MODULES WHOSE SUBMODULE LATTICE IS LOWER FINITE

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Abstract. A bounded poset $\mathbf{P} := (P, 0, 1, \leq)$ is said to be *lower finite* if P is infinite and for all $1 \neq x \in P$, there are but finitely many $y \in P$ such that $y \leq x$. In this paper, we classify the modules M over a commutative ring R with identity with the property that the lattice $\mathbf{L}_R(M)$ of R-submodules M (under set-theoretic containment) is lower finite. Our results are summarized in Theorem 1 at the end of this note.

1. Introduction

If $\mathbf{A} := (A, \mathcal{F})$ is an algebra[1], then the collection of subuniverses of A (recall that A is called the *universe* of $\mathbf{A}$) forms a partially ordered set with respect to set inclusion. For instance, if R is a ring, then the collection of subrings of R is a poset (again, with respect to set inclusion). The following question is quite natural and has been explored in a variety of mathematical contexts:

Problem 1. Given a class $\mathcal{C}$ of algebras, find necessary and sufficient conditions on a poset $\mathbf{P}$ for which there is a member of $\mathcal{C}$ whose poset of subalgebras (subuniverses) is order-isomorphic to $\mathbf{P}$.

A classical result in discrete mathematics in this direction is the following: let $\mathbf{L}$ be a finite lattice[2]. Then $\mathbf{L}$ is the lattice of flats of a matroid if and only if $\mathbf{L}$ is geometric (for a proof and further reading, we refer the reader to [15]). Transitioning to ring theory, Irving Kaplansky considered the question of which partially ordered sets can be realized as the poset of prime ideals of a commutative unital ring ([9]); this is the so-called "Kaplansky problem" and is still not completely resolved. More recently, the same question was investigated in the context of Leavitt path algebras ([1]).

Another approach to relating the poset of subalgebras of an algebra $\mathbf{A}$ to the structure of $\mathbf{A}$ is to fix some poset property $\mathcal{P}$, and study the algebras $\mathbf{A}$ in some class $\mathcal{C}$ of algebras whose poset of subalgebras satisfies property $\mathcal{P}$. For example, if G is an infinite group whose poset of subgroups is totally ordered by $\subseteq$, then G is isomorphic to a Prüfer p-group $C(p^\infty)$ for some prime p (this result can be found, for example, in the classic text [4]). Several well-studied properties of a module M over a commutative ring R are (can be) phrased in terms of the lattice $\mathbf{L}_R(M)$ of R-submodules of M, ordered by set inclusion. For instance, M is called *uniserial* if $\mathbf{L}_R(M)$ is totally ordered.

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[1] We take the universal definition of "algebra", and thus A is a set and $\mathcal{F}$ is a collection of operations on A, each of finite arity.

[2] Ordered by inclusion, the collection of flats of a matroid forms a special kind of poset called a *lattice*; that is, a partially ordered set for which any two members have both an infimum and and a supremum.

To state two more examples, M is *Artinian* if every nonempty subset of $\mathbf{L}_R(M)$ has a minimal element; M is *Noetherian* if every nonempty subset of $\mathbf{L}_R(M)$ has a maximal element.

Recall that a poset $\mathbf{P} := (P, \leq)$ is *bounded* provided there are (necessarily unique) elements $0, 1 \in P$ such that $0 \leq p \leq 1$ for all $p \in P$. We pause to give a relevant example.

Example 1. *Let R be a ring and let M be a left R-module. Then in the submodule poset $\mathbf{L}_R(M)$, we have $0 = \{0\}$ and $1 = M$.*

An infinite bounded poset $\mathbf{P} := (P, 0, 1, \leq)$ is called *lower finite* provided that for every $x \neq 1$ in P, there are but finitely members y of P such that $y \leq x$. Again, we pause to illustrate with a simple example.

Example 2. *Let X be an infinite set, and let $\mathcal{S} := \{A \subseteq X \colon A = X$ or A is finite$\}$. Then with respect to set inclusion, S is a bounded lattice which is lower finite.*

The purpose of this note is to classify the modules M over a commutative ring R with identity such that $\mathbf{L}_R(M)$ is lower finite. This work extends earlier results of Hirano and Mogami ([7]) as well as the second author's work on countable Jónsson modules ([12]). Theorem 1 gives a summary of our results. We then close the paper with directions for further research.

2. Preliminaries

We begin this section by giving a terse treatment of some fundamental definitions. Recall that a (non-strict) *partial order* on a set P is a binary relation $\leq$ on P which is reflexive, antisymmetric, and transitive. If P is a set and $\leq$ is a partial order on P, then we call the structure $\mathbf{P} := (P, \leq)$ a *partially ordered set* or *poset*. Next, suppose that $(P, \leq)$ is a poset, and let $x, y \in P$. Then an element $z \in P$ is called a *supremum* or *sup* of x and y provided $x \leq z$, $y \leq z$, and if $p \in P$ satisfies $x \leq p$ and $y \leq p$, then $z \leq p$. Similarly, $z \in P$ is called an *infimum* or *inf* of x and y provided $z \leq x$, $z \leq y$, and if $p \in P$ satisfies $p \leq x$ and $p \leq y$, then $p \leq z$. It is easy to see from the definition of partial order that if an inf or sup of x and y exists, then it is unique. We denote this unique element, when it exists, by $\inf(x, y)$ and $\sup(x, y)$, respectively. A poset $(P, \leq)$ is called a *lattice* provided that both $\inf(x, y)$ and $\sup(x, y)$ exist for all $x, y \in P$. A lattice $(P, \leq)$ is *bounded* if it is bounded as a poset, that is (as above) there are elements $0, 1 \in P$ (necessarily unique) such that $0 \leq x \leq 1$ for all $x \in P$. Finally, if $\mathbf{P_1} := (P_1, \leq_1)$ and $\mathbf{P_2} := (P_2, \leq_2)$ are posets and $f \colon P_1 \to P_2$ is a function, then we say that f is an *isomorphism* between $\mathbf{P_1}$ and $\mathbf{P_2}$ provided f is bijective and for all $x, y \in P_1$: $x \leq_1 y$ if and only if $f(x) \leq_2 f(y)$. If such an f exists, then we say that $\mathbf{P_1}$ and $\mathbf{P_2}$ are (*order-*) *isomorphic*.

