INFINITE SUMS IN TOTALLY ORDERED ABELIAN GROUPS

GREG OMAN, CAITLIN RANDALL, AND LOGAN ROBINSON

Abstract. The notion of convergence is absolutely fundamental in the study of the calculus. In particular, it enables one to define the sum of certain infinite sets of real numbers as the limit of a sequence of partial sums, thus obtaining so-called *convergent series*. Convergent series, of course, play an integral role in real analysis (and, more generally, functional analysis) and the theory of differential equations. An interesting textbook problem is to show that there is no canonical way to "sum" uncountably many positive real numbers to obtain a finite (i.e. real) value. Plenty of solutions to this problem, which make strong use of the completeness property of the real line, can be found both online and in textbooks. In this note, we show that there is a more general reason for the non-finiteness of uncountable sums. In particular, we present a canonical definition of "convergent series", valid in any totally ordered abelian group, which extends the usual definition encountered in elementary analysis. We prove that there are convergent real series of positive numbers indexed by an arbitrary countable well-ordered set and, moreover, that any convergent series in a totally ordered abelian group indexed by an arbitrary well-ordered set has but countably many nonzero terms.

1. Introduction

The set $\mathbb{R}$ of real numbers is rich in both algebraic and topological structure. For example, the usual addition $+$ of real numbers is a continuous binary operation on $\mathbb{R}$. This enables one to define the sum $\sum_{i=0}^{n} r_i$ of real numbers $r_0, \ldots, r_n$ for any non-negative integer n by recursion as follows:

$$\sum_{i=0}^{0} r_i := r_0, \text{ and for } 0 \leq j < n, \sum_{i=0}^{j+1} r_i := \left(\sum_{i=0}^{j} r_i\right) + r_{j+1}. \tag{1.1}$$

Observe that it is quite difficult to formulate a *natural* definition of an infinite sum of real numbers by appealing solely to the usual algebraic (field) axioms of $+$ and $\times$; one wants some notion of "getting close to". This is usually formalized topologically. The relevant topology on $\mathbb{R}$ is the order topology determined by the usual complete linear order $<$ on $\mathbb{R}$. In this setting, there is a canonical way to formalize an infinite list of real numbers "getting arbitrarily close to" another real number; this familiar notion is often presented in a first course in calculus, and is instrumental in defining convergence of so-called infinite series. More formally, a countably infinite real sequence $(r_n : n \in \mathbb{N})$ *converges* to a real number r provided that

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(1.2) $\quad$ for every $\epsilon > 0,$ there exists $k \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq k,$ then $|r_n - r| < \epsilon$.[1]

In this case, r is called the *limit*[2] of the sequence $(r_n : n \in \mathbb{N})$, and we write

$$\lim_{n\to\infty} r_n = r. \tag{1.3}$$

We now give the usual definition of a convergent series encountered in calculus:

Definition 1. Suppose that $(r_n : n \in \mathbb{N})$ is a countably infinite sequence of real numbers. For each $n \in \mathbb{N}$, set $S_n := \sum_{i=0}^{n} r_i$. The sequence $(S_n : n \in \mathbb{N})$ is called the *infinite series determined by* $(r_n : n \in \mathbb{N})$, denoted $\sum_{n=0}^{\infty} r_n$. For $n \in \mathbb{N}$, S_n is the *nth partial sum of the series*. Finally, if the sequence $(S_n : n \in \mathbb{N})$ converges to a real number r, then we say that the series $\sum_{n=0}^{\infty} r_n$ converges and that the *sum* of the series $\sum_{n=0}^{\infty} r_n$ is r, denoted $\sum_{n=0}^{\infty} r_n = r$.

We now work toward generalizing the above definition to formalize the notion of a possibly uncountable sum[3] in a natural way. Toward this end, consider a collection $\mathcal{S} := \{r_i : i \in I\}$ of real numbers, where I is an infinite index set. We want to define the series determined by $\mathcal{S}$. In what ways might we proceed? Recall that a real series $\sum_{n=0}^{\infty} r_n$ *converges absolutely* if the series $\sum_{n=0}^{\infty} |r_n|$ converges. If a series converges but does not converge absolutely, then we say the series *converges conditionally.* Next, we recall the following classical result of Riemann:

Theorem 1 (Rearrangement Theorem; see [7], Theorem 8.8.9)**.** *Suppose that $\sum_{n=0}^{\infty} r_n$ converges conditionally but not absolutely, and let r be an arbitrary real number. Then there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that the rearranged series $\sum_{n=0}^{\infty} r_{f(n)}$ converges to r.*

The Rearrangement Theorem implies that, in general, the sum of a convergent series is determined not only by the terms being summed, but also the order in which they are added. Therefore, it is reasonable to assume that there is some order relation $<$ on the set I, where (above) $\mathcal{S} = \{r_i : i \in I\}$ is the set whose sum we have yet to define. But what kind of order relation? Well, recall that our definition of "nth partial sum" in Definition 1 above implicitly employed the Recursion Theorem on the index set $\mathbb{N}$ of natural numbers. It is reasonable that we may want to employ the so-called Transfinite Recursion Theorem in our definition of an uncountable sum of real numbers (more generally, an uncountable sum of elements of a totally ordered abelian group). But if this is indeed the case, we would like a well-order on I.[4]

The outline of this note is as follows. In the next section, give a precise formulation of a sum of real numbers relative to a well-ordered *countable* index set which extends Definition 1 above. We

[1] We assume that $0 \in \mathbb{N}$ throughout this note.

[2] Since $\mathbb{R}$ is dense in itself under $<$, it is easy to see that a convergent sequence has a unique limit.

[3] Various such notions exist in the literature; more on this shortly. Though our initial discussion will involve uncountable sums of real numbers, we will soon transition to the more general setting of totally ordered abelian groups.

[4] Very roughly (though definitely *not* exactly), the well-order is what enables one to prove the existence of recursively defined functions.

then establish the existence of convergent sums of positive real numbers relative to an arbitrary countable well-ordered set. Section 3 is devoted to extending some notions of "uncountable sum" appearing in the literature. In particular, we show that in any totally ordered abelian group, every convergent series indexed by an uncountable well-ordered set (this definition is forthcoming) has but countably many nonzero terms.

2. Countable Sums of Positive Reals

2.1. **Order-Theoretic Terminology Review.** We begin with a review of some basic order-theoretic terminology which will be utilized often throughout this note. To wit, let S be a set. Recall that a *binary relation* on S is simply a subset of $S \times S$. If R is a binary relation on S and $(a, b) \in R$, then it is customary to write aRb to denote this fact. Next, recall that if R is a binary relation on S, then R is *irreflexive* on S if xRx holds for no $x \in S$. If for all $x, y, z \in S$: xRy and yRz imply xRz, then we say that R is *transitive* on S. A relation on S which is both irreflexive and transitive on S is called a *partial order* on S. A partial order $<$ on S for which either $x < y$ or $y < x$ for distinct $x, y \in S$ is called a *total order* or a *linear order* on S. If $<$ is a total order on S with the property that for every nonempty $X \subseteq S$, there exists $x_0 \in X$ such that $x_0 \leq x$ for all $x \in X$, then we say that $<$ is a *well-order* on S.[5] Finally, suppose that $\mathbf{P}_1 := (P_1, <_1)$ and $\mathbf{P}_2 := (P_2, <_2)$ are totally ordered sets. A one-to-one function $f\colon P_1 \to P_2$ with the property that $x <_1 y$ implies $f(x) <_2 f(y)$ for all $x, y \in P_1$ is said to be an *order embedding* of $\mathbf{P}_1$ into $\mathbf{P}_2$.[6] If in addition f is bijective, then f is called an *order isomorphism* between $\mathbf{P}_1$ and $\mathbf{P}_2$. If such an isomorphism exists, we write $\mathbf{P}_1 \cong \mathbf{P}_2$, and say that $\mathbf{P}_1$ and $\mathbf{P}_2$ are *order isomorphic*. We conclude this subsection with several natural examples.

