover, two summands of R_k are isomorphic (by Proposition 3.2). It follows by the induction hypothesis that each summand of R_k is a finite extension of some isomorph of F . It follows that each E_i also contains an isomorphic copy of F , and by (2), each such extension is finite. \square

In the following result we show that for semisimple rings, condition (1) implies condition (3). We also give a condition under which the converse is valid.

Theorem 3.16. *For all positive integers n : If E_1, E_2, \dots, E_n are fields each with only finitely many maximal subrings, then every descending chain $\cdots \subsetneq R_2 \subsetneq R_1 \subsetneq R_0 := E_1 \times \cdots \times E_n$ is finite, where R_i is a maximal subring of R_{i-1} for $i \geq 1$. Conversely, suppose that E_1, E_2, \dots, E_n are fields and set $R := E_1 \times \cdots \times E_n$. Suppose further that for $i \neq j$: if $E_i \cong E_j$, then E_i is finite. Then if every descending chain of maximal subrings in R is finite, then R has only finitely many maximal subrings.*

Proof. Suppose the first assertion fails, and let n be least for which the claim is false. Proposition 3.15, part (1) implies that each R_i is a direct product of k fields, $k \leq n$, each of which has only finitely many maximal subrings. It follows from the leastness of n that each R_i is a direct product of n fields, each of which has only finitely many maximal subrings. But then by repeated application of (1) of Proposition 3.15, we see that some E_i has an infinite descending chain of maximal subrings, contradicting Theorem 3.13. As for the second claim, it is clear that each E_i has the property that every descending chain of maximal subrings of E_i is finite. Thus by Theorem 3.13, we see that E_i has only finitely many maximal subrings. Theorem 3.9 now implies that R has only finitely many maximal subrings. \square

Example 3.17. *Let p and q be primes, n a positive integer, and set $R := \mathbb{F}_{p^{qn}} \times \mathbb{F}_{p^{qn}}$. Then R has a saturated chain of maximal subrings of length $2n + 1$ and a saturated chain of maximal subrings of length $n + 1$.*

Although as we see in the previous example the lengths of the chains are not equal for the semisimple rings with only finitely many maximal subrings, but as we see in the next result (which is our main result in this article) the last terms in these chains are all isomorphic to a (non submaximal) ring, say S , where S is unique (up to isomorphism) with respect to the property that R is finitely generated as an S -module.

Theorem 3.18. *For all positive integers n : suppose that E_1, \dots, E_n are fields each have only finitely many maximal subrings. For each i , $1 \leq i \leq n$, let F_i be the largest non submaximal subfield of E_i such that E_i/F_i is a finite field extension (whose existence is guaranteed by Theorem 3.13). If $\{K_1, \dots, K_m\}$ is a maximal subset of non-isomorphic elements of $\{F_1, \dots, F_n\}$, then the following hold:*

- (1) *If $R_l \subsetneq \cdots \subsetneq R_2 \subsetneq R_1 \subsetneq R_0 := E_1 \times \cdots \times E_n$ is a descending saturated chain of maximal subrings, then $R_l \cong K_1 \times \cdots \times K_m$. Moreover, R_0 is finitely generated over R_l .*
- (2) *If R' is a subring of R_0 which is not submaximal and R_0 is finitely generated over R' , then $R' \cong K_1 \times \cdots \times K_m$.*

Proof. (1) First note that by Corollary 1.5 each E_i is an absolutely algebraic field of prime characteristic which by (2) of Proposition 3.15 immediately implies that R_0 is finitely generated over R_l . Hence it remains to prove the first claim of (1). We show it by induction on the positive integers. The base case of the induction follows immediately from Theorem 3.13. Now let $n > 1$ and suppose (1) holds for $n - 1$. We will prove that (1) holds for n . Thus let E_1, \dots, E_n and F_1, \dots, F_n be as stated in the theorem. Moreover, let $R_l \subsetneq \cdots \subsetneq R_2 \subsetneq R_1 \subsetneq R_0 := E_1 \times \cdots \times E_n$ be a descending saturated chain of maximal subrings. We consider two cases.

Case 1: $R_l = L_1 \times \cdots \times L_n$ for some subfields L_1, \dots, L_n of E_1, \dots, E_n , respectively (see Proposition 3.15). Since R_l is non-submaximal, the same is true of each L_i . But then each L_i is the final term of a saturated descending chain of maximal subrings of E_i . By Theorem 3.13, it follows that $L_i = F_i$. Hence $R_l = F_1 \times \cdots \times F_n$. If $F_i \cong F_j$ for some $i < j$, then $F_i \times F_j$ is submaximal by [1, Theorem 2.2]. But then R_l is submaximal, a contradiction.