Example 3. *We have seen that if M is a module over a ring R, then $\mathbf{L}_R(M)$ is a bounded poset with respect to $\subseteq$ with $0 = \{0\}$ and $1 = M$. Indeed, $\mathbf{L}_R(M)$ is also a lattice: if L and K are R-submodules of M, then* $\sup(L, K) := L + K$ (*the R-submodule of M generated by L and K*) *and* $\inf(L, K) := L \cap K$.

We now review some basic module-theoretic terminology to which we shall refer in the next section. First, we mention that throughout this paper, *all rings are assumed to be commutative* *with* $1 \neq 0$*, and all modules are assumed to be left and unitary.* Throughout, if R is a ring and M is an R-module, then we denote by Rm the cyclic R-submodule of M generated by $m \in M$. Next, recall the following fundamental definition:

Definition 1. Let R and S be rings, and suppose that M is simultaneously an R-module and an S-module. Then we say that the R-module structure of M and the S-module structure of M are *essentially the same* provided $Rm = Sm$ for every $m \in M$.

To mitigate any confusion, if M and N are simultaneously R-modules and S-modules, and M and N are isomorphic as (say) R-modules, then we shall write $M \cong_R N$. We pause to make an easy observation:

Observation 1. If R and S are rings and M is simultaneously an R-module and an S-module such that the respective module structures are essentially the same, then every R-submodule of M is an S-submodule of M and vice-versa.

To see why this is true, let us suppose that N is an R-submodule of M. To show that N is an S-submodule of M, it suffices to show that if $n \in N$ and $s \in S$, then $sn \in N$. So let $n \in N$ and let $s \in S$ be arbitrary. Since $Rn = Sn$ and $sn \in Sn$, it follows that $sn \in Rn$; hence $sn = rn$ for some $r \in R$. Because N is an R-submodule of M, we have $sn = rn \in N$, as required.

Next, let us suppose that M is an R-module. Recall that the *annihilator* of M in R, denoted $\mathrm{Ann}_R(M)$, is defined by $\mathrm{Ann}_R(M) := \{r \in R \colon rM = \{0\}\}$. It is straightforward to to check that $\mathrm{Ann}_R(M)$ is an ideal of R and that, moreover, M becomes an $R/\mathrm{Ann}_R(M)$-module via the scalar multiplication $\overline{r} \cdot m := rm$. Moreover, it is clear that the structure of M as an R-module is essentially the same as the structure of M as an $R/\mathrm{Ann}_R(M)$-module. Finally, note that M is *faithful* over $R/Ann_R(M)$: if $\overline{r} \in R/Ann_R(M)$ satisfies $\overline{r}M = \{0\}$, then $\overline{r} = \overline{0}$.

Moving on, let R be a ring and M be an R-module. Further, let J be a maximal ideal of R. Then M is said to be J*-primary* provided that for every $m \in M$, there is a positive integer n such that $J^n m = \{0\}$. We now pause to establish the following (well-known classical) result.

Lemma 1. *Let R be a ring, J be a maximal ideal of R, and suppose that M is J-primary. Then M has a natural module structure over the local ring R_J. Moreover, the structure of M as an R-module is essentially the same as the structure of M as an R_J-module.*

Proof. Let R, J, and M be as stated. First, we claim that for every $m \in M$ and $s \in R\backslash J$, there is a unique $m' \in M$ such that $m = sm'$. So let $m \in M$ and let $s \in R\backslash J$. By assumption, $J^n m = \{0\}$ for some positive integer n. The maximality of J implies that $(J^n, s) = R$. Thus there exists $j \in J^n$ and $x \in R$ such that $j + xs = 1$. Multiplying through by m, $jm + (xs)m = m$. Since $j \in J^n$, we have $jm = 0$. Thus $(xs)m = m$, and so $m = s(xm)$, completing the existence proof.

Now suppose that $m = sm' = sm''$ for some $m', m'' \in M$. Then $s(m' - m'') = 0$. Now, $J^k(m' - m'') = 0$ for some positive integer k. Also, $s(m' - m'') = 0$ with $s \in R\backslash J$. Thus $s \notin J^k$. It follows from the maximality of J that $(J^k, s)(m' - m'') = R(m' - m'') = \{0\}$. This shows that $1(m' - m'') = 0$, and so $m' = m''$.