Example 1. *Let S be a set and let $\mathcal{P}(S)$ be the power set of S. Then the usual proper subset relation $\subsetneq$ is a partial order on $\mathcal{P}(S)$.*

Example 2. *Suppose that $(X, <_1)$ and $(Y, <_2)$ are linearly ordered sets. Then the order $<$ on $X \times Y$ defined by $(x_1, y_1) < (x_2, y_2)$ if and only if $x_1 <_1 x_2$ or $x_1 = x_2$ and $y_1 <_2 y_2$ is a total order on $X \times Y$, called the* ***dictionary order*** *on $X \times Y$.*

Example 3. *The usual order $<$ on $\mathbb{N}$ is a well-order on $\mathbb{N}$.*

2.2. **Defining a Sum of Reals Indexed by a Countable Well-Ordered Set.** Let $\mathbf{W} := (W, <)$ be a well-ordered set. Let us call a function $f\colon W \to \mathbb{R}$ a *real-valued W-sequence.* We shall often denote such a sequence by $(r_i\colon i \in W)$. Mimicking (1.2), the following is a natural notion of convergence[7]:

Definition 2. Suppose that $\mathbf{W} := (W, <)$ is a nonempty well-ordered set, and let $(r_i\colon i \in W)$ be a real-valued W-sequence. Say that $(r_i\colon i \in W)$ *converges* to a real number r (or that r is the *limit* of

[5] As usual, the notation $x_0 \leq x$ abbreviates "$x_0 < x$ or $x_0 = x$."

[6] Observe that if $<_1$ is a total order, then any f with this property is necessarily injective.

[7] The reader may notice that our definition of convergence resembles the definition of a real-valued net in the order topology on $\mathbb{R}$. This is certainly the case, but in the interest of keeping the paper as self-contained as possible, we shall say no more about nets. We refer the reader instead to the popular text [5] for further reading.

the sequence $(r_i \colon i \in W)$ provided that for every $\epsilon > 0$, there exists $i \in W$ such that if $j \in W$ and $j \geq i$, then $|r_j - r| < \epsilon$.[8] If $(r_i \colon i \in W)$ converges to r, we shall denote this by $\varinjlim(r_i \colon i \in W) = r$. In case $W = \varnothing$, we set $\varinjlim(r_i \colon i \in W) := 0$.

As in the case where $(W, <) = (\mathbb{N}, <)$, a real-valued W-sequence can converge to at most one limit; the proof is goes through *mutatis mutandis* as follows: suppose by way of contradiction that $(r_i \colon i \in W)$ converges to real numbers r and s with $r < s$. The open balls $B_{\frac{s-r}{2}}(r)$ and $B_{\frac{s-r}{2}}(s)$ are disjoint, yet it is clear from the definition of convergence that there must exist a real number in both open balls, a contradiction.

We now make a trivial yet useful observation.

Lemma 1. *Suppose that $(W, <)$ is a well-ordered set. Suppose, moreover, that W has a largest element w^* relative to $<$. Then every real-valued W-sequence $(r_i \colon i \in W)$ converges to r_{w^*}.*

Proof. Let $(W, <)$ and w^* be as stated, and consider a real-valued W-sequence $(r_i \colon i \in W)$. Let $\epsilon > 0$ be given. Then note that if $w \in W$ and $w \geq w^*$, then $w = w^*$. Consequently, if $w \geq w^*$, then $|r_w - r_{w^*}| = |r_{w^*} - r_{w^*}| = 0 < \epsilon$. $\square$

Next, we address the problem of formalizing a sum of real numbers indexed by a countable well-ordered set. Toward this end, we use Definition 1 as a template, but choose a countable well-ordered index set W which is ordered much differently than $\mathbb{N}$ in order to guide us to a "natural" definition of a sum of reals relative to an arbitrary countable well-ordered index set. To wit, consider the set $\mathbb{N} \times \mathbb{N}$ of ordered pairs of natural numbers. Recall from Example 2 that the dictionary order $<$ is a total order on $\mathbb{N} \times \mathbb{N}$. We claim that it is a well-order. To see this, suppose that $S \subseteq \mathbb{N} \times \mathbb{N}$ is nonempty. Let $a_0 \in \mathbb{N}$ be least (relative to the usual order on $\mathbb{N}$) such that there exists $b \in \mathbb{N}$ with $(a_0, b) \in S$. Now let $b_0 \in \mathbb{N}$ be least such that $(a_0, b_0) \in S$. It is easy to see that (a_0, b_0) is the $<$-least element of S, and thus $<$ well-orders $\mathbb{N} \times \mathbb{N}$. We pause to contrast the "shapes" of the well-ordered structures $(\mathbb{N}, <)$ and $(\mathbb{N} \times \mathbb{N}, <)$; see Figure 1. Observe that the dictionary order on $\mathbb{N} \times \mathbb{N}$ is, as an ordered structure, obtained by simply laying off countably infinitely many copies of $\mathbb{N}$ from left to right.

Before defining a convergent series of reals relative to a countable well-ordered index set, we recall that if $(W, <)$ is a well-ordered set and $w \in W$, then $\operatorname{seg}(w) := \{x \in W \colon x < w\}$. Observe that $\operatorname{seg}(w)$ is well-ordered via the well-order $<$ restricted to $\operatorname{seg}(w)$. We call $\operatorname{seg}(w)$ the *initial segment up to w*.

We now motivate our "bottom-up" definition of a convergent series of real numbers relative to $(\mathbb{N} \times \mathbb{N}, <)$, where $<$ is the dictionary order on $\mathbb{N} \times \mathbb{N}$. From this, we shall derive a natural definition of convergence of a sum of reals relative to an arbitrary countable well-ordered index set. Suppose that $(r_i \colon i \in \mathbb{N} \times \mathbb{N})$ is a real-valued $\mathbb{N} \times \mathbb{N}$-sequence. We may define finite partial sums recursively as before:

(1) $S_{(0,0)} := r_{(0,0)} = 0 + r_{(0,0)} =$ (by Definition 2) $\varinjlim(S_i \colon i \in \operatorname{seg}((0,0)) + r_{(0,0)}$.

[8] By abuse of notation, we use the symbol $<$ to denote both the order on W and the usual order on $\mathbb{R}$.

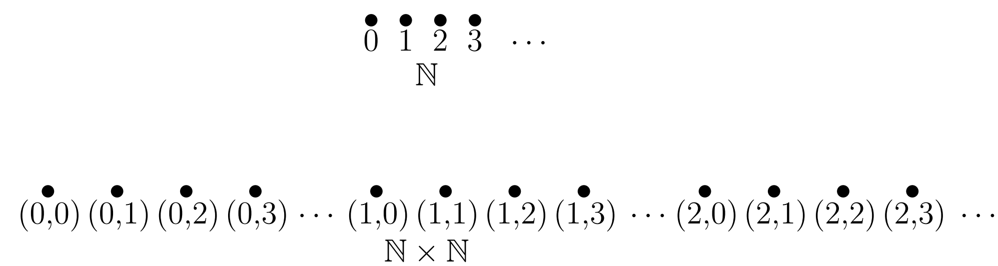

Figure 1. The set $\mathbb{N}$ of natural numbers relative to the usual order (above) and the set $\mathbb{N} \times \mathbb{N}$ with the dictionary order (below).

(2) $S_{(0,1)} := S_{(0,0)} + r_{(0,1)} =$ (by Lemma 1) $\underrightarrow{\lim}(S_i\colon i \in \text{seg}((0,1)) + r_{(0,1)}$.

(3) $S_{(0,2)} := S_{(0,1)} + r_{(0,2)} = \underrightarrow{\lim}(S_i\colon i \in \text{seg}((0,2)) + r_{(0,2)}$.

