FACTORIZATION THEORY OF ROOT CLOSED MONOIDS OF SMALL RANK

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Abstract. In this note, we investigate ideal and factorization-theoretic properties of some root closed cancellative commutative monoids of rank at most two.

1. Introduction

Factorization theory of commutative integral domains has a long, rich history. In recent years, there has been increasing interest in "porting over" ring-theoretic theorems whose statements are purely multiplicative in nature (that is, the statements make no reference to the additive structure of a ring) to the setting of cancellative commutative monoids, often with zero (the multiplicative monoid of a commutative domain, not coincidentally, has these properties). It is not our purpose to give a comprehensive account of such results in this paper; we refer the reader instead to [2]–[8], [10]–[16], and [18]–[28] for a partial list of relevant literature.

Before motivating this note, we require some preliminary definitions. If M is a submonoid of a commutative monoid N, then the *root closure of M in N* is the monoid $\{x \in N \colon x^n \in M \text{ for some } n \in \mathbb{Z}^+\}$. A commutative monoid M is *cancellative* provided whenever $a, b, c \in M$ with $ab = ac$, then $b = c$. Every cancellative commutative monoid M embeds naturally into its *group of fractions* $\mathcal{Q}(M)$ defined by $\mathcal{Q}(M) := \{\frac{a}{b} \colon a, b \in M\}$ (with the canonical multiplication inherited from M). We naturally identify M with its image in $\mathcal{Q}(M)$. Say that a cancellative commutative monoid M is *root closed* (or *normal*) provided the root closure $\overline{M}$ of M in $\mathcal{Q}(M)$ coincides with M. Note that this can be expressed internally as follows: for all $a, b \in M$: if $a^n \in b^nM$, then $a \in bM$. Note that we have used multiplicative notation in the previous definitions. When we use a different operation, we shall use appropriate notation without further explanation. Thus, for example, if we denote

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the operation on a monoid M additively (by $+$), then $\mathcal{Q}(M)$ is the group of differences of M. For additional background in commutative semigroup fundamentals, we refer the reader to [17], [20], and [21].

Recall that the rank of an abelian group G is the cardinal number (which may be infinite) of indecomposable divisible summands of the injective hull $E(G)$ of G. The *rank* of a cancellative commutative monoid M is simply the rank of its group of fractions $\mathcal{Q}(M)$. Note that groups of small rank can be immensely complicated. For example, the subgroups of $\mathbb{Q}$ have all been determined. However, classifying the subgroups of $\mathbb{Q} \times \mathbb{Q}$ is notoriously difficult. Even now, the subgroup structure of $\mathbb{Q} \times \mathbb{Q}$ is not well-understood ([9], [29]).

The impetus for writing this article is to initiate a factorization-theoretic study of root closed monoids of finite rank from a geometric viewpoint. The purpose of adding the root closed condition is two-fold. For one, the root closed monoids form a very broad and well-studied subclass of the class $\mathcal{M}$ of all monoids (this class properly includes the classes of GCD monoids and Krull monoids, for example). Further, certain structural complications are somewhat mitigated by this additional property. In this note, we present a catalog of ideal and factorization-theoretic properties of some root closed monoids of rank at most two. Space limitations preclude us from doing an exhaustive study (even in the rank two case). Our motivation is simply to provide evidence that the class of root closed finite rank monoids is fertile ground for harvesting interesting factorization-theoretic results. Our hope is that this paper spurs additional investigations of the class of root closed monoids of finite rank, ultimately culminating not only in new theorems, but new examples and counterexamples (of both semigroups and possibly rings as well via the semigroup ring construction).

We conclude the introduction by mentioning that throughout the paper, all monoids are assumed cancellative and commutative.[1]

2. Preliminaries

In this terse section, we present definitions and results to which we shall refer throughout the paper. We remark that space limitations limit our study to a proper subset of ideal and factorization-theoretic properties. The reader is encouraged to augment our results by considering additional notions not studied in this paper (such as almost Schreier monoids, elasticity, finite factorization monoids, etc.).

Let M be a monoid. Recall that M is *Noetherian* provided every congruence on M is finitely generated (see Section 5 of [17]). A nonempty subset $I \subseteq M$ is an *ideal* of M provided $x \in I$ and $m \in M$ imply that $mx \in I$. A proper ideal P of M is a *prime ideal* if for all $x, y \in M$: whenever $xy \in P$, then either $x \in P$ or $y \in P$.

[1]A nontrivial cancellative monoid cannot possess a zero element. However, the results of this article can easily be reformulated for monoids with 0. For brevity's sake, we leave this task to the interested reader.

The *radical* of an ideal I is the ideal $\sqrt{I} := \{x \in M : x^n \in I \text{ for some } n > 0\}$. We call M a *principal ideal monoid* if every ideal of M is of the form Mx for some $x \in M$. More generally, M satisfies *the ascending chain condition on ideals* (acc) if every ideal of M is finitely generated (equivalently, every ascending chain of ideals stabilizes). A monoid M is a *valuation monoid* if the ideals of M are linearly ordered by set inclusion. Finally, M is *coherent* provided the intersection of any two principal ideals of M is finitely generated (as an ideal).

Let M be a monoid. As usual, we denote the group of units of M by $M^\times$, and we let $M_{\text{red}} := M/M^\times$ be the associated reduced monoid. An element $a \in M \backslash M^\times$ is said to be an *atom* provided whenever $a = bc$, either $b \in M^\times$ or $c \in M^\times$. The monoid M is called *atomic* provided every $m \in M \backslash M^\times$ is a finite product of atoms. We say that M is *half-factorial* provided M is atomic, and whenever $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ are atoms with $a_1 \cdots a_r = b_1 \cdots b_s$, then $r = s$. A monoid M is *factorial* provided M is half-factorial and given $m = a_1 \cdots a_r = b_1 \cdots b_r$ with each a_i, b_j atoms, then (after reordering the index set, if necessary) for each i, there is a unit u such that $a_i = ub_i$ (in other words, a_i and b_i are *associates*). We now recall that M is a *GCD monoid* provided for all $x, y \in M$, there exists $z \in M$ such that $Mx \cap My = Mz$. Before stating the final definition of this paragraph, we recall that if $x, y \in M$, then "$x|y$" means there exists $z \in M$ such that $xz = y$. Lastly, M is a *Schreier monoid* if for all $a, b, c \in M$: whenever $a|bc$, then $a = rs$ for some $r|b$ and $s|c$.

A monoid homomorphism $\varphi\colon M \to N$ is called a *divisor map* provided for all $x, y \in M$: if $\varphi(x)|\varphi(y)$ in N, then $x|y$ in M. A monoid M is called *inside factorial* if there exists a factorial monoid D and divisor map $\varphi\colon D \to M$ such that for every $x \in M$, there exists some $n > 0$ such that $x^n \in \varphi(D)$. Analogously, a monoid M is *outside factorial* provided there is a factorial monoid D and divisor map $\varphi\colon M \to D$ such that for every $x \in D$, there exists some $n > 0$ such that $x^n \in \varphi(M)$. Finally, M is a *Krull monoid* if there exists a free monoid F and a divisor map $\varphi\colon M_{\text{red}} \to F$.[2]

We conclude this section with a proposition which collects various containment relations between the classes of monoids defined above. We caution the reader that we do not attempt to find all such relations. Rather, we list only those needed to prove the main results of this paper.