Case 2: R_l is the product of fewer than n fields. Let $i \geq 0$ be greatest such that R_i is the product of n fields, say $R_i = L_1 \times \cdots \times L_n$. Proposition 3.15 implies that each L_i is a subfield of E_i . Moreover, each F_i is the largest non submaximal subfield of L_i such that L_i/F_i is a finite field extension. It follows from Proposition 3.2 and Proposition 3.15 that there exists $i < j$ such that $L_i \cong L_j$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Further, $R_{i+1} \cong L_2 \times \cdots \times L_n$. The induction hypothesis now yields that $R_l \cong K_2 \times \cdots \times K_m$, where $\{K_2, \dots, K_m\}$ is a maximal subset of non-isomorphic elements of $\{F_2, \dots, F_n\}$. We claim that $\{K_2, \dots, K_m\}$ is a maximal subset of non-isomorphic elements of $\{F_1, \dots, F_n\}$. To see this, simply recall that $L_1 \cong L_2$. Thus $F_1 \cong F_2 \cong K_i$ for some i with $2 \leq i \leq m$.

We now prove (2). Thus suppose that R' is a subring of R_0 which is not submaximal and assume that R_0 is finitely generated over R' . Since R_0 is Noetherian, Eakin's Theorem implies that R' is Noetherian. Also note that R_0 is integral over R' , whence R_0 and R' have the same Krull dimension. R_0 is Artinian, whence has dimension 0. We conclude that R' has dimension 0, and thus R' is also Artinian. Since R' is reduced, it follows that $R' = L_1 \times \cdots \times L_k$ for some fields L_1, \dots, L_k . Thus $R_0 = S_1 \times \cdots \times S_k$ for

some rings S_1, \dots, S_k , where each S_i is a ring extension of L_i . Since R_0 is finitely generated over R' , we see that each S_i is a finite-dimensional vector space over L_i . Thus there exists a finite descending chain of maximal subrings from S_i to L_i . An easy application of the first part of Remark 3.3 shows that there exists a finite descending chain of maximal subrings from R_0 to R' . We are now done by (1). \square

The next example is in order now. In this example we see that the finitely generated condition in the final part of the previous theorem can not be omitted.

Example 3.19. Let q_1, q_2, q_3 and q_4 be distinct prime numbers and p be any prime number. Now define the following subfields of the algebraic closure of \mathbb{F}_p :

$$F = \bigcup_{m=0}^{\infty} \mathbb{F}_{p^{q_1^m}},$$

$$F \subset F_1 = \bigcup_{m,n=0}^{\infty} \mathbb{F}_{p^{q_1^m q_2^n}} \subset E_1 = \bigcup_{\substack{m,n=0 \\ 0 \leq k \leq 1}}^{\infty} \mathbb{F}_{p^{q_1^m q_2^n q_4^k}},$$

and

$$F \subset F_2 = \bigcup_{m,n=0}^{\infty} \mathbb{F}_{p^{q_1^m q_3^n}} \subset E_2 = \bigcup_{\substack{m,n=0 \\ 0 \leq k \leq 1}}^{\infty} \mathbb{F}_{p^{q_1^m q_3^n q_4^k}}.$$

Then F_i is the unique maximal subring of E_i , by [3, Theorem 2.6]. Therefore $[E_i : F_i]$ is finite, and since F_i is non submaximal, by [4, Theorem 1.8 and Proposition 1.11], we conclude that F_i is the largest non submaximal subring of E_i , by Theorem 3.13. Now note that $E_1 \times E_2$ has only finitely many maximal subrings by Corollary 3.7. Now put $R' = \{(x, x) \mid x \in F\}$. It is clear that R' is a subring of $E_1 \times E_2$ and since $R' \cong F$, we infer that R' is not submaximal, by [4, Theorem 1.8 and Proposition 1.11]. But clearly R' is not isomorphic to $F_1 \times F_2$.

We conclude this article by the next theorem and its corollary.

Theorem 3.20. Let R be a semilocal reduced ring. Assume that R has a subring S such that S is not submaximal and R is finitely generated as an S -module. Then $R \cong E_1 \times \dots \times E_n$, where each E_i is a field with only finitely many maximal subrings, for $1 \leq i \leq n$.