Finally, we may define a scalar multiplication on M over the ring R_J by $\frac{r}{s} \cdot m :=$ the unique $m \in M$ such that $rm = sm'$. This yields a module structure for M as an R_J-module. It remains to check that the structure of M as an R-module is essentially the same as the structure of M as an R_J-module. To see this, let $m \in M$. We must check that $Rm = R_J m$. Let $r \in R$ be arbitrary. Then by definition, $rm = \frac{r}{1}m$, and so $Rm \subseteq R_J m$. Conversely, consider $\frac{r}{s}m$, where $r \in R$ and $s \in R \backslash J$. By definition, $\frac{r}{s}m$ is the unique $m' \in M$ such that $rm = sm'$. By the first paragraph of the proof above, there is $x \in R$ such that $rm = s(xrm)$. Hence $\frac{r}{s}m = xrm \in Rm$, and the proof is complete. $\square$

Next, recall that an R-module M is *Artinian* provided every infinite, decreasing sequence of R-modules stabilizes. We shall make essential use of the following classical structure theorem for Artinian modules. First, observe that if J is a maximal ideal of a ring R and M is an R-module, then $M[J] := \{m \in M \colon J^n m = \{0\}$ for some positive integer $n\}$ is an R-submodule of M, called the *J-primary component* of M.

Proposition 1 ([16], Lemma 1.7)**.** *Let R be a ring and let M be an Artinian R-module. Then there is a finite, nonempty collection $\{J_1, \ldots, J_n\}$ of n (distinct) maximal ideals of R such that $M = M[J_1] \oplus \cdots \oplus M[J_n]$.*

3. Results

3.1. **Introduction of the problem.** Let us consider a ring R and an R-module M. Next, let $\mathbf{L}_R(M)$ denote the collection of R-submodules of M. Then as we have seen, $(\mathbf{L}_R(M), \subseteq)$ is a bounded lattice. Our goal, as stated in the Introduction, is to classify the R-modules M such that $\mathbf{L}_R(M)$ is lower finite (recall that this means that for every $N \in \mathbf{L}_R(M) \backslash \{M\}$, there are but finitely many $K \in \mathbf{L}_R(M)$ such that $K \leq N$). Phrased completely in module-theoretic terminology, we study the following problem:

Problem 2. Let R be a ring. Classify all R-modules M with the property that M has infinitely many R-submodules, but for every proper R-submodule N of M, N has but finitely many R-submodules.

For the sake of brevity, let us call an R-module M as described in Problem 2 *lower finite*. We pause to present two examples.

Recall that if p is a prime number, then the direct limit $\lim_{n\to\infty} \mathbb{Z}/(p^n)$ of the cyclic groups $\mathbb{Z}/(p^n), n \in \mathbb{N}$, is an infinite group called the *quasicyclic p-group* or the *Prüfer p-group*, denoted by $C(p^\infty)$. This group can be realized more concretely as the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of those elements with additive order a power of p. It is well-known (and easy to prove) that $C(p^\infty)$ is infinite, yet every proper subgroup of $C(p^\infty)$ is isomorphic to $\mathbb{Z}/(p^n)$ for some $n \in \mathbb{N}$. Moreover, $C(p^\infty)$ contains an isomorphic copy of $\mathbb{Z}/(p^n)$ for every $n \in \mathbb{N}$. It follows that $C(p^\infty)$ has infinitely many subgroups, yet every proper subgroup of $C(p^\infty)$, being finite, has but finitely many subgroups. Therefore,

Example 4. *$C(p^\infty)$ is a lower finite $\mathbb{Z}$-module for every prime number p.*

Next, consider $\mathbb{R}^2$ as a vector space over $\mathbb{R}$. Clearly there exist infinitely many lines through the origin, and every line through the origin is a subspace of $\mathbb{R}^2$. Therefore, $\mathbb{R}^2$ has infinitely many subspaces. Now, if V is any proper subspace of $\mathbb{R}^2$, then V is either a line through the origin or trivial. We conclude that V has but two subspaces or one subspace, respectively. Thus,

Example 5. *$\mathbb{R}^2$ is a lower finite $\mathbb{R}$-module.*

3.2. **Preliminary results.** We now equip ourselves to determine the structure of lower finite modules over an arbitrary ring. We start with a simple lemma. The techniques used and most of the results established in the course of proving Lemma 2 below are standard in the theory of Artinian modules. Since the argument required is short, we include it for completeness.

Lemma 2. *Let R be a ring and let M be an Artinian R-module. Further, suppose $M = \bigoplus_{i=1}^n M[J_i]$ is a primary decomposition of M as in Proposition 1. If each $M[J_i]$ has but finitely many R-submodules, then M has but finitely many R-submodules.*

Proof. We suppose that R, M, and the J_i are as stated. Let N be an arbitrary R-submodule of M. Since M is Artinian, so is N. Thus we may decompose N as follows:

$$N = N[Q_1] \oplus \cdots \oplus N[Q_m] \text{ for some distinct maximal ideals } Q_1, \ldots, Q_m \text{ of } R. \tag{3.1}$$