Proposition 1. *Let M be a monoid. The following implications hold:*

(1) *If M is a principal ideal monoid, then M is a factorial valuation monoid.*
(2) *If M is a factorial monoid, then M is a GCD monoid.*
(3) *If M is an outside factorial monoid, then M is a Krull monoid.*
(4) *If M is a GCD monoid, then M is a root closed Schreier monoid.*
(5) *If M is a valuation monoid, then M is a GCD monoid.*
(6) *If M is Noetherian, then M satisfies acc on ideals.*

[2]Equivalently, M is Krull if M is completely integrally closed and satisfies ACC on divisorial ideals.

(7) *If M is Krull, then M is atomic and root closed.*
(8) *M is finitely generated if and only if M is Noetherian.*

Sketch of Proof. Let M be a monoid.

(1) Suppose M is a principal ideal monoid. The proof that M is factorial follows *mutatis mutandis* from the usual proof that a principal ideal domain is a unique factorization domain. To show M is a valuation monoid, let $a, b \in M$ be arbitrary. Then $Ma \cup Mb = Mc$ for some $c \in M$. Without loss of generality, a and c are associates. It now follows that $a|b$.

(2) The proof is straightforward.

(3) This assertion is Theorem 23.3 of [21].

(4) Assume that M is a GCD monoid, and suppose that $b^n|a^n$. Since M a GCD monoid, we may assume that a and b are relatively prime (that is, $Ma \cap Mb = Mab$). Since $b|a^n$ and a and b are relatively prime, $b|a$ follows, and M is root closed. To show that M is Schreier, suppose $a|bc$. Set $x := gcd(a, b) = \frac{ab}{z}$, where $Ma \cap Mb = Mz$. Then $x\alpha = a$ for some $\alpha \in M$. One checks easily that $x|b$ and $\alpha|c$.

(5) The proof is trivial.

(6) This follows immediately from Theorem 5.1 of [17].

(7) Let $M := \mathbb{N}^I$ be a free monoid[3]. For $(x_i) \in M$, let $\sum x_i$ be the sum of the entries of (x_i). One proves easily by induction on the cardinality of $\sum x_i$ that every submonoid of a free monoid is atomic. If M is Krull, then M_{red} is isomorphic to a submonoid of a free monoid, and is thus atomic. It is easy to see that this forces M to be atomic as well. Furthermore, it is well-known that Krull monoids are root closed (see, for example, Corollary 4 of [7]).

(8) This is an amalgam of Theorems 5.10 and 7.8 of [17]. □

3. Root closed rank one monoids

We begin this section by commenting on notation. If M is an additive monoid, then "$a|b$" (in M) means that there exists $x \in M$ such that $a + x = b$. We now prove that the lattice of ideals of a root closed rank one monoid is totally ordered.

Proposition 2. *Every root closed rank one monoid is a valuation monoid.*

Proof. Let M be a root closed rank one monoid, and let $E(\mathcal{Q}(M))$ be the injective hull of $\mathcal{Q}(M)$. Since M is rank one, either $E(\mathcal{Q}(M)) = \mathbb{Z}(p^\infty)$ for some prime p or $E(\mathcal{Q}(M)) = \mathbb{Q}$. In the former case, for every $a \in M$, there exists some positive integer n such that $p^n a = 0$. But then M is a group, and is thus trivially a valuation monoid. Suppose now that M is a submonoid of the additive group $\mathbb{Q}$ of rational numbers. If M contains both positive and negative members, then it is easy to see (and is well-known; cf. [17], Theorem 2.9) that M is a group. Thus we may assume that M is a nontrivial submonoid of the non-negative rational numbers. To show

[3]Throughout this note, we include 0 as a member of $\mathbb{N}$.

that M is a valuation monoid, it suffices to prove that $\mathcal{Q}(M)^+ \subseteq M$, where $\mathcal{Q}(M)^+$ is the positive cone of the group of differences of M. Let $r, s \in M$ be arbitrary, and suppose that $r - s > 0$. Clearly M contains some positive integer m. Moreover, there exists a positive integer n such that $n(r-s) \in \mathbb{N}$. It follows that $nm(r-s) \in M$. But M is root closed, whence $r - s \in M$. This concludes the proof. □

Corollary 1. *Every root closed rank one monoid is a GCD monoid, hence coherent and (by (4) of Proposition 1) Schreier.*

Corollary 2. *Every root closed rank one monoid is inside factorial.*

Proof. Let M be a root closed rank one monoid. We may suppose that M is not a group, and hence M is (up to isomorphism) a nontrivial additive submonoid of $\mathbb{Q}_{\geq 0}$. Now choose any $x \in M \backslash \{0\}$, and let $\varphi \colon \mathbb{N} \to M$ be defined by $\varphi(n) := nx$. Observe that if $\varphi(m) | \varphi(n)$ in M, then $mx \leq nx$. We deduce that $m \leq n$, and thus $m|n$ (in the semigroup $(\mathbb{N}, +)$). Finally, let $z \in M$ be arbitrary. We must show that there exists $n > 0$ such that $nz = mx$ for some $m \in \mathbb{N}$. But clearly this is equivalent to $\frac{z}{x} \in \mathbb{Q}_{\geq 0}$, which is true. This concludes the proof. □

Having recorded several positive corollaries (i.e. several statements of the form, "every root closed rank one monoid is an X monoid"), we change gears and establish some negative results. Before proceeding to our next corollary, we pause to state the following lemma, whose easy proof we omit.

Lemma 1. *Suppose that M is a root closed submonoid of $\mathbb{Q}_{\geq 0}$. Then $M \cong (\mathbb{N}, +)$ if and only if M possesses a least positive element.*

Proposition 3. *Let M be a root closed rank one monoid which is not a group. Then the following are equivalent:*

(1) $M \cong (\mathbb{N}, +)$.
(2) *M is atomic.*
(3) *M is half-factorial.*
(4) *M is factorial.*
(5) *M satisfies acc on ideals.*
(6) *M is finitely generated.*
(7) *M is Krull.*
(8) *M is outside factorial.*
(9) *M is a principal ideal monoid.*

Proof. Clearly (1) implies (2)–(9). Moreover, Proposition 3 shows that each of (3), (4), (7), (8), and (9) implies (2), and (6) implies (5). Thus it remains only to prove that (2) implies (1) and (5) implies (1). We may assume that M is a nontrivial valuation submonoid of $\mathbb{Q}_{\geq 0}$.

(2) ⇒ (1). Assume by way of contradiction that M is atomic but $M \not\cong (\mathbb{N}, +)$. Then by Lemma 1, we see that M does not possess a least positive element. Now let

a be an arbitrary nonzero element of M. There exists $b \in M$ with $0 < b < a$. Since M is a valuation monoid, there exists $c \in M$, $c > 0$, such that $b + c = a$. But this implies that a is not an atom, and therefore M does not possess any atoms. We now obtain a contradiction to the assumption that M is atomic (and not a group).

(5) $\Rightarrow$ (1). Now suppose that M satisfies acc on ideals. Then the maximal ideal $J := M \setminus \{0\}$ is finitely generated. The least member of a generating set for J is easily seen to be the least positive element of M, and we are done by Lemma 1. □

The following corollary is immediate.