$\vdots$

Clearly we may continue recursively to define $S_{(0,n)}$ for every natural number n. Our "next" partial sum to define is thus $S_{(1,0)}$ (observe that from Figure 1, $(1,0)$ is the "next" element which appears after all of the pairs $(0,n)$). Intuitively, we want $S_{(1,0)}$ to be the sum of all of the r_i's, where $i \leq (1,0)$. Assuming the limit (4) below exists, we naturally define

(4) $S_{(1,0)} := \underrightarrow{\lim}(S_i\colon i \in \text{seg}((1,0)) + r_{(1,0)}$.

(5) $S_{(1,1)} := S_{(1,0)} + r_{(1,1)} = \underrightarrow{\lim}(S_i\colon i \in \text{seg}((1,1)) + r_{(1,1)}$.

(6) $S_{(1,2)} := S_{(1,1)} + r_{(1,2)} = \underrightarrow{\lim}(S_i\colon i \in \text{seg}((1,2)) + r_{(1,2)}$.

$\vdots$

Continuing in this manner, we may define what it means for the series $(S_i\colon i \in \mathbb{N} \times \mathbb{N})$ to converge. Roughly, this simply means that the limits defined above exist at each "stage" and that, finally, $\underrightarrow{\lim}(S_i\colon i \in \mathbb{N} \times \mathbb{N})$ exists (as a real number). With these observations in mind, we make the following definition:

Definition 3. Suppose that $(W, <)$ is a countable, nonempty well-ordered set and that $(r_i\colon i \in W)$ is a real-valued W-sequence. Choose any $e \notin \mathbb{R}$. We now present our definition of *the series*

$(S_i \colon i \in W)$ *determined by* $(r_i \colon i \in W)$ by recursion on W.[9] Suppose that $i \in W$ and that S_j has been defined for every $j < i$. Next, define S_i as follows:[10]

$$S_i := \begin{cases} e & \text{if } S_j = e \text{ for some } j < i \text{ or } \varinjlim(S_j \colon j \in \operatorname{seg}(i)) \text{ does not exist;} \\ \varinjlim(S_j \colon j \in \operatorname{seg}(i)) + r_i & \text{otherwise.} \end{cases}$$

We say that the series $(S_i \colon i \in W)$ *converges* provided that both $S_i \in \mathbb{R}$ for each $i \in W$ and $\varinjlim(S_i \colon i \in W) \in \mathbb{R}$. In this case, we say that S is the *sum* of the series $(S_i \colon i \in W)$.

2.3. **Countable Sums of Positive Real Numbers.** Our next goal is to show that for every countable well-ordered set $(W, <)$, there is a positive real-valued W-sequence $(r_i \colon i \in W)$ such that the corresponding series $(S_i \colon i \in W)$ converges. We require another definition.

Definition 4. Let $(L, <)$ be a linearly ordered set.

(1) $(L, <)$ is *dense* (in itself) if for any $x, z \in L$ such that $x < z$, there exists $y \in L$ such that $x < y < z$.
(2) $(L, <)$ is *without endpoints* if for every $y \in L$, there exist $x, z \in L$ such that $x < y < z$.

We now recall a result of Cantor which we shall soon utilize.

Lemma 2 ((Cantor); see [2], Theorem 26H)**.** *Any two nonempty, countable, dense linearly ordered sets without endpoints are isomorphic.*

Our next result is well-known. We give a short proof.

Lemma 3. *If $(W, <)$ is a countable well-ordered set, then there is $X \subseteq (0, 1)$ such that $(W, <) \cong (X, <)$, where the second occurrence of $<$ denotes the usual order on $\mathbb{R}$ restricted to X.*

Proof. Let $(W, <)$ be a countable well-ordered set. If $W = \varnothing$, the result is patent, so assume that W is nonempty. Now consider the set $W \times \mathbb{Q}$ with the dictionary order. It is clear that $w \mapsto (w, 0)$ is an order embedding of W into $W \times \mathbb{Q}$. It is also easy to verify that $W \times \mathbb{Q}$ is a dense linearly ordered set without endpoints (relative to the dictionary order). By Lemma 2, there is an order isomorphism $\varphi \colon W \times \mathbb{Q} \to \mathbb{Q}$. Let $i \colon \mathbb{Q} \hookrightarrow \mathbb{R}$ be the inclusion map. The function $\psi(x) := \frac{\tanh(x)+1}{2}$ yields an order isomorphism between the ordered structures $(\mathbb{R}, <)$ and $((0, 1), <)$. Composing the above maps furnishes us with a sequence of order embeddings

$$W \to W \times \mathbb{Q} \to \mathbb{Q} \to \mathbb{R} \to (0, 1).$$

Now let X be the image of W in the above composition. Then clearly $(W, <) \cong (X, <)$, concluding the argument. □

[9] We are employing the Transfinite Recursion Theorem implicitly; see [2] or [4] for the precise statement of the theorem and its proof.

[10] The reader should interpret $S_i = e$ to mean, intuitively, that the ith partial sum of the series doesn't exist.

Recall that a countably infinite real-valued sequence $(r_n : n \in \mathbb{N})$ is *bounded* if there is $M \in \mathbb{R}$ such that $|r_n| \leq M$ for every $n \in \mathbb{N}$. The natural generalization is:

Definition 5. Let $(W, <)$ be a well-ordered set, and let $(r_i : i \in W)$ be a real-valued W-sequence. Say that $(r_i : i \in W)$ is *bounded* provided there is a real number M such that $|r_i| \leq M$ for every $i \in W$.

Our next definition generalizes the notion of a countably infinite monotonic real-valued sequence.

Definition 6. Let $(W, <)$ be a well-ordered set, and let $(r_i : i \in W)$ be a real-valued W-sequence.

(1) $(r_i : i \in W)$ is *monotonically increasing* if whenever $i, j \in W$ and $i < j$, then $r_i \leq r_j$.
(2) $(r_i : i \in W)$ is *monotonically decreasing* if whenever $i, j \in W$ and $i < j$, then $r_i \geq r_j$.
(3) If $(r_i : i \in W)$ is either monotonically increasing or monotonically decreasing, then we say that $(r_i : i \in W)$ is *monotonic.*

We now generalize the familiar result that every bounded monotonic countably infinite real sequence converges. The proof is a *mutatis mutandis* adaptation of the proof in the countably infinite case.

Lemma 4. *Let $(W, <)$ be a well-ordered set, and suppose that $(r_i : i \in W)$ is a monotonic real-valued bounded W-sequence. Then $(r_i : i \in W)$ converges. Moreover, if $W \neq \varnothing$, then $(r_i : i \in W)$ converges to* $\sup\{r_i : i \in W\}$ *if $(r_i : i \in W)$ is increasing. Respectively, $(r_i : i \in W)$ converges to* $\inf\{r_i : i \in W\}$ *if $(r_i : i \in W)$ is decreasing.*

Proof. Let $(W, <)$ be a well-ordered set, and assume that $(r_i : i \in W)$ is real-valued, monotonic, and bounded. We assume that $(r_i : i \in W)$ is monotonically increasing, as a similar argument handles the case where $(r_i : i \in W)$ is monotonically decreasing. If $W = \varnothing$, then $(r_i : i \in W)$ converges to 0 by Definition 2. Thus we assume that W is nonempty. Let M be the least upper bound of $\{r_i : i \in W\}$. We claim that $\lim_{\rightarrow}(r_i : i \in W) = M$. To see this, let $\epsilon > 0$ be given. Then $M - \epsilon$ is not an upper bound of $\{r_i : i \in W\}$. Hence there is $i \in W$ such that $M - \epsilon < r_i$. Now if $j \in W$ and $j \geq i$, then $M - \epsilon < r_i \leq r_j \leq M < M + \epsilon$. Therefore, $|r_j - M| < \epsilon$, as required. □

We are almost equipped to present our first theorem. Our proof will rely upon the Principle of Transfinite Induction. Recall first the Principle of Strong Induction for the set $\mathbb{N}$ of natural numbers:

Principle of Strong Induction on $\mathbb{N}$. Suppose that $S \subseteq \mathbb{N}$ has the property that for every $n \in \mathbb{N}$: if every natural number less than n is in S, then $n \in S$. Then $S = \mathbb{N}$.

