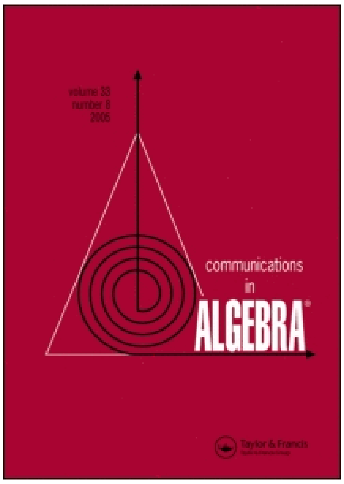

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Jónsson Modules over Noetherian Rings

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Online publication date: 30 September 2010

To cite this Article Oman, Greg(2010) 'Jónsson Modules over Noetherian Rings', Communications in Algebra, 38: 9, 3489 — 3498

To link to this Article: DOI: 10.1080/00927870902936943

URL: http://dx.doi.org/10.1080/00927870902936943

Communications in Algebra®, 38: 3489–3498, 2010

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870902936943

JÓNSSON MODULES OVER NOETHERIAN RINGS

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Let R be a commutative ring with identity, and let M be an infinite unitary R-module. M is said to be a Jónsson module provided every proper submodule of M has strictly smaller cardinality than M. Utilizing earlier results of the author [11] as well as results of Gilmer/Heinzer, Weakley, and Heinzer/Lantz [8, 10, 14], we study Jónsson modules over Noetherian rings. After a brief introduction, we classify the countable Jónsson modules over an arbitrary ring up to quotient equivalence. We then give a complete description of the Jónsson modules over a 1-dimensional Noetherian ring, extending W. R. Scott's classification over ℤ. We show that these results may be extended to Jónsson modules over an arbitrary Noetherian ring if one assumes The Generalized Continuum Hypothesis. Finally, we close with a list of open problems.

Key Words: Cardinality; Discrete valuation ring; Generalized continuum hypothesis; Jónsson module; Noetherian ring.

2000 Mathematics Subject Classification: 13C05; 03E50.

All rings in this paper are assumed to be commutative with identity, and all modules are assumed to be unitary.

1. INTRODUCTION

An old problem posed by Kurosh was to determine if there exists a group of cardinality $\aleph_1$ in which all proper subgroups are countable. In the mid 1970s, Shelah constructed such a group [13]. This spurred more interest from the logic community in so-called Jónsson algebras, which are algebras with countably many finitary operations in which every proper subalgebra has smaller cardinality (see [2] for an excellent survey).

These notions piqued the interest of commutative algebraists Robert Gilmer and Bill Heinzer, who translated these ideas to the context of unitary modules over a commutative ring with identity [8]. They define an infinite module M over a ring R to be a *Jónsson module* if every proper submodule of M has smaller cardinality than M. Using various ideal-theoretic techniques, they give a complete description of all countable Jónsson modules over a Prüfer domain, and prove several propositions

Received August 15, 2008. Communicated by R. Wiegand.
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about general Jónsson modules. They applied and extended these results in several subsequent papers [5–7]. More recently, further results were obtained by the author [11]. As a precursor to our work, mathematician W.R. Scott classified all Jónsson modules over $\mathbb{Z}$ [12].

We begin by providing the reader with 2 canonical examples of Jónsson modules.

Example 1. Let F be an infinite field, and consider F as a module over itself. The submodules of F are precisely the ideals of F. Since F has only trivial ideals, it is easy to see that F is a Jónsson module over itself.

More generally, if R is any ring with an infinite residue field R/M, then R/M becomes a Jónsson module over R.

Example 2. Let p be a prime number. The direct limit of the cyclic groups $\mathbb{Z}/(p^n)$ is the so-called quasi-cyclic group of type p^∞, denoted by $C(p^\infty)$. It is well known that every proper subgroup of $C(p^\infty)$ is finite, whence $C(p^\infty)$ is a Jónsson module over $\mathbb{Z}$.

Scott proved some time ago that the quasi-cyclic groups are the only abelian Jónsson groups ($\mathbb{Z}$-modules). We close the introduction by giving a new proof of this fact. Before doing so, we recall 2 results from abelian group theory. The first is exercise 2, p. 67 of [3].

Proposition 1. *Every abelian group is either divisible or contains a maximal subgroup.*

Proposition 2 (Structure Theorem for Divisble Abelian Groups). *Every divisible abelian group is a direct sum of copies of $\mathbb{Q}$ and copies of $C(p^\infty)$ for various primes p.*

Proof. See p. 64 of [3]. □

We now provide a new and simple proof of Scott's result.