Corollary 3. *Let M be a rank one monoid which is not finitely generated. Then M is Krull (hence by (7) of Proposition 1, root closed) if and only if M is a dense subgroup of $\mathbb{Q}$ or $M = C(p^\infty)$ for some prime number p.*

4. Root closed rank two monoids I: the mixed case

We now begin our analysis of root closed rank two monoids. Let M be such a monoid. If $\mathcal{Q}(M)$ is torsion, then M is a group, and this case is not interesting from our point of view. Thus we assume that $\mathcal{Q}(M)$ is not torsion. In this section, we treat the case where M is of rank two and $\mathcal{Q}(M)$ is *mixed*, that is, $\mathcal{Q}(M)$ contains both elements of infinite order and nonzero elements of finite order.

Our first task is to determine the subgroups of $\mathbb{Q} \times \mathbb{Z}(p^\infty)$, where p is an arbitrary prime. This is an exercise in elementary abelian group theory, but to keep the paper self-contained, we sketch the details.

Lemma 2. *Every mixed rank two abelian group is of the form $G \times H$, where G is a nontrivial subgroup of $\mathbb{Q}$, and H is either a nontrivial cyclic group of prime power order or $H = C(p^\infty)$ for some prime p.*

Sketch of Proof. Let K be rank two and mixed. Then we may assume that $K < \mathbb{Q} \times C(p^\infty)$ for some prime p. Now, let $T(K)$ be the torsion subgroup of K. Then $T(K)$ is isomorphic to either $\mathbb{Z}/\langle p^n \rangle$ for some positive integer n or to $\mathbb{Z}(p^\infty)$. In the former case, $T(K)$ is a cyclic pure subgroup of K of prime power order, and it follows from Theorem 24.1 of [14] that $T(K)$ is a direct summand of K. In the latter, $T(K)$ is divisible, whence also a direct summand of K by Theorem 18.1 of [14]. The result follows. □

We now describe the root closed rank two monoids whose quotient groups are mixed.

Proposition 4. *Let M be a monoid which is not a group and is such that $\mathcal{Q}(M)$ is mixed of rank two. Then M is root closed if and only if $M \cong V \times H$ for some nontrivial valuation monoid $V \subseteq \mathbb{Q}_{\geq 0}$ and group H which is either quasicyclic or cyclic of prime power order.*

Proof. Let M be as stated. If $M \cong V \times H$ (with properties listed above), then it is easy to see that M is a valuation monoid, hence root closed. Conversely, suppose that M is root closed. By Lemma 2, we may assume that $\mathcal{Q}(M) = G \times H$ for some nontrivial subgroup G of $\mathbb{Q}$ and group H which is either quasicyclic or cyclic of prime power order.

We claim that

$$\{0\} \times H \subseteq M. \tag{4.1}$$

To see this, let $h \in H$ be arbitrary. There exists a positive integer n such that $nh = 0$. But then $n(0, h) = (0, nh) = (0, 0) \in M$. Since M is root closed, $(0, h) \in M$.

Now, let $\pi_1 \colon M \to \mathbb{Q}$ be projection onto the first coordinate. If $\pi_1(M)$ contains both positive and negative rationals, then (as noted in the proof of Proposition 2) $\pi_1(M)$ is a group. But then this fact along with (4.1) implies that M is a group, and this is a contradiction. Thus we may assume that $\pi_1(M) \subseteq \mathbb{Q}_{\geq 0}$. It follows easily from (4.1) that $M = \pi_1(M) \times H$. We now deduce that $\pi_1(M)$ is root closed. An application of Proposition 2 concludes the proof. □

The next corollary follows immediately from Corollary 1, Corollary 2, Proposition 3, and Proposition 4. As such, we omit the proof.

Corollary 4. *Let M be a rank two monoid which is not a group and such that $\mathcal{Q}(M)$ is mixed. Then M a GCD monoid, hence coherent and (by (4) of Proposition 1) Schreier. Further, M is inside factorial. Moreover, the following are equivalent:*

(1) *$M \cong \mathbb{N} \times H$ for some group H which is either quasicyclic or cyclic of prime power order.*
(2) *M is atomic.*
(3) *M is half-factorial.*
(4) *M is factorial.*
(5) *M satisfies acc on ideals.*
(6) *M is Krull.*
(7) *M is outside factorial.*
(8) *M is a principal ideal monoid.*

We close with an analog of Corollary 3.

Corollary 5. *Let M be a rank two monoid which is not a group and whose group of fractions is mixed. Suppose further that M is not finitely generated. Then M is Krull if and only if $M \cong \mathbb{N} \times \mathbb{Z}(p^\infty)$ for some prime number p.*

5. Root closed rank two monoids II: the torsion-free discrete case

We begin by remarking that the following applies to all propositions P in this section: if P asserts that M is an X monoid, where X is one of the monoids mentioned in Proposition 1, and if every X monoid is a Y monoid (again, by Proposition 1), then for brevity we do not explicitly mention that M is a Y monoid in the statement of P.

We now analyze root closed monoids M for which $\mathcal{Q}(M)$ is a free abelian group on two generators. We first set up some notation (defining so-called *pointed cones* in $\mathbb{R}^2$).

Definition 1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^2$.

(1) $(\mathbf{u}, \mathbf{v}) := \{s\mathbf{u} + t\mathbf{v} \colon s, t \in (0, \infty)\} \cup \{(0, 0)\}$,
(2) $[\mathbf{u}, \mathbf{v}) := \{s\mathbf{u} + t\mathbf{v} \colon s \in (0, \infty),\ t \in [0, \infty)\} \cup \{(0, 0)\}$, *and*
(3) $[\mathbf{u}, \mathbf{v}] := \{s\mathbf{u} + t\mathbf{v} \colon s, t \in [0, \infty)\}$.

Let us agree to call cones defined by (1) *open cones*, the cones in (2) *half open cones*, and the cones in (3) *closed cones*. By *cone*, we mean a subset of $\mathbb{R}^2$ which is either an open, half open, or closed cone as defined above. The vectors $\mathbf{u}$ and $\mathbf{v}$ will be called *bounding vectors* of the cone.

We begin our analysis with the following lemma. All three assertions are obvious upon reflection; we therefore omit the easy proof.

Lemma 3. *Let* C *be a cone bounded by* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

(1) $C \cap \mathbb{Z}^2$ *is a root closed submonoid of* $\mathbb{Z}^2$.
(2) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent over* $\mathbb{R}$, *then* $\mathcal{Q}(C \cap \mathbb{Z}^2)$ *has rank at most one.*
(3) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly independent over* $\mathbb{R}$, *then* $\mathcal{Q}(C \cap \mathbb{Z}^2) = \mathbb{Z}^2$ *(set-theoretic equality is what is meant here, not simply isomorphism).*

Our next lemma collects some additional information about closed cones.

Lemma 4. *The following hold:*

(1) *If* $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$, *then* $[\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2$ *is a finitely generated submonoid of* $\mathbb{Z}^2$.
(2) *If* M *is a root closed submonoid of* $\mathbb{Z}^2$ *such that* $\mathcal{Q}(M) = \mathbb{Z}^2$, *then for any* $\mathbf{u}, \mathbf{v} \in M$, $[\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2 \subseteq M$.