A more general version of the above induction principle holds for any well-ordered set. As a bonus, it is quite easy to prove.

Principle of Transfinite Induction. Let $(W, <)$ be a well-ordered set. Suppose further that $S \subseteq W$ has the property that for every $w \in W$: if $\text{seg}(w) \subseteq S$, then $w \in S$. Then $S = W$.

Proof. Suppose that $(W, <)$ is a well-ordered set and that $S \subseteq W$ has the above property. Assume by way of contradiction that $S \neq W$. Then $W \backslash S$ is nonempty; let $w \in W \backslash S$ be least. By leastness of w, $\mathrm{seg}(w) \subseteq S$. But then by the condition on S, $w \in S$. This contradiction concludes the proof. □

We are now sufficiently equipped to prove the main result of this section.

Theorem 2. *Let $(W, <)$ be a countable, nonempty well-ordered set. Then there is a positive, real-valued W-sequence $(r_i : i \in W)$ such that the corresponding series $(S_i : i \in W)$ converges.*

Proof. Let $(W, <)$ be an arbitrary countable, nonempty well-ordered set. By Lemma 3, there is a sequence $(x_i : i \in W)$ in the open unit interval $(0, 1)$ such that

$$\text{for all } i, j \in W : i < j \text{ iff } x_i < x_j. \tag{2.1}$$

Since W is countable, there is an injective map $f : W \to \mathbb{N}$. For each $i \in W$, set

$$T_i := x_i + \sum_{j \leq i} \frac{1}{2^{f(j)}}. \tag{2.2}$$

Clearly each $T_i \in \mathbb{R}$. Now suppose $i, j \in W$ with $i < j$. Then by (2.1),

$$x_i < x_j. \tag{2.3}$$

Moreover, since $i < j$,

$$\sum_{k \leq i} \frac{1}{2^{f(k)}} < \sum_{k \leq j} \frac{1}{2^{f(k)}}. \tag{2.4}$$

It is immediate from (2.3) and (2.4) that $T_i < T_j$. Summarizing,

$$\text{for all } i, j \in W : i < j \text{ iff } T_i < T_j. \tag{2.5}$$

Recall above that for each $i \in W$, $0 < x_i < 1$. It is immediate from the definition of T_i that $0 < T_i < 3$ for all $i \in W$. Therefore, $(T_i : i \in W)$ is monotonically increasing and bounded, hence converges by Lemma 4. Even more, for any $i \in W$, $(T_j : j \in \mathrm{seg}(i))$ is also monotonically increasing and bounded. Summarizing, we have

$$(T_i : i \in W) \text{ converges. Moreover,} \tag{2.6}$$

$$(T_j : j \in \mathrm{seg}(i)) \text{ converges for every } i \in W. \tag{2.7}$$

Our next claim is that

$$\lim_{\to}(T_j\colon j \in \operatorname{seg}(i)) < T_i \text{ for all } i \in W. \tag{2.8}$$

To see this, let $i \in W$ be arbitrary. If i is the least element of W, then the result is patent. So let us assume this is not the case. Let $j \in \operatorname{seg}(i)$ be arbitrary. Since $j < i$, also $x_j < x_i$ (see (2.1)). Observe too that $\sum_{k\le j} \frac{1}{2^{f(k)}} \le \sum_{k<i} \frac{1}{2^{f(k)}}$. Therefore, $x_j + \sum_{k\le j} \frac{1}{2^{f(k)}} < x_i + \sum_{k<i} \frac{1}{2^{f(k)}}$. Set $M := x_i + \sum_{k<i} \frac{1}{2^{f(k)}}$. We have shown that

$$T_j < M \text{ for all } j \in \operatorname{seg}(i). \tag{2.9}$$

Finally,

$$\lim_{\to}(T_j\colon j \in \operatorname{seg}(i)) = \sup\{T_j\colon j \in \operatorname{seg}(i)\} \le M < T_i, \tag{2.10}$$

establishing (2.8).

Now, for each $i \in W$, set

$$r_i := T_i - \lim_{\to}(T_j\colon j \in \operatorname{seg}(i)). \tag{2.11}$$

Invoking (2.8), we see that $r_i > 0$ for every $i \in W$. Let $(S_i\colon i \in W)$ be the series determined by $(r_i\colon i \in W)$. Next, we demonstrate that

$$S_i = T_i \text{ for all } i \in W. \tag{2.12}$$

Let $i \in W$ and suppose that $S_j = T_j$ for all $j < i$. By the Principle of Transfinite Induction, it suffices to prove that $S_i = T_i$. Toward this end, simply note that

$$S_i = \lim_{\to}(S_j\colon j \in \operatorname{seg}(i)) + r_i = \lim_{\to}(T_j\colon j \in \operatorname{seg}(i)) + r_i = T_i \text{ (by definition of } r_i).$$

and we have proven (2.12). It follows that $S_i \in \mathbb{R}$ for every $i \in W$. Applying (2.6) and (2.12), $(S_i\colon i \in W)$ converges, and the proof of Theorem 2 is complete. □

3. Uncountable Sums in Totally Ordered Abelian Groups

3.1. **Transitioning to Uncountable Sums.** The question of how to define an uncountable sum of positive real numbers is quite a natural one, given the applications of (standard) convergent series to calculus. Indeed, this question has received attention both online and in the literature (see [8] (Exercise 0.0.1), [6] (Chapter 1), and [3] (Chapter 0)). A common way (which circumvents the need to delve more deeply into set theory and order theory) is to define the sum of an uncountable set S of positive real numbers as follows:

$$\sum_S := \sup\{s_1 + s_2 + \cdots s_n \colon n \in \mathbb{Z}^+, s_i \in S, s_i \neq s_j \text{ for } i \neq j\}.^{11} \tag{3.1}$$

Interestingly, regardless of the uncountable set S of positive reals, $\sum_S$ as defined in (3.1) will never be finite. To see why, let S be an arbitrary uncountable set of positive real numbers. For every positive integer n, let $S_n := S \cap (\frac{1}{n}, \infty)$. One checks immediately that $S = \bigcup_{n \in \mathbb{Z}^+} S_n$. Because S is uncountable, it follows that some S_k must be infinite (uncountable, even). But then there are infinitely many elements of S which are larger than $\frac{1}{k}$. Given a real number N, choose (by the Archimedean property of the real line[12]) a positive integer m such that $m > kN$. Now choose m distinct elements $s_1, \ldots, s_m$ of S which are larger than $\frac{1}{k}$. Then observe that $s_1 + \cdots + s_m > \frac{m}{k} > N$. We deduce that $\sum_S$ does not exist (as a real number).

There is a sense in which the reason that $\sum_S$ (as defined in (3.1) above) is never finite for uncountable $S \subseteq (0, \infty)$ is because the real line isn't "long enough" to accommodate such a phenomenon. To give a related example, a well-known topological property of the real line is that there does not exist an uncountable collection of pairwise-disjoint open intervals. The usual proof is by contradiction: if such a collection existed, simply pick a rational number from each interval, and you get an uncountable set of rational numbers, which is absurd. We would like to offer a somewhat different argument. The argument is *much* less elegant than the proof just given; our purpose is to make a connection between the nonexistence of such a collection and the real line's lack of "length" to which we alluded above. This phenomenon will inspire many of the results of Section 3.