We assume of course that each $N[Q_j]$ is nontrivial. Our first claim is that

$$\text{each } Q_j \text{ is equal to some } J_l. \tag{3.2}$$

To see this, let $n \in N[Q_j]$ be nonzero. Then $\mathrm{Ann}_R(n)$ is a proper ideal of R, and so $\mathrm{Ann}_R(n) \subseteq I$ for some maximal ideal I of R. Because $n \in N[Q_j]$, $Q_j^k n = \{0\}$ for some positive integer k. We conclude that $Q_j^k \subseteq \mathrm{Ann}_R(n) \subseteq I$. As I is maximal, I is also a prime ideal. The primeness of I and the fact that $Q_j^k \subseteq I$ imply that $Q_j \subseteq I$. As Q_j is maximal, we deduce that $Q_j = I$. On the other hand, $n \in M$. It follows from the J_i-primary decomposition of M above that there exist positive integers $k_1, \ldots, k_n$ such that $(J_1^{k_1} \cap \cdots \cap J_n^{k_n})n = \{0\}$. We conclude as above that $J_1^{k_1} \cap \cdots \cap J_n^{k_n} \subseteq I$. The primeness of I implies that $J_l^{k_l} \subseteq I$ for some l with $1 \leq l \leq n$. Then as above, $J_l \subseteq I$ and $J_l = I$. We now have $Q_j = I = J_l$, establishing (3.2). Without loss of generality, $Q_i = J_i$ for $1 \leq i \leq m$. It is immediate from the definition of primary components that

$$N[Q_i] \subseteq M[Q_i = J_i] \text{ for } 1 \leq i \leq m. \tag{3.3}$$

It is now clear from (3.1) and (3.3) that

$$N = M_1 \oplus \cdots \oplus M_m \text{ for some } M_1, \ldots, M_m \leq M \text{ such that } M_i \subseteq M[J_i] \text{ for } 1 \leq i \leq m. \tag{3.4}$$

Because each $M[J_i]$ has but finitely many R-submodules, it follows from (3.4) and the fact that $N \leq M$ was arbitrary that M itself has but finitely many R-submodules, as claimed. ☐

Now let R be a ring, M be an R-module, and J be a maximal ideal of R. If every element of M is annihilated by some power of J, then we say that M is *J-primary*. We are ready to prove our first result on lower finite modules.

Proposition 2. *Let R be a ring, and suppose that M is a lower finite R-module. Then the following hold:*

(1) *M is Artinian, and*
(2) *M is J-primary for some maximal ideal J of R.*

Proof. We suppose that M is a lower finite R-module. Since every proper R-submodule of M has but finitely many R-submodules, it is clear that any strictly decreasing sequence of R-submodules of M is finite. Therefore, M is Artinian. As for (2), since M is Artinian, M has a primary decomposition, say $M = \bigoplus_{i=1}^{n} M_i[J_i]$ for some maximal ideals $J_1, \ldots, J_n$ of R. If each $M[J_i]$ has finitely many R-submodules, then by Lemma 2, M itself has but finitely many R-submodules, a contradiction. Thus $M[J_i]$ has infinitely many R-submodules for some j, $1 \leq j \leq n$. Because M is lower finite, we deduce that $M = M[J_i]$, and hence M is J_i-primary. □

Our first corollary shows that no ring is lower finite as a module over itself, and thus lower-finiteness is strictly a module-theoretic property.

Corollary 1. *Let R be a ring. Then R is not a lower finite R-module.*

Proof. Suppose by way of contradiction that there exists a ring R which is lower finite as a module over itself. Then by (1) of Proposition 2, R is an Artinian ring, and therefore has but finitely many maximal ideals: let $M_1, \ldots, M_k$ be the maximal ideals of R. By definition of lower finite, R has infinitely many ideals, thus infinitely many proper ideals. As every proper ideal is contained in a maximal ideal and because R has but finitely many maximal ideals, we conclude that some M_i contains infinitely many ideals. But M_i is a proper ideal, and this contradicts the definition of lower finite. □

Remark 1. Suppose that M is lower finite module over a ring R. Then by Proposition 2, M is J-primary for some maximal ideal J of R. By Lemma 1, M is naturally an R_J-module with essentially the same module structure as M as an R-module. Thus without loss of generality, we may assume that lower finite modules are modules over local rings. Now, let M be a lower finite module over the local ring (R, J). Recall from earlier that M has essentially the same structure over R as over $R/Ann_R(M)$, and $R/Ann_R(M)$ remains local. Hence without loss of generality, we may consider only faithful lower finite modules over local rings.

We conclude this subsection with a straightforward lemma and a final proposition of which we shall make heavy use throughout the remainder of the paper.

Lemma 3. *Let F be an infinite field. Then the F-vector space F^2 has infinitely many subspaces.*

Proof. Suppose that F is an infinite field. For each $a \in F$, let $\mathbf{v}_a := (1, a)$. Now set $\ell_a := \{t\mathbf{v}_a : t \in F\}$. Then of course, each ℓ_a is an F-subspace of F^2. It suffices to prove that $\ell_a \neq \ell_b$ for $a \neq b$. Indeed, suppose that $a \neq b$. Then it is clear that $(1, a) \in \ell_a \backslash \ell_b$, and the proof is complete. □

Proposition 3. *Suppose that (R, J) is a local ring and M is a faithful lower finite R-module. Assume further that the residue field R/J is infinite. Then either $M \cong_R R \oplus R$ and R is a field or there is a simple R-submodule N of M which is essential in M.*

Proof. Let R/J and M be as stated, and suppose that R/J is infinite. We consider two cases.