Proof. Assertion (1) is well-known and an immediate consequence of Gordon's Lemma (cf. p. 59 of [5]). As for (2), assume that M is a root closed submonoid of $\mathbb{Z}^2$ such that $\mathcal{Q}(M) = \mathbb{Z}^2$, and let $\mathbf{u}, \mathbf{v} \in M$ be arbitrary. Now suppose that $\mathbf{w} := (w_1, w_2) \in [\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2$. Then $(w_1, w_2) = s\mathbf{u} + t\mathbf{v}$ for some real numbers $s, t \geq 0$. It follows from elementary linear algebra that we may take $s, t \in \mathbb{Q}_{\geq 0}$. But then there exists a positive integer n such that $n\mathbf{w} \in \mathbb{N}\mathbf{v} + \mathbb{N}\mathbf{w} \subseteq M$. Since M is root closed, we conclude that $\mathbf{w} \in M$, completing the proof. □

Finally, we shall make use of the following result.

Lemma 5 ([20], Theorem 8.7)**.** *Let* M *be a reduced submonoid of* $\mathbb{Z}^n$ *such that* $\mathcal{Q}(M) = \mathbb{Z}^n$. *Then* M *is Krull if and only if* M *is finitely generated and root closed.*

We are now ready to study half-planes bounded by a line ℓ through the origin (intersected with $\mathbb{Z}^2$). The terms "closed half-plane," "open half-plane," and "half open half-plane" have obvious meanings analogous to (1), (2), and (3) of Definition 1. To streamline terminology in what follows, we refer to the intersection of a (closed, open, or half open) half-plane with $\mathbb{Z}^2$ simply as an *integral half-plane*.

To begin, note that the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi((x, y)) := (y, -x)$ induces a one-to-one isomorphic correspondence between the collection of integral half-planes bounded by $x = 0$ and the collection of integral half-planes bounded by $y = 0$. Thus we may restrict our study to integral half-planes bounded by a non-vertical line ℓ without loss of generality. We first consider the closed case where ℓ has rational slope.

Proposition 5. *Let M be a closed integral half-plane bounded by $\ell(x) = rx$ for some rational number r. Then M is a finitely generated principal ideal monoid.*

Proof. Let M be as stated. It is clear that for any $\mathbf{u} \in \mathbb{Z}^2$, either $\mathbf{u} \in M$ or $-\mathbf{u} \in M$. Therefore, M is a valuation monoid. Now pick nonzero opposite vectors $\mathbf{u}$,$\mathbf{v} \in M \cap \ell$. Further, choose $\mathbf{w} \in M\backslash\ell$. Setting $M_1 := [\mathbf{u}, \mathbf{w}] \cap \mathbb{Z}^2$ and $M_2 := [\mathbf{v}, \mathbf{w}] \cap \mathbb{Z}^2$, it is clear that $M = M_1 \cup M_2$. By (1) of Lemma 4, we conclude that both M_1 and M_2 are finitely generated, and hence M too is finitely generated. We have shown that M is a finitely generated valuation monoid. Invoking (6) and (8) of Proposition 1, we deduce that M is a principal ideal monoid, and the proof is complete. □

We now analyze the half open integral half-planes bounded by a line with rational slope.

Proposition 6. *Let M be a half open integral half-plane bounded by $\ell(x) = rx$ for some rational number r. Then M is a reduced valuation monoid whose maximal ideal is principal, yet M does not satisfy acc on ideals. Moreover, M is neither atomic nor inside factorial.*

Proof. Suppose that M is as defined above. It is clear that M is a reduced. As with closed integral half-planes, for any $\mathbf{u} \in \mathbb{Z}^2$, either $\mathbf{u} \in M$ or $-\mathbf{u} \in M$. Thus M is a valuation monoid. Now, the submonoid $M_\ell := M \cap \ell$ is cyclic; say $M_\ell = \mathbb{N}\mathbf{a}$. One checks easily that

$$\text{for any } \mathbf{x} \in M\backslash\{\mathbf{0}\},\ \mathbf{x} - \mathbf{a} \in M. \tag{5.1}$$

From this observation, we deduce that $M + \mathbf{a}$ is the set of nonzero elements of M. Therefore $M + \mathbf{a}$ is the (principal) maximal ideal of M.

Now let $\mathbf{j} \in M\backslash\{\mathbf{0}\}$ be arbitrary. It follows from (5.1) and induction that $\mathbf{j} - n\mathbf{a} \in M$ for every positive integer n. Moreover, one checks at once that $M + (\mathbf{j} - \mathbf{a}) \subsetneq M + (\mathbf{j} - 2\mathbf{a}) \subsetneq M + (\mathbf{j} - 3\mathbf{a}) \subsetneq \cdots$ and M does not satisfy acc on ideals.

It is easy to see that M is not atomic. To wit, M is a reduced valuation monoid. Thus M has at most one atom (in fact M does have one atom, namely $\mathbf{a}$). Since M has rank two, M cannot be atomic.

Finally, we show that M is not inside factorial. Suppose by way of contradiction that there exists a factorial monoid F and divisor map $\varphi\colon F \to M$ such that for every $\mathbf{x} \in M$, there exists some positive integer n such that $n\mathbf{x} \in \varphi(F)$. Since M is reduced, $\varphi(u) = 0$ for all units $u \in F$. But then φ induces an embedding of F_{red} into M defined by $\varphi([x]) := \varphi(x)$. As F is factorial, F_{red} is free. Because F_{red} embeds into M, we see that F_{red} has rank at most two. Therefore, there exist vectors $\mathbf{u}, \mathbf{v} \in M$ such that $\varphi(F) = \varphi(F_{\text{red}}) = \mathbb{N}\mathbf{u} + \mathbb{N}\mathbf{v}$. However, it is easy to see that there exists $\mathbf{w} \in M$ such that $n\mathbf{w} \notin \mathbb{N}\mathbf{u} + \mathbb{N}\mathbf{v}$ for any positive integer n. This contradicts the assumption that M is inside factorial, and the proof is concluded. □

We complete our analysis of integral half-planes bounded by lines with rational slope by considering those which are open.

Proposition 7. *Let M be an open integral half-plane bounded by $\ell(x) = rx$ for some rational number r. Then M is a reduced half-factorial monoid. However, M is neither Schreier, coherent, Krull, nor inside factorial.*

Proof. First, observe that either $(0,1) \in M$ or $(0,-1) \in M$; we may assume without loss of generality that $(0,1) \in M$. We now define a function $D\colon \mathbb{Z}^2 \to \mathbb{Q}$ as follows:

$$D((x,y)) := y - rx. \tag{5.2}$$

Since $(0,1) \in M$, it follows that

$$M = \{\mathbf{u} \in \mathbb{Z}^2\colon D(\mathbf{u}) > 0\} \cup \{\mathbf{0}\}. \tag{5.3}$$

Next, it is obvious that

$$D \text{ is a group homomorphism.} \tag{5.4}$$

Therefore, $D(\mathbb{Z}^2)$ is a finitely generated subgroup of $\mathbb{Q}$. We deduce that

$$D(\mathbb{Z}^2) \text{ is cyclic; let } g \text{ be the positive generator.} \tag{5.5}$$