Example 4 (well-known)**.** *There does not exist an uncountable pairwise-disjoint collection of open intervals on the real line.*

Proof. Suppose by way of contradiction that there exists a collection $\mathcal{C} := \{(a_i, c_i) \colon i \in I\}$ of pairwise-disjoint (nonempty) open intervals on the real line, where I is an uncountable set enumerating $\mathcal{C}$. For each $i \in I$, set $b_i := \min(c_i, a_i + 1)$. Then one checks at once that the map $(a_i, c_i) \mapsto (a_i, b_i)$ is injective. Set $\mathcal{B} := \{(a_i, b_i) \colon i \in I\}$. As the members of $\mathcal{C}$ are pairwise-disjoint, it is immediate that

$$\text{the members of } \mathcal{B} \text{ are also pairwise-disjoint.} \tag{3.2}$$

For each $i \in I$, let $\ell(a_i, b_i) := b_i - a_i$ be the length of the interval (a_i, b_i). By construction of the b_is,

$$\ell(a_i, b_i) \leq 1 \text{ for every } i \in I. \tag{3.3}$$

Because the members of $\mathcal{B}$ are pairwise-disjoint, it is clear that

[11]Note that, in some sense, any "natural" definition of the sum of an uncountable set of positive real numbers must be at least as big as the sup given above.

[12]The Archimedean property is simply that for every $r \in \mathbb{R}$ and $x \in (0, \infty)$, there is $n \in \mathbb{Z}^+$ such that $nx > r$.

$$a_i \neq a_j \text{ for } i \neq j \text{ in } I. \tag{3.4}$$

Now, since $\{a_i : i \in I\}$ is uncountable, there is some integer n for which $[n, n+1] \cap \{a_i : i \in I\} := J$ is uncountable. Applying (3.3), $(a_j, b_j) \subseteq [n, n+2]$ for all $j \in J$. Thus we may assume without loss of generality that

$$(a_i, b_i) \subseteq [n, n+2] \text{ for every } i \in I. \tag{3.5}$$

Consider the length function $\ell\colon \mathcal{B} \to [0,2]$ defined above. Because each $(a_j, b_j) \subseteq [n, n+2]$ and the intervals $(a_i, b_i), i \in I$ are pairwise-disjoint, we conclude that

$$\ell^{-1}(y) \text{ is finite for every } y \in [0,2]. \tag{3.6}$$

Noting that $\{\ell^{-1}(y) : y \in \operatorname{ran}(\ell)\}$ partitions the uncountable set $\mathcal{B}$, we deduce from (3.6) that

$$\operatorname{ran}(\ell) \text{ is uncountable.} \tag{3.7}$$

For each $y \in \operatorname{ran}(\ell)$, choose $(a_y, b_y) \in \mathcal{B}$ such that $y = \ell(a_y, b_y) = b_y - a_y$. Next, set $S := \{b_y - a_y : y \in \operatorname{ran}(\ell)\}$. Note that S is uncountable by (3.7). Consider again the definition of $\sum_S$ given in (3.1) for S an uncountable set of positive real numbers. We argued that $\sum_S$ is not a real number. However, observe that if $b_{y_1} - a_{y_1}, \ldots, b_{y_k} - a_{y_k}$ are distinct members of S, then we deduce from the pairwise-disjointness of the intervals (a_y, b_y) and (3.5) that $\sum_{i=1}^{k} b_{y_i} - a_{y_i} \leq 2$. But then certainly $\sum_S$ is real, a contradiction. $\square$

Several questions and comments are now in order. First, there are plenty of examples of conditionally convergent (countably infinite) real series. Is it possible to define the sum of a real series with uncountably many nonzero terms which has both positive *and* negative terms? If we can find such a way, is convergence to a finite value possible? Next, observe that in our proof that $\sum_S$ (defined in (3.1)) is infinite, we applied the Archimedean property of the ordering on $\mathbb{R}$. What if we replace the totally ordered additive abelian group of real numbers by an arbitrary totally ordered abelian group (the definition of which is forthcoming in Section 3.2)? The notions of convergence we have discussed can easily be translated to this more general context (we settle on such a notion in Section 3). Are there any uncountable sums of nonzero elements of a totally ordered abelian group which are "finite" (that is, have values in the group)? Finally[13], recall that the cumulative hierarchy of sets V_α, α an ordinal, is generated from the empty set by transfinite applications of the union and power set operations. By definition, $V_{\omega_1} = \bigcup_{i<\omega_1} V_i$, so in a sense, V_{ω_1} is a "limit" of the V_is for $i < \omega_1$. Moreover, there is no countable $S \subseteq \omega_1$ such that V_{ω_1} is the "limit" of $\{V_i : i \in S\}$ in the sense that for any countable such S, $\bigcup_{i\in S} V_i \subsetneq V_{\omega_1}$.[14] Thus V_{ω_1} cannot be obtained as the union of a countable collection of V_js, where each $j < \omega_1$. Supposing we have settled on a definition of "uncountable sum" in a totally ordered abelian group, as in the cumulative hierarchy example

[13] This example requires some knowledge of axiomatic set theory.

[14] On the other hand, $\bigcup_{i\in S} V_i = V_{\omega_1}$ if S is uncountable.

above, can we find a convergent series of positive terms whose sum is *not* the sup of the collection of finite sums (in contrast to the definition of an uncountable sum of positive reals given in (3.1))? We investigate these questions in the remainder of this article. As a precursor, we give a self-contained introduction to ordered abelian groups to be utilized shortly.

3.2. **Preliminaries: Totally Ordered Abelian Groups.** The purpose of this subsection is to give a gentle introduction to the theory of totally ordered abelian groups, as we shall cast our remaining results in terms of these structures. For the reader already comfortable with ordered groups, we recommend skipping to the definition of co-initiality below.

To begin, consider the set $\mathbb{Z}$ of integers. The usual addition $+$ on $\mathbb{Z}$ enjoys many nice properties; we single out several below:

(1) ($+$ is associative on $\mathbb{Z}$) $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{Z}$,
(2) ($+$ is commutative on $\mathbb{Z}$) $x + y = y + x$ for all $x, y \in \mathbb{Z}$,
(3) (existence of an additive identity) there exists an element $0 \in \mathbb{Z}$ such that $x + 0 = x$ for all $x \in \mathbb{Z}$, and
(4) (existence of additive inverses) for all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $x + y = 0$.

Suppose now that G is a set and $\oplus$ is a *binary operation* on G (that is, $\oplus\colon G\times G \to G$ is a function). Then observe that the properties of associativity and commutativity, the existence of an additive identity, and the existence of additive inverses enjoyed above by $\mathbb{Z}$ with the usual addition can all be translated *mutatis mutandis* to this more abstract setting. For example, $\oplus$ is commutative on G exactly when $x \oplus y = y \oplus x$ for all $x, y \in G$. If G is a set and $\oplus$ is a binary operation on S such that (1) – (4) above hold (with $+$ replaced with $\oplus$, of course), then we say that the structure $(G, \oplus)$ is an *abelian group*. We pause to give two additional examples. The uninitiated reader is encouraged to verify that the following are, in fact, abelian groups.

Example 5. *Let X be a set, and let $\mathcal{F}(X, \mathbb{R})$ be the collection of all functions $f\colon X \to \mathbb{R}$. Define $\oplus$ on $\mathcal{F}(X, \mathbb{R})$ by $(f\oplus g)(x) := f(x)+g(x)$, where $+$ is the usual addition on $\mathbb{R}$. Then $(\mathcal{F}(X, \mathbb{R}), \oplus)$ is an abelian group.*

Example 6. *Let S be a set, and let $\mathcal{P}(S)$ denote the power set of S. Define $\oplus$ on $\mathcal{P}(S)$ by $A \oplus B := (A\backslash B) \cup (B\backslash A)$. Then $(\mathcal{P}(S), \oplus)$ is an abelian group.*

Remark 1. We now adopt the usual convention of denoting the operation on an arbitrary abelian group by $+$ instead of $\oplus$.