Theorem 1 ([12, Remark 1]). *The only abelian Jónsson groups are the quasi-cyclic groups $C(p^\infty)$.*

Proof. Let G be an abelian Jónsson group. It is easy to see that G is indecomposable: for if $G = M \bigoplus N$, then since G is infinite, either $|M| = |G|$ or $|N| = |G|$. Since G is Jónsson, this forces either $M = G$ or $N = G$. If G is divisible, it follows from the structure theorem for divisible abelian groups and the fact that G is indecomposable that $G \cong \mathbb{Q}$ or $G \cong \mathbb{C}(p^\infty)$ for some prime p. As $\mathbb{Q}$ is clearly not a Jónsson group, we have $G \cong \mathbb{C}(p^\infty)$ and we're done. Suppose now by way of contradiction that G has a maximal subgroup M. Since G is Jónsson, $|M| < |G|$. It is well known that $G/M \cong \mathbb{Z}/(p)$ for some prime p and thus M must be infinite. Let $g \in G - M$. Then $(M, g) = G$ by maximality of M. However, $|G| = |(M, g)| = |M| < |G|$ and this is a contradiction. This completes the proof. □

It is a curious fact that the Jónsson modules over $\mathbb{Z}$ are all countable and all isomorphic to well-known groups. Surprisingly, it turns out that these 2 properties are essentially true of all Jónsson modules over an arbitrary Noetherian ring if one assumes the generalized continuum hypothesis.

2. COUNTABLE JÓNSSON MODULES

In this section, we provide a complete description of the countable Jónsson modules over an arbitrary ring up to quotient equivalence. This classification relies heavily on ideas from Gilmer, Heinzer, Weakley, and Lantz. We begin with some fundamental results from the literature.

Proposition 3 ([8, Proposition 2.5]). *Suppose that M is a Jónsson module over a ring R. Then the following hold*:

(1) *For every $r \in R$, either $rM = M$ or $rM = \{0\}$.*
(2) *$Ann(M) := \{r \in R : (\forall m \in M) rm = 0\}$ is a prime ideal of R.*

Hence by modding out the annihilator, there is no loss of generality in restricting our study to faithful Jónsson modules over an integral domain.

Lemma 1 ([11, Lemma 2]). *Suppose that M is a Jónsson module over R. If N is any proper submodule of M, then M/N is also a Jónsson module over R.*

We now recall the following important result on countable Jónsson modules due to Robert Gilmer and Bill Heinzer.

Proposition 4 ([8, Theorem 3.1]). *Suppose that M is a countably infinite Jónsson module over the ring R and that M is not finitely generated. Then M is a torsion R-module, and there exists a maximal ideal Q of R such that the following hold*:

(1) *$Ann(x)$ is a Q-primary ideal of finite index for every $x \in M - \{0\}$.*
(2) *R/Q is finite.*
(3) *The powers of Q properly descend.*
(4) *$\bigcap_{i=1}^{\infty} Q^i = Ann(M)$.*
(5) *If $H_i = \{x \in M : Q^i x = 0\}$, then $\{H_i\}_{i=1}^{\infty}$ is a strictly ascending sequence of submodules of M such that $M = \bigcup_{i=1}^{\infty} H_i$.*

Heinzer and Lantz [10] and Weakley [14] call a module M *almost finitely generated* if M is not finitely generated, but all proper submodules of M are finitely generated. We recall 2 other important definitions as well as some results to be used shortly.

Definition 1. Let D be a domain. D is called an almost DVR provided that the integral closure $\overline{D}$ of D is a DVR which is finitely generated as a D-module.

Definition 2 (Matlis). Two modules M and N are said to be quotient equivalent if each module is a homomorphic image of the other. In this case, we write $M \sim^e N$.

Proposition 5 ([10, Proposition 2.2]). *Let D be a domain with quotient field K and suppose that (S, M) is an almost DVR between D and K (here M is the maximal ideal of S). Then K/S is an almost finitely generated D-module if S/M is a D-module of finite length.*

Proposition 6 ([14, Proposition 2.2]). *Let M be an Artinian almost finitely generated R-module, and let $q = Ann(M)$ (q must be prime). Then there is a discrete valuation ring V between R/q and $Q(R/q)$ (the quotient field of R/q) such that $M \sim^e Q(R/q)/V$.*

Using these results, we are able to characterize the faithful countable Jónsson modules over an arbitrary domain D. We first prove 2 easy lemmas.

Lemma 2. *Suppose that M is a faithful torsion module over the domain D and N is a finitely generated submodule of M. Then M/N is also faithful over D.*

Proof. Let $N = Rn_1 + \cdots + Rn_k$. Since M is torsion, we may pick nonzero elements $d_1, \ldots, d_k$ in D such that for each i, $d_i n_i = 0$. Suppose now that $d \in D$ and d annihilates all of M/N. We will show that $d = 0$. Let $m \in M$ be arbitrary. Then as d annihilates M/N, $dm \in N$. It follows easily that $d_1 \cdots d_k dm = 0$. This shows that $d_1 \cdots d_k d$ annihilates M. Since M is faithful, $d_1 \cdots d_k d = 0$. Since D is a domain and each d_i is nonzero, this forces $d = 0$. Hence M/N is faithful, as required. □