We now describe the atoms of M as follows:

$$\text{for all } \mathbf{u} \in M, \mathbf{u} \text{ is an atom if and only if } D(\mathbf{u}) = g. \tag{5.6}$$

To prove the forward implication, suppose that $\mathbf{u} \in M$ is an atom, and assume by way of contradiction that $D(\mathbf{u}) \neq g$. Then we deduce from (5.3) and (5.5) that $D(\mathbf{u}) > g$. Let $\mathbf{x} \in M$ be such that $D(\mathbf{x}) = g$. Observe that $D(\mathbf{u}) = D(\mathbf{x}+(\mathbf{u}-\mathbf{x})) =$

$g + D(\mathbf{u} - \mathbf{x})$. Therefore $D(\mathbf{u} - \mathbf{x}) > 0$, and by (5.3) above, $\mathbf{u} - \mathbf{x} \in M$. Since $\mathbf{u}$ is an atom and M is clearly reduced, we conclude that $\mathbf{u} = \mathbf{x}$; hence $D(\mathbf{u}) = D(\mathbf{x}) = g$. We now have a contradiction to our assumption that $D(\mathbf{u}) > g$. Conversely, suppose by way of contradiction that $D(\mathbf{u}) = g$ but $\mathbf{u}$ is not an atom. Then $\mathbf{u} = \mathbf{x} + \mathbf{y}$ for some nonzero $\mathbf{x}, \mathbf{y} \in M$. Thus $g = D(\mathbf{u}) = D(\mathbf{x}) + D(\mathbf{y}) \geq g + g = 2g$, and we have another contradiction. This proves (5.6).

We now prove that M is half-factorial, and begin by showing that M is atomic. To do this, it suffices by (5.3) and (5.5) to prove that for all $n > 0$: if $\mathbf{u} \in M$ and $D(\mathbf{u}) = ng$, then $\mathbf{u}$ is the sum of atoms. The base case of the induction is immediate from (5.6) above. So assume that claim is true for some $n > 0$, and suppose $D(\mathbf{u}) = ng + g$. Let $\mathbf{v} \in M$ be such that $D(\mathbf{v}) = ng$. Now observe that $ng + g = D(\mathbf{u}) = D(\mathbf{v} + (\mathbf{u} - \mathbf{v})) = ng + D(\mathbf{u} - \mathbf{v})$. Therefore, $D(\mathbf{u} - \mathbf{v}) = g$, and we deduce from (5.3) and (5.6) that $\mathbf{u} - \mathbf{v}$ is an atom of M. Since $\mathbf{u} = \mathbf{u} - \mathbf{v} + \mathbf{v}$, we are done by the inductive hypothesis. Now suppose that $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_r = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_s$ and each $\mathbf{u}_i, \mathbf{v}_j$ is an atom. Applying D to both sides of the equation and using (5.6), we see that $rg = sg$, whence $r = s$.

Our next task is to show that M is not Schreier. Toward this end, we note first that M is clearly not finitely generated. Since M is atomic and reduced, it follows that

(5.7) the set $\mathcal{A}$ of atoms is infinite.

Now, let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be distinct atoms. Then $D(\mathbf{v} + \mathbf{w} - \mathbf{u}) = g$. Therefore, $\mathbf{u} | \mathbf{v} + \mathbf{w}$. But since M is reduced and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are atoms, it is clear that $\mathbf{u} \neq \mathbf{x} + \mathbf{y}$ for any $\mathbf{x} | \mathbf{v}$ and $\mathbf{y} | \mathbf{w}$.

To prove that M is not coherent, begin by choosing distinct atoms $\mathbf{u}$ and $\mathbf{v}$ of M. We shall prove that $(M + \mathbf{u}) \cap (M + \mathbf{v})$ is not finitely generated. Suppose by way of contradiction that

$$(M + \mathbf{u}) \cap (M + \mathbf{v}) = (M + \mathbf{x}_1) \cup (M + \mathbf{x}_2) \cdots \cup (M + \mathbf{x}_n) \tag{5.8}$$

for some $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n \in M$. It is patent upon reflection that

$$(M + \mathbf{u}) \cap (M + \mathbf{v}) = \{\mathbf{y} \in M \colon D(\mathbf{y}) \geq 2g\}. \tag{5.9}$$

Since $\mathcal{A}$ is infinite, so is $2\mathcal{A} := \{2\mathbf{a} \colon \mathbf{a} \in \mathcal{A}\}$. Pick an atom $\mathbf{a}$ such that $2\mathbf{a} \notin \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. As $D(2\mathbf{a}) = 2g$, it follows from (5.8) and (5.9) above that $2\mathbf{a} \in M + \mathbf{x}_i$ for some i. But then $D(2\mathbf{a}) = D(\mathbf{x}_i) = 2g$, and we see that $D(2\mathbf{a} - \mathbf{x}_i) = 0$. Applying observation (5.3), we conclude that $2\mathbf{a} - \mathbf{x}_i = 0$, but this contradicts how $\mathbf{a}$ was chosen.

To conclude the proof, we must establish that M is neither Krull nor inside factorial, but this is easy. Since M is not finitely generated, Lemma 5 implies that M is not Krull. The verification that M is not inside factorial is analogous to the proof presented in Proposition 6; as such we omit it. $\square$

Finally, we consider integral half-planes which are bounded by a line with irrational slope.

Proposition 8. *Let M be an integral half-plane bounded by $\ell(x) = \alpha x$, where $\alpha \in \mathbb{R}$ is irrational. Then M is a reduced valuation monoid whose maximal ideal is not finitely generated. Moreover, M is neither atomic nor inside factorial.*

Proof. Consider a line $\ell(x) = \alpha x$, where α is an irrational number. We may assume that ℓ bounds M from below. Analogous to the proof of Proposition 7, $D : \mathbb{Z}^2 \to \mathbb{R}$ defined by $D((x, y)) := y - \alpha x$ is a homomorphism. Since $\{1, \alpha\} \subseteq D(\mathbb{Z}^2)$ is linearly independent over $\mathbb{Q}$, we see that the image of $\mathbb{Z}^2$ under D is a non-cyclic subgroup of $\mathbb{R}$. Therefore,

$$D(\mathbb{Z}^2) \text{ is dense in } \mathbb{R}. \tag{5.10}$$

As in the previous proposition, we have

$$M = \{\mathbf{u} \in \mathbb{Z}^2 \colon D(\mathbf{u}) > 0\} \cup \{\mathbf{0}\}. \tag{5.11}$$

It is easy to see that M is a reduced valuation monoid. Thus to show that the maximal ideal $J = M \backslash \{\mathbf{0}\}$ is not finitely generated, it suffices to show that J is not principal. Toward this end, let $\mathbf{u} \in J$ be arbitrary. Now pick $\mathbf{v} \in J$ such that $D(\mathbf{v}) < D(\mathbf{u})$. Then $\mathbf{v} \notin M + \mathbf{u}$, and J is not principal. A similar argument (invoking (5.10)) shows that every nonzero member of M has infinitely many divisors. Therefore, M has no atoms and is not atomic. The proof that M is not inside factorial proceeds as in the proof of Proposition 6. $\square$

Having completed our analysis of integral half-planes, we proceed to study monoids of the form $C \cap \mathbb{Z}^2$, where C is a cone generated by two linearly independent vectors in $\mathbb{R}^2$. We shall require the following simple lemma.