In Section 2.1, we defined various binary relations on a set S. Consider instead an abelian group $(G, +)$ and a total order $\prec$ on G. If we are to define a a totally ordered abelian group structure, we would expect some compatibility between the order $\prec$ and the addition on G. This compatibility is in the form of so-called *translation invariance*, which simply means that if $x, y, z \in G$ with $x \prec y$, then $x + z \prec y + z$. If $(G, +)$ is an abelian group and $\prec$ is a translation invariant total order on G, then we call the structure $(G, +, \prec)$ a *totally ordered abelian group*. More examples are now in order. Again, the reader is encouraged to check the details for her or himself.

Example 7. *Let $G := \mathbb{Z} \times \mathbb{Z}$ with the usual component-wise addition. Then the dictionary order $\prec$ on G is total and translation invariant.*

Example 8. *Consider the set $\mathbb{R}[X]$ of polynomial functions with real coefficients. Then $\mathbb{R}[X]$ becomes an abelian group under the usual addition of polynomial functions. Define $\prec$ on $\mathbb{R}[X]$ by $f \prec g$ if and only if there is $N \in \mathbb{R}$ such that $f(x) < g(x)$ for all $x \geq N$. It is straightforward to verify that $(\mathbb{R}[X], +, \prec)$ is a totally ordered abelian group.*

Our final preliminary result concerns convergence in ordered abelian groups, though for now, we have no need to complicate things by introducing a formal definition of convergence just yet (though as the reader may expect, it will mirror Definition 3). First, we present the ordered group analog of Definition 6.

Definition 7. Let $(G, +, \prec)$ be a totally ordered abelian group, and $(W, <)$ be a well-ordered set. Further, suppose that $(g_i : i \in W)$ is a W-indexed sequence in G.

(1) $(g_i : i \in W)$ is *increasing* (respectively, *strictly increasing*) if whenever $i, j \in W$ and $i < j$, then $g_i \preceq g_j$ (respectively, $g_i \prec g_j$).
(2) $(g_i : i \in W)$ is *decreasing* (respectively, *strictly decreasing*) if whenever $i, j \in W$ and $i < j$, then $g_i \succeq g_j$ (respectively, $g_i \succ g_j$).
(3) If $(g_i : i \in W)$ is either (strictly) increasing or (strictly) decreasing, then we say that $(g_i : i \in W)$ is (strictly) *monotonic*.

To close this subsection, we introduce an important definition and lemma of which we shall shortly make use.

Definition 8. Suppose that $(G, +, \prec)$ is a totally ordered abelian group and $(W, <)$ is a well-ordered set. Further, let $g^* \in G$.

(1) Suppose that $(g_i : i \in W)$ is a strictly increasing sequence in G such that $g_i \prec g^*$ for all $i \in W$. Then we say that $(g_i : i \in W)$ is *left g^*-coinitial* provided that for every $g \in G$: if $g \prec g^*$, then there is $i \in W$ such that $g \preceq g_i$.
(2) Suppose that $(g_i : i \in W)$ is a strictly decreasing sequence in G such that $g^* \prec g_i$ for all $i \in W$. Then we say that $(g_i : i \in W)$ is *right g^*-coinitial* provided that for every $g \in G$: if $g^* \prec g$, then there is $i \in W$ such that $g_i \preceq g$.

We conclude with the following observation:

Lemma 5. *Let $(G, +, \prec)$ be a totally ordered abelian group and $(W, <)$ be a well-ordered set. Suppose that $(g_i : i \in W)$ is a W-sequence of elements of G and that $g^* \in G$. If $(g_i : i \in W)$ is either left or right g^*-coinitial, then there is a right 0-coinitial G-valued W-sequence $(h_i : i \in W)$.*

Proof. Let $(G, +, \prec)$, $(W, <)$, $(g_i : i \in W)$, and g^* be as stated.

Case 1. $(g_i : i \in W)$ is right g^*-coinitial. Then $(g_i : i \in W)$ is strictly decreasing. For $i \in W$, let $h_i := g_i - g^*$. We claim that $(h_i : i \in W)$ is right 0-coinitial. First, suppose that $i, j \in W$ and $i < j$. Then $g_j \prec g_i$. By translation invariance, $g_j - g^* \prec g_i - g^*$, that is, $h_j \prec h_i$. Thus $(h_i : i \in W)$ is strictly decreasing. We claim that $h_i \succ 0$ for all $i \in W$. It is clear that this reduces to $g_i \succ g^*$ for all $i \in W$, which is true since $(g_i : i \in W)$ is right g^*-coinitial. It remains to show that if $g \succ 0$, then there is $i \in W$ such that $h_i \preceq g$. So suppose that $g \succ 0$. Then $g + g^* \succ g^*$.

Because $(g_i \colon i \in W)$ is right g^*-coinitial, there is $i \in W$ such that $g_i \preceq g + g^*$. Subtracting g^* from both sides of the inequality, we obtain $g_i - g^* \preceq g$, that is, $h_i \preceq g$. We have shown that $(h_i \colon i \in W)$ is right 0-coinitial, completing the proof in this case.

Case 2. $(g_i \colon i \in W)$ is left g^*-coinitial. Then it is clear that $(-g_i \colon i \in W)$ is right $(-g^*)$-coinitial. Therefore, by Case 1, there is a right 0-coinitial G-valued W-sequence $(h_i \colon i \in W)$. The proof is now concluded. □

For further reading on ordered groups, we refer the reader to the text [1].

3.3. **Defining and Exploring Uncountable Sums in Totally Ordered Abelian Groups.** Our first task is to give a rigorous definition of "infinite sum" in a totally ordered abelian group. We begin by defining the convergence of a sequence. To wit, let $(G, +, \prec)$ be a totally ordered abelian group. We may define the absolute value of $g \in G$ by mimicking the usual definition on the real line:

$$|g| := \begin{cases} g & \text{if } g \succeq 0; \\ -g & \text{otherwise.} \end{cases}$$

We are now prepared to define convergence of a G-valued W-sequence, where $(G, +, \prec)$ is a totally ordered abelian group and $(W, <)$ is a well-ordered set (compare to Definition 2).

Definition 9. Let $(G, +, \prec)$ be a totally ordered abelian group, $(W, <)$ a nonempty well-ordered set, $(g_i \colon i \in W)$ a G-valued W-sequence, and $g \in G$. Say that $(g_i \colon i \in W)$ *converges* to g (or that g is the *limit* of $(g_i \colon i \in W)$) provided that for every $\epsilon \succ 0$, there exists $i \in W$ such that if $j \in W$ and $j \geq i$, then $|g_j - g| \prec \epsilon$. In this case, we write $\lim\limits_{\rightarrow}(g_i \colon i \in W) = g$. If $W = \varnothing$, then we set $\lim\limits_{\rightarrow}(g_i \colon i \in W) := 0$.

We must pause to address a potential difficulty. Notice the appearance of the word 'the' in the phrase, "g is *the* limit of $(g_i \colon i \in W)$" above. To justify our choice of article, we must confirm that a convergent G-valued W-sequence has a unique limit.

Lemma 6. *Let $(G, +, \prec)$, $(W, <)$, and $(g_i \colon i \in W)$ be as in Definition 9. Then $(g_i \colon i \in W)$ converges to at most one $g \in G$.*

Proof. Suppose that $(g_i \colon i \in W)$ is a convergent G-valued W-sequence. If G is trivial, then the result is trivial as well, so we assume that G is nontrivial. Then there is some nonzero $g \in G$. Now, either $g \succ 0$ or $-g \succ 0$, so G has positive elements relative to $\prec$.