Case 1. There exist nonzero R-submodules N_1 and N_2 of M such that $N_1 \cap N_2 = \{0\}$. Because M is Artinian, both N_1 and N_2 are also Artinian. It follows that there exist simple R-modules M_1 and M_2 contained in N_1 and N_2, respectively. Because $N_1 \cap N_2 = \{0\}$, also $M_1 \cap M_2 = \{0\}$. Let $M_1 := Rm_1$ and $M_2 := Rm_2$. Because Rm_1 is simple, it follows that $\mathrm{Ann}_R(m_1)$ is a maximal ideal of R, and thus $\mathrm{Ann}_R(m_1) = J$. Similarly $\mathrm{Ann}_R(m_2) = J$. We deduce that $Rm_1 \cong Rm_2 \cong R/J$. Thus M contains an isomorphic copy of $R/J \oplus R/J$. Observe that the structure of $R/J \oplus R/J$ as an R-module is essentially the same as the structure of $R/J \oplus R/J$ as an R/J-vector space. Invoking Lemma 3, $R/J \oplus R/J$ has infinitely many R/J-subspaces, and therefore infinitely many R-subspaces. Because M is lower finite, we deduce that $M \cong_R R/J \oplus R/J$. Recall that M is also faithful, and hence $J = \{0\}$. It follows that R is a field, and $M \cong_R R \oplus R$.

Case 2. M is uniform. Because M is a nontrivial Artinian module, as above, M contains a simple R-submodule N. It is clear that N is essential in M. □

3.3. **A classification of the infinitely generated lower finite R-modules.** We begin this subsection with a brief review of the fundamentals of discrete valuation rings (DVRs), as they will play a significant role in what follows. If D is a domain, then $K := \{\frac{a}{b} \colon a \in D, b \in D\backslash\{0\}\}$ (with addition and multiplication carried out formally) is a field, called the *quotient field* of D. The map $d \mapsto \frac{d}{1}$ is an embedding of D into K, and we identify D with its image in K. Next, we recall that an integral domain V is a *valuation domain* if the ideals of V are linearly ordered by set inclusion. A *discrete valuation ring* (DVR) is a principal ideal domain (PID) V with a unique nonzero prime ideal $\mathfrak{m} := (m)$. Since V is a PID, V is also a unique factorization domain (UFD). As V has a unique prime element m (up to units), it follows that every nonzero, nonunit of V has the form um^k for some unit $u \in V$ and some positive integer k. From this fact, it is immediate that the ideals of V are linearly ordered (that is, V is in fact a valuation domain). We now collect some well-known facts about the V-module K/V, where K is the quotient field of the discrete valuation ring V.

Lemma 4. *Let V be a DVR with maximal ideal $\mathfrak{m} := (m)$ and quotient field K. Further, for $k \in \mathbb{Z}^+$, we define the following V-submodule of K/V: $M_k := \{V + \frac{v}{m^k} \colon v \in V\}$. Then the following hold:*

1. *the V-submodules M_k as defined above strictly ascend,*
2. *the V-submodules M_k are exactly the proper, nontrivial V-submodules of K/V,*
3. *K/V is not finitely generated over V, and*
4. *K/V is faithful over V.*

Sketch of Proof. We let V, $\mathfrak{m} := (m)$, and K be as stated.

(1) First, it is clear that $M_k \subseteq M_{k+1}$ for every positive integer k. We claim that $V + \frac{1}{m^{k+1}} \notin M_k$. If so, then $V + \frac{1}{m^{k+1}} = V + \frac{v}{m^k}$ for some $v \in V$. Therefore, $\frac{1}{m^{k+1}} - \frac{v}{m^k} \in V$. Multiplying through by m^k (a member of V), we have $\frac{1}{m} - v \in V$. Since $v \in V$, we deduce that $\frac{1}{m} \in V$. This contradicts the fact that m is not a unit of V.

(2) Well-known; see [14], Lemma 4.

(3) First, observe that any nonzero element of K/V has the form $V + \frac{u}{m^k}$ for some unit $u \in V$ and positive integer k, hence is a member of M_k. Hence by (1), it is clear that K/V is not finitely generated.

(4) Suppose by way of contradiction that there is some nonzero $v \in V$ such that $v(K/V) = \{0\}$. Then clearly v is not a unit, and so $v = um^k$ for some unit u of V and some positive integer k. Since $v(K/V) = \{0\}$, we have $vK \subseteq V$. But then $\frac{um^k}{m^{k+1}} \in V$. This implies that m is a unit of V, a contradiction as above. □

We are almost ready to analyze the lower finite J-primary R-modules M in case the residue field R/J is infinite and M is infinitely generated over R (by *infinitely generated*, we mean *not finitely generated*). We first recall the following results of Hirano and Mogami:

Lemma 5 ([7], Theorems 8 and 10)**.** *Let R be a ring and let M be an R-module. Suppose further that $\mathbf{L}_R(M)$ is isomorphic to the ordinal $\omega + 1$.*[3] *Let $S := \mathrm{End}_R(M)$ be the endomorphism ring of M over R. Then the structure of M as an S-module is essentially the same as the structure of M as an R-module. Moreover, S is a DVR, and if K is the quotient field of S, then $M \cong_S K/S$.*

Proposition 4. *Let (R, J) be a local ring with infinite residue field, and suppose that M is an infinitely generated faithful lower finite R-module. Set $S := \mathrm{End}_R(M)$. Then S is a discrete valuation ring, the structure of M as an S-module is essentially the same as the structure of M as an R-module, and $M \cong_S K/S$, where K is the quotient field of S.*