Lemma 6. *Let $M := C \cap \mathbb{Z}^2$, where C is a cone in $\mathbb{R}^2$ bounded by two linearly independent vectors. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are distinct atoms of M, then $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent over $\mathbb{R}$.*

Proof. Suppose not. Then $\mathbf{a}_1$ and $\mathbf{a}_2$ both lie on a line ℓ through the origin. Note further that since M is root closed, $\ell \cap \mathbb{Z}^2 = \mathbb{Z}\mathbf{m}$ for some $\mathbf{m} \in M$. But since $\mathbf{a}_1$ and $\mathbf{a}_2$ are atoms and M is reduced, this forces $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{m}$, a contradiction. $\square$

Proposition 9. *Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ are linearly independent over $\mathbb{R}$, and set $M := [\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2$. Then M is a reduced finitely generated inside and outside factorial Krull monoid. Moreover, the following are equivalent:*

(1) *M is a free monoid on two generators.*
(2) *M is factorial.*
(3) *M is a GCD monoid.*
(4) *M is Schreier.*

Lastly, M is half-factorial if and only if there is a line which contains all the atoms of M.

Proof. Let M be as stated above. We may assume that

(5.12) $\mathbf{u}$ is a generator of the cyclic group $\mathbb{R}\mathbf{u}\cap\mathbb{Z}^2$ and $\mathbf{v}$ is a generator of $\mathbb{R}\mathbf{v}\cap\mathbb{Z}^2$.

It is clear that M is reduced, and Lemma 4 tells us that M is finitely generated. We now apply Lemma 5 to conclude that M is a Krull monoid. Since M is reduced, there exists a divisor map $\varphi : M \to \mathbb{N}^{(I)}$ for some index set I. But then it is easy to see that $\varphi : M \to \mathbb{Q}_{\geq 0}^{(I)}$ is also a divisor map. Proposition 2 of [7] yields that M is a *rational generalized Krull monoid* (stating this definition would take us too far afield; we refer the interested reader to [7] for details). We now prove that every minimal prime ideal of M is the radical of a principal ideal. Toward this end, let P be a prime ideal of M, and let $p \in P$ be arbitrary (by definition, P is a proper ideal, and thus $p \neq 0$). There exists a positive integer n such that $np \in \mathbb{N}\mathbf{u} + \mathbb{N}\mathbf{v}$. Since P is prime, clearly this implies that $\mathbf{u} \in P$ or $\mathbf{v} \in P$; assume that $\mathbf{u} \in P$. Now,

(5.13) for any $\mathbf{x} \in M\backslash\mathbb{N}\mathbf{v}$, there exists $m \in \mathbb{Z}^+$ such that $m\mathbf{x} \in M + \mathbf{u}$.

But then $\mathbf{x} \in P$, and we have shown that $M\backslash\mathbb{N}\mathbf{v} \subseteq P$. It is not hard to see that $M\backslash\mathbb{N}\mathbf{v}$ is a prime ideal of M. Thus $M\backslash\mathbb{N}\mathbf{u}$ and $M\backslash\mathbb{N}\mathbf{v}$ are the minimal prime ideals of M. It follows from (5.13) above that every minimal prime ideal of M is the radical of a principal ideal (namely, either $M + \mathbf{u}$ or $M + \mathbf{v}$). By Corollary 1 and Theorem 1 of [7], we conclude that M is outside factorial. Invoking Theorem 3 of [7], M is inside factorial as well.

We now prove the equivalence of (1)–(4). Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to prove that (4) implies (1). So assume that M is Schreier. Recall that M is Krull, and thus atomic by (7) of Proposition 1. It follows from (5.12) that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent atoms of M. It suffices to prove that there are no other atoms. Suppose by way of contradiction that $\mathbf{w}$ is an atom distinct from $\mathbf{u}$ and $\mathbf{v}$. There are positive integers a, b, and c such that $a\mathbf{w} = b\mathbf{u}+c\mathbf{v}$. Therefore, $\mathbf{w}|b\mathbf{u}+c\mathbf{v}$. Since $\mathbf{w}$ is an atom and M is reduced and Schreier, we deduce that $\mathbf{w}|b\mathbf{u}$ or $\mathbf{w}|c\mathbf{v}$. Continuing inductively, we see that $\mathbf{w}|\mathbf{u}$ or $\mathbf{w}|\mathbf{v}$. But then $\mathbf{w} = \mathbf{u}$ or $\mathbf{w} = \mathbf{v}$, and this is a contradiction.

As for the final assertion, assume that M is half-factorial. We claim that all atoms are on the line ℓ through $\mathbf{u}$ and $\mathbf{v}$. Suppose that $\mathbf{w}$ is any atom distinct from $\mathbf{u}$ and $\mathbf{v}$. Then as we have seen, $a\mathbf{w} = b\mathbf{u} + c\mathbf{v}$ for some positive integers a,b, and c. Since

M is half-factorial, we deduce that $(b+c)\mathbf{w} = b\mathbf{u} + c\mathbf{v}$. Thus $\mathbf{w} = (1 - \frac{c}{b+c})\mathbf{u} + \frac{c}{b+c}\mathbf{v}$, and the proof of the claim is complete. Conversely, suppose that

$$\text{all atoms are on a line } \ell(t) := \mathbf{x} + t\mathbf{y}. \tag{5.14}$$

As M has at least two atoms (namely, $\mathbf{u}$ and $\mathbf{v}$), and since any two distinct atoms are linearly independent over $\mathbb{R}$ (Lemma 6), it follows that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent over $\mathbb{R}$ as well. Recall above that M is Krull, hence atomic. Assume that $\mathbf{a}_1 + \cdots + \mathbf{a}_r = \mathbf{b}_1 + \cdots + \mathbf{b}_s$ and each $\mathbf{a}_i, \mathbf{b}_j$ is an atom. Then we deduce from (5.14) that there exist real numbers $t_1, \ldots, t_{r+s}$ such that $\mathbf{x} + t_1\mathbf{y} + \cdots + \mathbf{x} + t_r\mathbf{y} = \mathbf{x} + t_{r+1}\mathbf{y} + \cdots + \mathbf{x} + t_{r+s}\mathbf{y}$. Rearranging the algebra, this equation becomes $(r-s)\mathbf{x} = (t_{r+1} - t_1 + \cdots + t_{r+s} - t_r)\mathbf{y}$. Because $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent over $\mathbb{R}$, we deduce that $r = s$. Therefore, M is atomic, and the proof is concluded. □

We now present an example to show that there exist monoids M as in the statement of Proposition 9 which are half-factorial but not factorial.

Example 1. *Let M be the submonoid of $\mathbb{Z}^2$ generated by the atoms $\mathbf{u} := (-1, 1)$, $\mathbf{v} := (1, 1)$, and $\mathbf{w} := (0, 1)$. Then $M = [\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2$. Moreover, it is clear that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are the only atoms of M. Since the atoms are collinear, the previous proposition implies that M is half-factorial. However, $2\mathbf{w} = \mathbf{u} + \mathbf{v}$; therefore M is not factorial.*

We now collect several facts which will allow us to conclude our analysis of root closed monoids M such that $\mathcal{Q}(M)$ is free on two generators.