Proof. First note that since R is reduced, we conclude that S is reduced too. Thus by Corollary 1.10, we infer that $J(S) = 0$, since S is not submaximal. Now since R is finitely generated as an S -module, we conclude that R is integral over S . Therefore S is semilocal too, since R is semilocal. Hence we may assume that $S = F_1 \times \dots \times F_m$, where each F_i is a non submaximal field (and by Corollary 1.5, each field F_i is absolutely algebraic over \mathbb{F}_{p_i} for some prime p_i) and $R = R_1 \times \dots \times R_m$, where each R_i is a ring which contains F_i . Since R is finitely generated as an S -module we conclude that each R_i is a finite dimensional F_i -vector space. Thus each R_i is an artinian ring. Now since R is reduced, we infer that each R_i is a reduced ring. Therefore each R_i is semisimple ring. Thus $R_i \cong E_{i1} \times \dots \times E_{in_i}$, where E_{ij} is a field. Since F_i has characteristic p_i and $F_i \subseteq R_i$, we conclude that R_i has characteristic p_i . But then each E_{ij} also has characteristic p_i . Since R_i is finite-dimensional over F_i , it follows that R_i is an integral extension of F_i . Since F_i is algebraic over \mathbb{F}_{p_i} , we conclude that R_i is integral over \mathbb{F}_{p_i} . This implies that each E_{ij} is algebraic over \mathbb{F}_{p_i} , whence each E_{ij} is absolutely algebraic of characteristic p_i . Thus by (3) of Proposition 3.15, we deduce that each E_{ij} contains a copy of F_i and is finite-dimensional over F_i . Now, Theorem 3.13, implies that each E_{ij} is a field with only finitely many maximal subrings (and F_i is the largest non submaximal subfield of E_{ij}). Thus each R_i is a finite direct product of fields which have only finitely many maximal subrings; and therefore R is a finite direct product of fields which have only finitely many maximal subrings and we are done. \square

Finally, in the next corollary we give a generalization of Theorem 3.13.

Corollary 3.21. Let R be a semilocal reduced ring. Then the following statements are equivalent:

- (1) $R \cong E_1 \times \dots \times E_n$, where each E_i is a field with only finitely many maximal subrings.
- (2) R is a semisimple ring and each descending chain $\dots \subset R_2 \subset R_1 \subset R_0 = R$ is finite, where each R_i is a maximal subring of R_{i-1} for every $i > 0$.
- (3) There exists a non submaximal subring S of R such that R is finitely generated as an S -module.

Moreover, if one of the above equivalent conditions holds for R , then the following are true:

- (i) S is unique up to isomorphism; and the last terms of all chains in (2) are isomorphic to S .
(ii) R has only finitely many maximal subrings up to isomorphism.

Furthermore, if R' is a subring of R such that R is a finitely generated as R' -module, then R satisfies one of the above equivalent conditions if and only if R' satisfies one of them.

Proof. (1) implies (2), by the first part of Theorem 3.16; and by the previous theorem (3) implies (1). Now suppose (2) holds and let $S := R_m \subset \cdots \subset R_2 \subset R_1 \subset R_0 = R$ be a saturated chain of maximal subrings, then clearly S is not submaximal and by (2) of Proposition 3.15, R is finitely generated as an S -module. Thus (3) holds. Now assume that one of the equivalent conditions (1)-(3) holds, then (i) holds by Theorem 3.18. Also note that (ii) holds by (1) of Proposition 3.15. For the final claim, first assume that R' satisfies one of the conditions. Since R is integral over R' and R is reduced, we infer that R' is a semilocal reduced ring. Hence all conditions are equivalent for R' . Thus by condition (2), R' has a non submaximal subring S' such that R' is a finitely generated S' -module. It is now clear that R is a finitely generated S' -module and therefore R satisfies condition (2). Conversely, assume that R satisfies one of the equivalent conditions. We prove that that the same holds for R' . Similar to the previous proof, since R is semilocal reduced ring and R is integral over R' , we infer that R' is a semilocal reduced ring. Hence by the first part of the theorem, it suffices to show that R' satisfies at least one of the conditions. Now, since R is a zero dimensional ring, we infer that R' is zero dimensional too. Thus $J(R') = 0$, and therefore we may assume that $R' = F_1 \times \cdots \times F_n$, where each F_i is a field and $R = R_1 \times \cdots \times R_n$, where each R_i is a ring which contains F_i . Now since R is finitely generated as an R' -module, we infer that each R_i is a finite dimensional F_i -vector space. Therefore there exists a finite descending chain of maximal subrings from R_i to F_i . Therefore we infer that there exists a finite descending saturated chain of maximal subrings from R to R' . Now since R satisfies condition (2), we conclude that R' satisfies condition (2), and hence we are done. \square

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