Proof. Assume that (R, J) is a ring with infinite residue field and M is an infinitely generated faithful lower finite R-module. By Lemma 5, it suffices to prove that $\mathbf{L}_R(M)$ is isomorphic to $\omega + 1$. Applying Proposition 3, either $M \cong_R R \oplus R$ and R is a field or there is a simple R-submodule N of M which is essential in M. The former is impossible because M is infinitely generated; let N_1 be a simple essential R-submodule of M. Then note that $N_1 \subseteq N$ for every nontrivial R-submodule N of M. Because every proper R-submodule of M has but finitely many R-submodules, clearly the same is true of M/N_1. As M is lower finite, M has infinitely many R-submodules, hence certainly has infinitely many nontrivial R-submodules. As N_1 is contained in every nontrivial R-submodule of M, we deduce that M/N_1 has infinitely many R-submodules. Because M is infinitely generated and N_1 is finitely generated, M/N_1 is infinitely generated over R. We have established that M/N_1 is an infinitely generated lower finite R-module. By what we proved above[4], there is a finitely generated R-submodule N_2 of M strictly containing N_1 with the property that every R-submodule of M which strictly contains N_1 contains N_2. Proceeding recursively, we obtain a strictly increasing sequence $N_0 := \{0\} \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots$ of submodules of M with the property that

$$\text{for all } i \in \mathbb{N}: \text{ if } K \text{ is an } R\text{-submodule of } M \text{ such that } N_i \subsetneq K, \text{ then } N_{i+1} \subseteq K. \tag{3.5}$$

[3] As an ordered set, $\omega + 1$ is isomorphic to $\mathbb{N} \cup \{\infty\}$, where we extend the usual order $\leq$ on $\mathbb{N}$ by defining $a \leq \infty$ for all $a \in \mathbb{N} \cup \{\infty\}$.

[4] M/N_1 may not be faithful over R, but if M/N_1 has no simple essential submodule, it is two-generated, a contradiction.

Now, set $N := \bigcup_{i\in\mathbb{N}} N_i$. Since the N_i strictly ascend, N is an R-submodule of M. Moreover, each N_i is a submodule of N, and we deduce that N has infinitely many R-submodules. Because M is lower finite, it follows that $M = N$. To finish the proof, it suffices to prove that $\{N_i : i \in \mathbb{N}\}$ is the set of proper R-submodules of M. For any $i \in \mathbb{N}$, $N_i \subsetneq N_{i+1}$, and so N_i is proper. Conversely, let K be any proper R-submodule of M. We cannot have $N_i \subseteq K$ for arbitrarily large $i \in \mathbb{N}$, lest $M = \bigcup_{i\in\mathbb{N}} N_i \subseteq K$, and K not be proper. Choose the largest i such that $N_i \subseteq K$. We claim that $N_i = K$. Otherwise, $N_i \subsetneq K$; by (3.5), $N_{i+1} \subseteq K$, contradicting the maximality of i. This concludes the proof. □

Next, we turn our attention to classifying the infinitely generated, lower finite, (R, J)-modules in case the residue field R/J is finite. Toward this end, we shall invoke the theory of Jónsson modules. We recall that a module M over a ring R is a *Jónsson module* provided M is infinite and every proper R-submodule of M has smaller cardinality than M. It is well-known that the Jónsson $\mathbb{Z}$-modules are precisely the quasi-cyclic groups $C(p^\infty)$, p a prime number (this result can be found in the classic text [4], for example). We collect the following results to be utilized in the sequel.

Lemma 6. *Let R be a ring and M be an R-module.*

(1) ([5], *Corollary 2.3*) *If M is a finitely generated Jónsson module, then M is cyclic.*
(2) ([5], *Proposition 2.5(2)*) *If M is a Jónsson R-module, then* $\mathrm{Ann}_R(M)$ *is a prime ideal of R.*
(3) ([13], *Theorem 4.1 (partial)*) *If M is a faithful, infinitely generated Artinian Jónsson module, then M is countable.*
(4) ([12], *Theorem 2 (partial)*) *If M is infinitely generated, countable, and faithful over R, then M is a Jónsson R-module if and only if R is a domain (say with quotient field K) and there exist both a DVR overring $(V, \mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$.*

We now determine the infinitely generated faithful lower finite (R, J)-modules when R/J is finite.

Proposition 5. *Suppose that R is a ring and M is an infinitely generated faithful lower finite (R, J)-module, where R/J is finite. Then R is a domain; let K be the quotient field of R. Moreover, there is a DVR overring $(V, \mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$.*

Proof. We let (R, J) and M be as stated. Because M is lower finite, by definition, M has infinitely many R-submodules. Hence M is infinite. Because M is Artinian, there is an R-submodule N of M which is minimal with repect to being infinite. Hence by minimality, every proper R-submodule of N is finite, and so N is a Jónsson module. We claim that

$$M = N. \tag{3.6}$$

Suppose not. Because M is lower finite, it follows that N has but finitely many R-submodules, and is thus Noetherian. By (1) of Lemma 6, N is cyclic. Invoking (2) of Lemma 6, $Q := \mathrm{Ann}_R(N)$ is a prime ideal of R. On the other hand, M is J-primary, and so $J^k N = \{0\}$ for some positive integer k. Thus $J^k \subseteq Q$, and the primeness of Q along with the maximality of J implies that

$J = Q$. But now $N \cong_R R/J$, and so N is finite, a contradiction. This establishes (3.6). As M is faithful over R, it follows from (2) of Lemma 6 that R is a domain. Invoking (3) of Lemma 6, M is countable. Finally, an immediate application of (4) of Lemma 6 completes the argument. $\square$

3.4. **A classification of the finitely generated lower finite R-modules.** Finally, we consider lower finite modules which are finitely generated. For the purposes of the remainder of the article, we introduce some final terminology. Let M be an R-module and let K be an R-submodule of M. Recall that K is called a *waist* of M provided that for every R-submodule N of M, either $K \subseteq N$ or $N \subseteq K$.[5] Observe trivially that if M itself is uniserial, then every R-submodule of M is a waist.