Proposition 10. *Let C be a cone bounded by two linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ which is not closed, and set $M := C \cap \mathbb{Z}^2$. Then M is atomic, but neither finitely generated, Krull, Schreier, nor inside factorial.*

Proof. First, note that there exist $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ such that M is a submonoid of the monoid $M' := [\mathbf{x}, \mathbf{y}] \cap \mathbb{Z}^2$. Proposition 9 implies that M' is Krull. But M' is reduced, whence there is an embedding of M' into a free monoid, and hence there is an embedding of M into a free monoid. Therefore, M is atomic.

Now, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be arbitrary vectors in M. We shall prove that $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ does not generate M. Since C is not closed, we may assume that $\mathbb{R}\mathbf{v} \cap M = \{\mathbf{0}\}$. For each $\mathbf{x} \in M$, we let $\theta_{\mathbf{x}}$ be the angle between $\mathbf{x}$ and $\mathbf{v}$. Now pick $\mathbf{x} \in M$ such that $\theta_{\mathbf{x}} < \min\{\theta_{\mathbf{x}_1}, \theta_{\mathbf{x}_2}, \ldots, \theta_{\mathbf{x}_n}\}$. It is clear that

$$r\mathbf{x} \notin \mathbb{N}\mathbf{x}_1 + \mathbb{N}\mathbf{x}_2 + \cdots + \mathbb{N}\mathbf{x}_n \text{ for any positive integer } r, \tag{5.15}$$

and this proves that M is not finitely generated (of course, we have proved something even stronger; we will shortly make use of (5.15)). We apply Lemma 5 to conclude that M is not Krull.

We now show that M is not Schreier. Toward this end, suppose by way of contradiction that M is Schreier. Recall that M is atomic but not finitely generated. Thus M possesses infinitely many atoms. Choose distinct atoms $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. By Lemma 6, we see that the angles $\theta_{\mathbf{a}_1}$, $\theta_{\mathbf{a}_2}$, and $\theta_{\mathbf{a}_3}$ are distinct; say $\theta_{\mathbf{a}_1} < \theta_{\mathbf{a}_2} < \theta_{\mathbf{a}_3}$. It follows that there exist positive integers r, s, and t such that $s\mathbf{a}_2 = r\mathbf{a}_1 + t\mathbf{a}_3$. We now obtain a contradiction as in the proof of Proposition 9.

Lastly, we verify that M is not inside factorial. Observe from (5.15) above that for any $\mathbf{x}$ and $\mathbf{y}$ in M, there exists $\mathbf{z} \in M$ such that $n\mathbf{z} \notin \mathbb{N}\mathbf{x} + \mathbb{N}\mathbf{y}$ for any positive integer n. The remainder of the verification unfolds as in the proof of Proposition 6. ☐

It remains to study coherence and half-factoriality for monoids M as defined in the previous proposition.

Proposition 11. *Let C be a cone bounded by linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$, and assume that $\mathbb{R}\mathbf{u} \cap \mathbb{Z}^2 \neq \{\mathbf{0}\}$ and $\mathbf{u} \notin C$. Set $M := C \cap \mathbb{Z}^2$. Then M is not coherent. Moreover, M is half-factorial if and only if the following hold:*

(1) $\mathbb{R}\mathbf{v} \cap M \neq \{\mathbf{0}\}$, *and*
(2) *all atoms of M lie on the line $\ell(t) := \mathbf{a} + t\mathbf{u}$, where $\mathbf{a}$ is the unique atom of M which lies on $\mathbb{R}\mathbf{v}$.*

Proof. By applying a rotation by $\frac{\pi}{2}$ if necessary, we may assume that $\mathbf{u}$ does not lie on the y-axis. Then $\mathbf{v}$ lies in an open integral half-plane H bounded by the line $\mathbb{R}\mathbf{u}$. We may assume without loss of generality that $(0, 1) \in H$. Note trivially that M is a submonoid of H. Further, recall that the function D defined in Proposition 7 has the property that there exists $g \in \mathbb{Q}_{>0}$ such that for all $\mathbf{h} \in H$:

$$\mathbf{h} \text{ is an atom of } H \text{ if and only if } D(\mathbf{h}) = g. \tag{5.16}$$

Note that all atoms of H lie on a line parallel to $\mathbb{R}\mathbf{u}$. Thus there are infinitely many atoms of H which belong to M. Showing that M is not coherent proceeds exactly as in the proof of Proposition 7.

Suppose now that (1) and (2) above hold. Recall from Proposition 10 that M is atomic. Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, clearly the same is true of $\mathbf{u}$ and $\mathbf{a}$. That M is half-factorial is now obvious (the proof is the same as the corresponding proof of Proposition 9).

Conversely, assume that M is half-factorial. We first show that

$$\text{the atoms of } M \text{ are collinear.} \tag{5.17}$$

Toward this end, it suffices to show that every triple of atoms forms a collinear set. Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be arbitrary (distinct) atoms, and assume that $\theta_{\mathbf{a}_1} < \theta_{\mathbf{a}_2} < \theta_{\mathbf{a}_3}$, where $\theta_{\mathbf{a}_i}$ is the angle between $\mathbf{a}_i$ and $\mathbf{u}$ (note that all three angles are distinct by

Lemma 6). One proves as in the proof of Proposition 9 that $\mathbf{a}_2$ is on the line through $\mathbf{a}_1$ and $\mathbf{a}_3$. Now let ℓ be the line containing all the atoms of M. Then

$$\ell \text{ does not pass through the origin,} \tag{5.18}$$

lest there exist two atoms of M which are linearly dependent, contradicting Lemma 6. We now claim that

$$\ell \cap \{t\mathbf{u} : t > 0\} = \varnothing. \tag{5.19}$$

Suppose by way of contradiction that $\ell \cap \{t\mathbf{u} : t > 0\} \neq \varnothing$. Choose an atom $\mathbf{a} \in M$ for which the angle θ between $\mathbf{a}$ and $\mathbf{u}$ is minimal (recall that $\mathbf{u} \notin C$, and therefore $\theta > 0$). Now pick $\mathbf{m} \in M \backslash \{\mathbf{0}\}$ such that the angle β between $\mathbf{m}$ and $\mathbf{u}$ is less than θ. Then $\mathbf{m}$ is not a sum of atoms, and this contradicts the fact that M as atomic. Finally, we deduce that

$$\ell \cap \{t\mathbf{v} : t > 0\} \neq \varnothing. \tag{5.20}$$

It follows by Lemma 6 and the argument just given that there is a unique atom $\mathbf{a} \in \ell \cap \{t\mathbf{v} : t > 0\}$ (lest M not be atomic). To finish the proof, it suffices to show that ℓ is parallel to $\mathbf{u}$. If not, then either there is an element of M (infinitely many, actually) which is not the sum of atoms (contradicting that M is atomic), or $\ell \cap \{t\mathbf{u} : t > 0\} \neq \varnothing$ (contradicting (5.19)). The proof is complete. □

We now present an example showing that there exist half-factorial monoids M as considered in the previous proposition.

Example 2. *Let* $\mathbf{u} := (0,1)$ *and* $\mathbf{v} := (1,0)$*. Further, consider the monoid* $M := (\mathbf{u}, \mathbf{v}] \cap \mathbb{Z}^2$*. Then the set* $\mathcal{A}$ *of atoms of* M *is given by* $\mathcal{A} = \{(1,m) : m \in \mathbb{N}\}$*.*

Finally, we consider monoids for which a bounding ray has irrational slope. We shall require a lemma.