Case 1. G has a least positive element, say ϵ_0. Let g be a limit of $(g_i \colon i \in W)$. There exists $i \in W$ such that for all $j \geq i$, $|g_j - g| < \epsilon_0$. By the leastness of ϵ_0, we must have $|g_j - g| = 0$, and hence $g_j = g$ for all $j \geq i$. If g' is any other limit of $(g_i \colon i \in W)$, then there is $i' \in W$ such that $g_j = g'$ for all $j \geq i'$. Let $k := \max(i, i')$. Then observe that $g_k = g = g'$, and we are done in this case.

Case 2. G has no least positive element. We claim that G is dense in itself with respect to $\prec$.[15] To see this, suppose $x \prec z$. By translation invariance, $z - x \succ 0$. Since G has no least positive element, there is $g \in G$ such that $0 \prec g \prec z - x$. Adding x throughout, we get $x \prec g + x \prec z$, completing the verification that G is dense in itself relative to $\prec$. Now, suppose by way of contradiction that $(g_i \colon i \in W)$ converges to both x and z for some distinct $x, z \in G$. Without loss of generality, we may suppose that $x \prec z$. By denseness, there is $y \in G$ such that $x \prec y \prec z$. Set $\epsilon := \min(y - x, z - y)$. Then the balls $B_\epsilon(x) := \{g \in G \colon |x - g| \prec \epsilon\}$ and $B_\epsilon(z)$ are disjoint. But by convergence, there is some $i \in W$ such that g_i is a member of both balls, a contradiction. □

Now that we have established the uniqueness of limits, we are ready to define infinite series in an arbitrary totally ordered abelian group. Our definition is the canonical extension of Definition 3.

Definition 10. Suppose that $(G, +, \prec)$ is a totally ordered abelian group and that $(W, <)$ is a nonempty well-ordered set. Next, let $(g_i \colon i \in W)$ be a G-valued W-sequence, and choose any $e \notin G$. We now define *the series* $(S_i \colon i \in W)$ *determined by* $(g_i \colon i \in W)$ by recursion on W. Suppose that $i \in W$ and that S_j has been defined for every $j < i$. We now define S_i as follows:

$$S_i := \begin{cases} e & \text{if } S_j = e \text{ for some } j < i \text{ or } \varinjlim(S_j \colon j \in \mathrm{seg}(i)) \text{ does not exist;} \\ \varinjlim(S_j \colon j \in \mathrm{seg}(i)) + g_i & \text{otherwise.} \end{cases}$$

Say that the series $(S_i \colon i \in W)$ *converges* if $S_i \in G$ for each $i \in W$ and $\varinjlim(S_i \colon i \in W) := S \in G$. In this case, we say that S is the *sum* of the series $(S_i \colon i \in W)$.

We now present a trivial example of a convergent series relative to the above definition.

Example 9. *Let $(W, <)$ be an uncountable well-ordered set, and let i_0 be the least element of W. Continuing recursively (on $\mathbb{N}$), define $i_{n+1} :=$ the least element of W larger than i_n. Because W is uncountable, it is clear that i_n is a well-defined member of W for every $n \in \mathbb{N}$. Next, for $n \in \mathbb{N}$, set $r_{i_n} := (\frac{1}{2})^n$. For $j \in W \backslash \{i_n \colon n \in \mathbb{N}\}$, set $r_j := 0$. Then (as one might expect) the corresponding series sums to 2.*

Notice that the sequence introduced above has but countably many nonzero terms (this is what we mean by "trivial"). The following question is natural:

Question 1. Does there exist a totally ordered abelian group $(G, +, \prec)$, a well-ordered set $(W, <)$, and a G-valued W-sequence $(g_i \colon i \in W)$ such that $g_i \neq 0$ for uncountably many $i \in W$, yet the corresponding series $(S_i \colon i \in W)$ converges?

The remainder of this paper is devoted to proving that Question 1 has a negative answer.

[15] This is a fundamental result in the theory of totally ordered abelian groups

3.4. **The Nonexistence of Convergent Series with Uncountably Many Nonzero Terms.** We begin by remarking that in order to show that there does not exist a totally ordered abelian group $(G, +, \prec)$, a well-ordered set $(W, <)$, and a G-valued W-indexed sequence $(g_i \colon i \in W)$ such that the corresponding series $(S_i \colon i \in W)$ converges and $g_i \neq 0$ for uncountably many $i \in W$, it suffices to show that if $(g_i \colon i \in W)$ is a G-valued W-sequence for which both $g_i \neq 0$ for every $i \in W$ and $(S_i \colon i \in W)$ converges, then W is countable. One can give a proof that this is sufficient to guarantee a negative answer to Question 1 via straightforward (transfinite) inductive arguments. As such, we omit the details.

Before proceeding, we shall require more terminology. Let $(W, <)$ be a nonempty well-ordered set, and let $S \subseteq W$ be nonempty. Since $<$ is a well-order on W, there exists a least element $s \in S$ (relative to $<$). Because $<$ is total, it follows that s is unique; we denote this s by $\inf(S)$ (the *infimum* of S). As in the real case, say that $S \subseteq W$ is *bounded above* if there is $i \in W$ such that $s \leq i$ for all $s \in S$. Such an i is called an *upper bound* of S. If S is bounded above, then the least upper bound of S is called the *supremum* of S and is denoted by $\sup(S)$. Next, we say that $i \in W$ is a *successor* if there is $j \in W$ such that $j < i$ and there is no $k \in W$ such that $j < k < i$. If this is the case, then we write $i = j^+$. Finally, if $i \in W$ is not a successor, then we say that i is a *limit*.[16] We present an example below.

Example 10. *Let ∞ be any object not in $\mathbb{N}$. Now extend the usual order $<$ on $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$ by declaring $n < \infty$ for all $n \in \mathbb{N}$. Setting $W := \mathbb{N} \cup \{\infty\}$, we see that $0 = \inf(W)$ is a limit, every nonzero natural number is a successor, and ∞ is a limit. Moreover, $\infty = \sup(\mathbb{N})$.*

We now state a more general version of Lemma 1. As the proof is essentially identical, we omit it.

Lemma 7. *Suppose that $(G, +, \prec)$ is a totally ordered abelian group and that $(W, <)$ is a well-ordered set with largest element w^*. Further, suppose that $(g_i \colon i \in W)$ is a G-valued W-sequence. Then $\varinjlim(g_i \colon i \in W) = g_{w^*}$.*

We are almost ready to prove the final result of this paper. We shall require two definitions and three additional lemmas. The first definition and lemma generalize the well-known fact that if $(r_n \colon n \in \mathbb{N})$ is a convergent real-valued sequence, then every subsequence of $(r_n \colon n \in \mathbb{N})$ converges to the same limit.

Definition 11. Suppose that $(W, <)$ is a nonempty well-ordered set and that $S \subseteq W$. Say that S is *cofinal* in W if for every $w \in W$, there is $s \in S$ such that $w \leq s$.

Lemma 8. *Let $(G, +, \prec)$ be a totally ordered abelian group and $(W, <)$ be a nonempty well-ordered set. Further, let $(g_i \colon i \in W)$ be a G-valued W-sequence. If $S \subseteq W$ is cofinal in W and $\varinjlim(g_i \colon i \in W) = g^* \in G$, then also $\varinjlim(g_i \colon i \in S) = g^*$.*

[16]The reader familiar with the class of ordinal numbers should appreciate the terminology chosen above.

Proof. Suppose S is cofinal in W and $\varinjlim(g_i\colon i \in W) = g^* \in G$. Now let $\epsilon \succ 0$. There exists $i \in W$ such that if $j \geq i$, then $|g_j - g^*| \prec \epsilon$. Since S is cofinal in W, there is $s \in S$ such that $s \geq i$. Hence if $j \in S$ and $j \geq s$, then also $j \geq i$. Therefore, $|g_j - g^*| \prec \epsilon$, and $\varinjlim(g_i\colon i \in S) = g^*$. □

Definition 12. Let $(G, +, \prec)$ be a totally ordered abelian group and $(W, <)$ be a non-empty well-ordered set. Further, let $(g_i\colon i \in W)$ be a G-valued W-sequence. Then $(g_i\colon i \in W)$ is called *ultimately constant* provided there is $i \in W$ such that $g_j = g_i$ for all $j \geq i$.