Proposition 6. *Suppose that (R, J) is a local ring and M is a finitely generated faithful lower finite R-module. Then the following hold:*

1. *R/J is infinite, and*
2. *there exists a finitely generated, uniserial, Artinian waist N of M such that $M/N \cong R/J \oplus R/J$.*

Proof. Assume that (R, J) is a ring and M is a finitely generated faithful lower finite R-module.

(1) Suppose by way of contradiction that R/J is finite. Now, M is finitely generated and every proper R-submodule N of M has but finitely many submodules, hence N is also finitely generated. It follows that M is Noetherian. We claim that

(3.7) every maximal R-submodule of M contains the R-submodule JM.

Suppose instead that there is a maximal R-submodule K of M such that $JM \nsubseteq K$. Then by maximality of K, we have $JM + K = M$. But then by Nakayama's Lemma, $K = M$, a contradiction. This proves (3.7). Now, recall that R/J was assumed finite and M is finitely generated. Hence M/JM is a finite dimensional vector space over the finite field R/J, and is thus finite. We deduce that there are but finitely many R-submodules of M which contain JM. Invoking (3.7), it follows that M has but finitely many maximal submodules. Because M is Noetherian, every proper submodule of M is contained in a maximal submodule of M. But as M is lower finite, it follows that some maximal submodule of M contains infinitely many submodules, contradicting lower finiteness. This establishes (1).

(2) If $M \cong_R R \oplus R$, and R is a field then we may take $N := \{0\}$, and we're done. So suppose not. By Proposition 3, there is a simple R-submodule N_1 of M which is essential in M. Observe that M/N_1 is also a lower finite R-module. If $M/N_1 \cong_R R/J \oplus R/J$, then we are done, as N_1 is certainly a finitely generated, uniserial, Artinian waist of M. Otherwise, there is a simple R-submodule N_2/N_1 of M/N_1 which is essential in M/N_1. If $(M/N_1)/(N_2/N_1) \cong_R R/J \oplus R/J$, then we have (by the Isomorphism Theorems) $M/N_2 \cong_R R/J \oplus R/J$. In this case, note that N_2 is a finitely generated, Artinian waist of M; thus we are done. Continue this process recursively. Because M is finitely generated, this algorithm must terminate after finitely many steps: otherwise, as M is lower finite,

[5] The term "waist" was introduced to Auslander, Green, and Reiten in [3]. We thank Gene Abrams for bringing this to our attention.

we would have $N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots$ and (because M is lower finite) $M = \bigcup_{i \in \mathbb{Z}^+} N_i$. But then M is not finitely generated, a contradiction. $\square$

Remark 2. Observe that we cannot dispense with the assumption that there is a finitely generated, uniserial, Artinian waist N of M (that is, this doesn't follow from our other assumptions). For example, suppose that (R, J) is a local ring, N is a finitely generated, nontrivial, uniserial, Artinian R-submodule of M and that $M = N \oplus R/J \oplus R/J$. Then $M/N \cong R/J \oplus R/J$, and yet N is not a waist of M.

3.5. **Summary of Results.** We conclude this section with our main result. Note that we simply must prove the converses of results already included in the paper to establish the "only if" implications. We shall require the following lemmas from the literature.

Lemma 7 ([12], Lemma 2)**.** *Suppose that M is a faithful torsion module over the domain R and N is a finitely generated submodule of M. Then M/N is also faithful over R.*

Lemma 8 ([12], Lemma 3(1))**.** *Let R be a ring and suppose that I is a finitely generated ideal of R. If R/I is finite, then R/I^n is finite for every positive integer n.*

Theorem 1. *Let (R, J) be a local ring and let M be a faithful R-module. Then M is lower finite if and only if one of the following holds:*

1. *$S := \mathrm{End}_R(M)$ is a discrete valuation ring, the structure of M as an S-module is essentially the same as the structure of M as an R-module, and $M \cong_S K/S$, where K is the quotient field of S,*
2. *R is a domain; let K be the quotient field of D. There is a DVR overring $(V, \mathfrak{m})$ of R with $V/\mathfrak{m}$ finite and an R-module N such that $V \subseteq N \subsetneq K$ and $M \cong_R K/N$, or*
3. *R/J is infinite and there exists a finitely generated, Artinian waist N of M such that $M/N \cong_R R/J \oplus R/J$.*

Proof. Assume R is a ring an M is an R-module.

($\Rightarrow$) By Propositions 4 - 6.

($\Leftarrow$) We must verify that the modules in classes (1) - (3) are lower finite. Toward this end, it is immediate from (1), (2), and (4) of Lemma 4 that if S is a DVR with quotient field K, then K/S is a faithful lower finite S-module. This disposes of the modules in class (1).