Lemma 7. *Consider the lines* $\ell_1(x) := \alpha x + b_1$ *and* $\ell_2(x) := \alpha x + b_2$*, where* $b_1 < b_2$ *are real numbers and* $\alpha \in \mathbb{R}$ *is irrational. Then for any* $r \in \mathbb{R}$*, there exists* $(m,n) \in \mathbb{Z}^2$ *with* $r < m$ *such that*

$$m\alpha + b_1 < n < m\alpha + b_2. \tag{5.21}$$

Moreover, there exists a pair (m,n) *satisfying* (5.21) *above with* $m < r$*.*

Proof. Assume that ℓ_1 and ℓ_2 are as defined above, and let $r \in \mathbb{R}$ be arbitrary. We shall prove the existence of a pair (m,n) with $r < m$ satisfying (5.21) (the existence of such a pair with $m < r$ follows by a similar argument). Clearly, we may assume that $0 < b_2 - b_1 < 1$ and $r > 0$. For a real number x, let $fr(x) \in [0,1)$ denote the fractional part of x. The group $G := \{n + m\alpha : m, n \in \mathbb{Z}\}$ is non-cyclic, thus dense in $\mathbb{R}$. Therefore, the set $\{fr(m\alpha) : m \in \mathbb{Z}\}$ is dense in $(0, 1)$. Then of course $\{fr(m\alpha) : m \in \mathbb{Z}^+\}$ is also dense in $(0, 1)$. Since there are but finitely many positive integers m satisfying $m \leq r$, we deduce that $\{fr(m\alpha) : m > r\}$ is dense in $(0, 1)$. Choose $m > r$ such that $fr(m\alpha) > 1 - (b_2 - b_1)$. Then $fr(m\alpha) + (b_2 - b_1) > 1$, and we obtain n satisfying (5.21). This completes the proof. □

We arrive at the final proposition of this article.

Proposition 12. *Let C be a cone bounded by linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$, and assume that $\mathbb{R}\mathbf{u} \cap \mathbb{Z}^2 = \{\mathbf{0}\}$. Set $M := C \cap \mathbb{Z}^2$. Then M is neither half-factorial nor coherent.*

Proof. Let M be as stated above. Since $\mathbb{R}\mathbf{u} \cap \mathbb{Z}^2 = \{\mathbf{0}\}$, it follows that

(5.22) the line $\mathbb{R}\mathbf{u}$ has irrational slope.

Suppose by way of contradiction that M is half-factorial. Then one shows as in the proof of Proposition 9 that all atoms of M lie on a line ℓ. It follows from the argument given in the proof of the previous proposition that ℓ is parallel to the line $\mathbb{R}\mathbf{u}$. But this is impossible since ℓ has rational slope (since it contains multiple lattice points), yet (by (5.22)) $\mathbb{R}\mathbf{u}$ does not.

We now prove that M is not coherent. Toward this end, we shall require additional notation. If $\mathbf{x} \in \mathbb{R}^2$, then let $\vec{\mathbf{x}} := \{t\mathbf{x} : t \geq 0\}$ denote the ray in the direction of $\mathbf{x}$. For $\mathbf{y} \in \mathbb{R}^2$, set $\vec{\mathbf{x}} + \mathbf{y} := \{t\mathbf{x} + \mathbf{y} : t \geq 0\}$. Proposition 1 and Proposition 10 imply that M is not a valuation monoid. Choose $\mathbf{a}, \mathbf{b} \in M$ such that $\mathbf{a} \nmid \mathbf{b}$ and $\mathbf{b} \nmid \mathbf{a}$. Then it is easy to see that there exists $\mathbf{c} \in \mathbb{R}^2$ such that either $(\vec{\mathbf{u}} + \mathbf{a}) \cap (\vec{\mathbf{v}} + \mathbf{b}) = \{\mathbf{c}\}$ or $(\vec{\mathbf{u}} + \mathbf{b}) \cap (\vec{\mathbf{v}} + \mathbf{a}) = \{\mathbf{c}\}$. We assume without loss of generality that

(5.23) $$(\vec{\mathbf{u}} + \mathbf{a}) \cap (\vec{\mathbf{v}} + \mathbf{b}) = \{\mathbf{c}\}.$$

We will show that the ideal $I := (M + \mathbf{a}) \cap (M + \mathbf{b})$ is not finitely generated. Note first that

(5.24) I consists of those lattice points bounded by the rays $\vec{\mathbf{u}} + \mathbf{c}$ and $\vec{\mathbf{v}} + \mathbf{c}$.

(whether the cone C is open or closed at $\mathbf{v}$ is irrelevant to the proof). Second, observe that since $\mathbf{b} \nmid \mathbf{a}$,

(5.25) $$\mathbf{a} \notin I.$$

Now, the line $\mathbb{R}\mathbf{u} + \mathbf{a}$ contains $\mathbf{a}$ as its *only* lattice point, lest we have a contradiction to (5.22). We conclude from (5.23) that $\mathbb{R}\mathbf{u} + \mathbf{c} \subseteq \mathbb{R}\mathbf{u} + \mathbf{a}$;

$$(5.26) \qquad (\mathbb{R}\mathbf{u} + \mathbf{c}) \cap \mathbb{Z}^2 = \{\mathbf{a}\}$$

is now patent. Suppose by way of contradiction that $I = (M + \mathbf{x}_1) \cup (M + \mathbf{x}_2) \cup \cdots \cup (M + \mathbf{x}_n)$ for some $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in I$. It follows from (5.25) and (5.26) above that $\mathbf{x}_i \notin \mathbb{R}\mathbf{u} + \mathbf{c}$ for all i. Now pick j such that the distance from $\mathbb{R}\mathbf{u} + \mathbf{x}_j$ to $\mathbb{R}\mathbf{u} + \mathbf{c}$ is minimal. Applying Lemma 7 and observation (5.24), we obtain a member of I which is not in any $M + \mathbf{x}_i$, and the proof is complete. □

The previous proposition completes our analysis of root-closed monoids M such that $\mathcal{Q}(M)$ is free on two generators. To see why this is so, let M be such a monoid. We may assume without loss of generality that $\mathcal{Q}(M) = \mathbb{Z}^2$. It is now clear from (2) of Lemma 4 that M must be one of the following monoids:

(a) $\mathbb{Z}^2$,
(b) an integral half-plane, or
(c) $C \cap \mathbb{Z}^2$ for some cone C bounded by linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$.

6. Directions for further research

We conclude the paper by outlining potential extensions of the work presented in this note. As stated in the introduction, the purpose of this article is to present evidence that a geometric analysis may yield nontrivial factorization-theoretic results for root-closed monoids of finite rank. The canonical next step in this analysis is to complete the study of root closed monoids of rank two by examining torsion-free rank two monoids M such that $\mathcal{Q}(M)$ is not discrete, using the strategy of Section 5 as a template. Moreover, it would be interesting to see how the results of this paper generalize to higher dimensions. Finally (as stated in Section 2), the list of ideal and factorization-theoretic properties considered in this note is far from exhaustive. We invite the motivated reader to expand upon the topics treated herein.

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