Lemma 9. *Suppose that $(G, +, \prec)$ is a totally ordered abelian group, $(W, <)$ a non-empty well-ordered set with no largest member, and that $(g_i\colon i \in W)$ is a G-valued W-sequence with nonzero terms. If the corresponding series $(S_i\colon i \in W)$ converges, then $(S_i\colon i \in W)$ is not ultimately constant.*

Proof. Fix an arbitrary $i \in W$. Now observe that $S_{i^+} = \varinjlim(S_j\colon j \in \text{seg}(i^+)) + g_{i^+} =$ (by Lemma 7) $S_i + g_{i^+}$. Because $g_{i^+} \neq 0$, we conclude that $S_i \neq S_i^+$. Recalling that W has no largest element, it is clear that $(S_i\colon i \in W)$ is not ultimately constant. □

Lemma 10. *Let $(G, +, \prec)$ be a totally ordered abelian group, and suppose that $(g_n\colon n \in \mathbb{N})$ is a convergent sequence in G which is not ultimately constant. Then there exists a right 0-coinitial G-valued sequence $(h_n\colon n \in \mathbb{N})$.*

Proof. We suppose that $(g_n\colon n \in \mathbb{N})$ is a convergent G-valued sequence which is not ultimately constant. Let $\varinjlim(g_n\colon n \in \mathbb{N}) := g$. Since $(g_n\colon n \in \mathbb{N})$ is nonconstant, there is some $n_0 \in \mathbb{N}$ such that $g_{n_0} \neq g$. Set $\epsilon_0 := |g_{n_0} - g|$. There is $k \in \mathbb{N}$ such that if $j \geq k$, then $|g_j - g| < \epsilon_0$. Because $(g_n\colon n \in \mathbb{N})$ is nonconstant, there is $n_1 \in \mathbb{N}$ such that $n_1 > n_0$, $g_{n_1} \neq g$, and $|g_{n_1} - g| < \epsilon_0$. Next, set $\epsilon_1 := |g_{n_1} - g|$. Similarly, there is $n_2 > n_1$ such that $g_{n_2} \neq g$ and $|g_{n_2} - g| < \epsilon_1$. Continuing recursively, we obtain a sequence $(g_{n_k}\colon k \in \mathbb{N})$ such that $n_0 < n_1 < n_2 \cdots$ and for every $k \in \mathbb{N}$, $|g_{n_{k+1}} - g| < |g_{n_k} - g|$. We deduce that

$$g_{n_i} \neq g_{n_j} \text{ for } i \neq j. \tag{3.8}$$

Via the same argument one uses to prove that every real-valued sequence has a monotonic subsequence, we conclude that $(g_{n_k}\colon k \in \mathbb{N})$ has a monotonic subsequence $(g_{n_{k_l}} : l \in \mathbb{N})$. Invoking (3.8), it follows that $(g_{n_{k_l}} : l \in \mathbb{N})$ is *strictly* monotonic. Since $\{n_{k_l}\colon l \in \mathbb{N}\}$ is cofinal in $\mathbb{N}$, we deduce from Lemma 8 that $(g_{n_{k_l}} : l \in \mathbb{N})$ is a strictly monotonic sequence which converges to g. It follows that $(g_{n_{k_l}} : l \in \mathbb{N})$ is either left or right g-coinitial. Applying Lemma 5 yields the desired conclusion. □

At long last, we are prepared to prove the concluding theorem of this note.

Theorem 3. *There does not exist a totally ordered abelian group $(G, +, \prec)$, an uncountable well-ordered set W, and a G-valued W-sequence $(g_i\colon i \in W)$ of nonzero terms such that the corresponding series $(S_i\colon i \in W)$ converges.*

Proof. We argue by contradiction. Thus suppose $(G, +, \prec)$ is totally ordered abelian group, W an uncountable well-ordered set, and $(g_i\colon i \in W)$ a G-valued W-sequence of nonzero terms such that the corresponding series $(S_i\colon i \in W)$ converges to $S \in G$. Let i_0 be the least element of W. Proceeding by recursion, set $i_{n+1} := \inf\{j \in W : i_n < j\}$. Because W is uncountable, the i_n's do not exhaust W. Let i_ω be the least element of W larger than every i_n. Since $(S_i\colon i \in W)$ converges, it follows by definition of convergence that the sequence $(S_j\colon j \in \text{seg}(i_\omega)) = (S_{i_n}\colon n \in \mathbb{N})$ also converges. By Lemma 9, $(S_{i_n}\colon n \in \mathbb{N})$ is not ultimately constant; invoking Lemma 10,

$$\text{there is a right 0-coinitial } G\text{-valued sequence } (h_n\colon n \in \mathbb{N}). \tag{3.9}$$

We now consider two cases to obtain a contradiction.

Case 1. For every $w \in W$, $\text{seg}(w)$ is countable. Because W is uncountable, it follows that W does not possess a largest element. Employing Lemma 9, we deduce that

$$(S_i\colon i \in W) \text{ is not ultimately constant.} \tag{3.10}$$

Because $(S_i\colon i \in W)$ converges to S, for each $n \in \mathbb{N}$, choose $j_n \in W$ such that if $w \in W$ and $w \geq j_n$, then $|S_w - S| < h_n$. We claim that

$$\text{there is } j \in W \text{ such that } j > j_n \text{ for all } n \in \mathbb{N}. \tag{3.11}$$

If not, then $W = \{j_n\colon n \in \mathbb{N}\} \cup (\bigcup\{\text{seg}(j_n)\colon n \in \mathbb{N}\})$. But now W is a countable union of countable sets, hence countable, a contradiction. Let $j \in W$ satisfy (3.11), and consider any $w \in W$ such that $w \geq j$. Then $w \geq j_n$ for all $n \in \mathbb{N}$, and so $|S_w - S| < h_n$ for every $n \in \mathbb{N}$. Because $(h_n\colon n \in \mathbb{N}$ is right 0-cofinal, we conclude that $|S_w - S| = 0$, and therefore $S_w = S$. But now $S_w = S$ for all $w \geq j$ and $(S_i\colon i \in W)$ is ultimately constant, a contradiction to (3.10).

Case 2. There is some $w \in W$ such that $\text{seg}(w)$ is uncountable. Choose the least such w. Then observe that $\text{seg}(w)$ is uncountable, but for every $i \in \text{seg}(w)$, $\text{seg}(i)$ is countable. Now simply consider the sequence $(g_j\colon j \in \text{seg}(i))$ and the corresponding series $(S_j\colon j \in \text{seg}(i))$. This restriction puts us back in Case 1, and so we obtain a contradiction again, as required. $\square$

Now that the smoke has cleared, we conclude the note with an informal description of why nontrivial uncountable sums don't exist in any totally ordered abelian group. With notation as in the previous theorem, we let $i \in W$ be the first nonzero limit of W. Then convergence of the partial sums at this stage, say to S, forces there to be (in some sense) "countably much space" on at least one side of S in that there is a countable S-coinitial sequence. By translation invariance, there is countably much space around any $g \in G$ (on both sides of g). Consequently, there simply isn't enough "room" for a nontrivial uncountable series to converge.

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(Greg Oman) Department of Mathematics, University of Colorado, Colorado Springs, CO 80918, USA
E-mail address: goman@uccs.edu

(Caitlin Randall) Department of Mathematics, University of Colorado, Colorado Springs, CO 80918, USA
E-mail address: crandal2@uccs.edu

(Logan Robinson) Department of Mathematics, University of Colorado, Colorado Springs, CO 80918, USA
E-mail address: 26.loganmr.5@gmail.com