Now suppose that M belongs to class (2). We claim that M is infinitely generated over R. Toward this end, recall from Lemma 4 that the proper, nontrivial V-submodules of K/V are the modules $M_k := \{V + \frac{v}{m^k} : v \in V\}$, where $k \in \mathbb{Z}^+$. Now, $M_k \cong_V V/\mathfrak{m}^k$. Since $V/\mathfrak{m}$ is finite, we apply Lemma 8 to conclude that M_k is also finite. Because K/V is the strictly increasing union of finite V-submodules, we conclude that K/V is countable. By (3) and (4) of Lemma 4, K/V is also infinitely generated and faithful over V. Because $R \subseteq V$, we see that

$$K/V \text{ is a countable, infinitely generated, faithful } R \text{ -module.} \tag{3.8}$$

By (4) of Lemma 6, K/V is a Jónsson R-module. Now, $M \cong_R K/N \cong_R (K/V)/(N/V)$. Because K/V is a countable Jónsson R-module and N/V is a proper R-submodule of K/V, we deduce that N/V is finite. Since K/V is infinitely generated over R and N/V is finite, it follows that $(K/V)/(N/V) \cong_R M$ is also infinitely generated over R, as claimed. Because M is an R-homomorphic image of the countable R-module K/V, we see that M is countable. So we have established that M is infinitely generated and countable over R. Finally, observe that K/V is a torsion V-module. Because V is a DVR overring of R, it follows from Lemma 7 that $(K/V)/(N/V) \cong_R M$ is also faithful over R. Applying (4) of Lemma 6, M is a Jónsson R-module. Because M is infinitely generated over R, M has infinitely many R-submodules. Moreover, every proper R-submodule of M is finite, hence has but finitely many R-submodules. Thus M is a lower finite R-module.

Finally, suppose that M belongs to class (3). Because N is finitely generated Artinian, N is also Noetherian. It follows that N has finite length as an R-module. Because N is also uniserial, we deduce that the factors of the (necessarily unique) composition series of N are precisely the R-submodules of N. Thus N has but finitely many R-submodules. Now, $M/N \cong_R R/J \oplus R/J$. Because J has infinite index in R, M/N has infinitely many R/J submodules (Lemma 3), and thus infinitely many R-submodules. It follows that M has infinitely many R-submodules. Now let K be a proper R-submodule of M. It remains to prove that K has but finitely many R-submodules. Because N is a waist of M, either $K \subseteq N$ or $N \subsetneq K$. In the first case, because N has but finitely many R-submodules, also K has but finitely many R-submodules. Now assume that $N \subsetneq K$. Then note that K/N is a one-dimensional R/J-subspace of M/N. It follows that there are no R-submodules of M which strictly contain N and are strictly contained in K. Because N is a waist, we deduce that if L is an R-submodule of K, then either $L = K$ or $L \leq N$. Since N has but finitely many R-submodules, we conclude that K has but finitely many R-submodules, as desired. □

Remark 3. Observe that if M is a finitely generated lower finite (R, J)-module, then M is two generated: we have $M/N \cong_R R/J \oplus R/J$ for some maximal ideal J of R such that N is a waist of M. Let m_1 and m_2 generate M modulo N. Since N is a waist, $N \subseteq Rm_1$ and $N \subseteq Rm_2$. Thus $M = Rm_1 + Rm_2$.

4. Directions for further research

We close the article with two natural lines of investigation for further research. The first question we leave completely open, though we have some comments on the latter.

Open Problem 1. Investigate lower finite modules over noncommutative rings.

A second question is the following "dual" of the question investigated in this article. We are agnostic on whether to assume commutativity of the operator ring, so as to state in more generality.

Open Problem 2. Let R be a ring and let M be an R-module. Say that M is *upper finite* provided M has infinitely many submodules, but for every nonzero R-submodule N of M, there are but finitely many R-submodules of M which contain N. Study the upper finite R-modules.

We have a bit to say about this problem. First, it is not hard to see that the ring $\mathbb{Z}$ of integers is upper finite as a module over itself; this follows more or less immediately from the fact that every nonzero integer has but finitely many divisors. In fact, more generally, if D is any Dedekind domain (a domain for which every proper, nonzero ideal factors as a product of prime ideals), then D is upper finite. A curious fact is that for "sufficiently large" commutative domains D, D is Dedekind if and only if D is upper finite. This completely characterizes the Dedekind domains of size strictly greater than $2^{\aleph_0}$. On the other hand, if κ is a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, then it can be shown (in ZFC) that there is an upper finite domain D of cardinality κ which is not Dedekind, and so the strict lower bound of $2^{\aleph_0}$ is sharp. These results can be found in [11]. These results show that unlike the case for lower finite modules, being upper finite is not strictly a proper module-theoretic property in the sense that there are rings which are upper finite as modules over themselves. We conclude the paper with an upper-finite characterization of the abelian group $(\mathbb{Z}, +)$.

Proposition 7. *Let G be an abelian group. Then G is infinite cyclic if and only if G is upper finite as a $\mathbb{Z}$-module.*

Proof. We have already explained that $\mathbb{Z}$ is upper finite. Conversely, suppose that G is an upper finite abelian group. Upper finiteness clearly implies that G is a Noetherian $\mathbb{Z}$-module, and so G is finitely generated. The fact that G is infinite along with the Fundamental Theorem of Finitely Generated Abelian Groups implies that $G \cong H \oplus F$, where H is a finite abelian group and F is a nontrivial free abelian group. Clearly F has rank one, lest a summand be contained in infinitely many subgroups of G. Hence $G \cong H \oplus \mathbb{Z}$. Now, H is a subgroup of $H \oplus K$ for every subgroup K of $\mathbb{Z}$. It follows from upper finiteness that H must be trivial, whence $G \cong \mathbb{Z}$, as claimed. □

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