Lemma 3. *Let R be a ring and I a finitely generated ideal of R. Then*:

(1) *If R/I is finite, then R/I^n is finite for every positive integer n.*
(2) *If R/I has infinite cardinality κ, then R/I^n has cardinality κ for every positive integer n.*

Proof. Induct on n, the case $n = 1$ being obvious. By the second isomorphism theorem, there is a surjective module homomorphism from R/I^n to R/I with kernel I/I^n, whence $|R/I^n| = |I/I^n||R/I|$. Thus it suffices to show that I/I^n is finite (or, analogously, that I/I^n has cardinality κ). Note that I/I^n is a finitely generated module over the ring R/I^{n-1}, which by our inductive hypothesis is finite (or has infinite cardinality κ). This completes the proof. □

We are now in position to prove the main result of this section.

Theorem 2. *Let D be a domain with quotient field K, and suppose that M is a countably infinite faithful D-module. Then M is a Jónsson module if and only if one of the following holds*:

(1) *D is a field and $M \cong D$.*
(2) *There is a discrete valuation overring V of D with finite residue field such that M is a homomorphic image of K/V.*

Proof. We assume that D is a domain with quotient field K. We first show that the modules in (1) and (2) are in fact faithful Jónsson modules. If D is a field, then D has no nontrivial submodules, and hence is trivially a Jónsson module. Suppose now that V is a discrete valuation overring of D with a finite residue field and that M is a homomorphic image of K/V. Let (g) be the maximal ideal of V. We first claim that every cyclic V-submodule of K/V is finite. Thus consider a nonzero element $x \in K/V$. Then $(x) \cong V/Ann(x)$ as V-modules. Since K/V is a torsion V-module, we see that $Ann(x)$ is a nonzero proper ideal of V. But since V is a DVR, it is well known that $Ann(x) = (g^k) = (g)^k$ for some positive integer k. Thus $|(x)| = |V/Ann(x)| = |V/(g)^k|$. Since $V/(g)$ is finite, it follows from Lemma 3 that (x) is also a finite V-module. It now follows trivially that every cyclic D-submodule of K/V is finite. Because $V/(g)$ is finite, it clearly has finite length as a D-module, so by Proposition 5 K/V is an almost finitely generated D-module. We claim that K/V is a Jónsson D-module. For suppose that N is a proper D-submodule of K/V. Then N is finitely generated. Since each cyclic D-submodule of K/V is finite, it follows that N is also finite, and hence K/V is a Jónsson module over D. Clearly K/V is faithful over V and hence faithful over D. As M is a homomorphic image of K/V, it follows from Lemmas 1 and 2 that M is a faithful Jónsson module over D.

Now suppose that M is an arbitrary countable faithful Jónsson module over D. If M is finitely generated, then Corollary 2.3 of [8] shows that D is a field and $M \cong D$. Thus we suppose that M is infinitely generated. Since every proper submodule of M is finite, M is almost finitely generated and Artinian. By Proposition 6, there is a discrete valuation overring V of D such that $M \sim^e K/V$. It remains to show that V has a finite residue field. Since $M \sim^e K/V$, K/V is a homomorphic image of M. By Lemma 1, K/V is a Jónsson module over D. But since $D \subseteq V$, clearly K/V is a Jónsson module over V. It now follows from (2) of Proposition 4 that V has a finite residue field. This completes the proof. □

Remark 1. If R is a ring and M is an infinite faithful module over R of cardinality κ, then $|R| \leq 2^\kappa$. To see this, first note that the endomorphism ring $End(M)$ of M has cardinality at most $\kappa^\kappa = 2^\kappa$. Since M is faithful, the map defined by $r \mapsto f_r$ ($f_r(m) := rm$) is injective. Hence if R is a ring admitting a faithful countable Jónsson module, then R has cardinality at most $2^{\aleph_0}$.

We also note that the bound $2^{\aleph_0}$ given in the previous remark is sharp. It is well known that the endomorphsim ring of the quasi-cyclic group $C(p^\infty)$ is isomorphic to the ring J_p of p-adic integers. Hence $C(p^\infty)$ is naturally a faithful Jónsson module over J_p, a ring of cardinality $2^{\aleph_0}$.

Now let M be an R-module, and let $S \subseteq M - \{0\}$. We recall that S is *linearly independent* over R provided that whenever $m_1, \ldots, m_k$ are distinct elements of S and $r_1, \ldots, r_k \in R$ satisfy $r_1m_1 + \cdots + r_km_k = 0$, then each $r_im_i = 0$. Burns, Okoh, Smith, and Wiegold showed in Theorem 1 of [1] that if M is an infinite module over the Noetherian ring R and $|M| > |R|$, then M possesses a linearly independent subset S with $|S| = |M|$. Using their result, the following lemma is proved easily and will be used several times in the sequel.

Lemma 4. *Let R be a Noetherian ring. There does not exist a Jónsson module M over R with $|M| > |R|$.*

Proof. We assume that M is an infinite R-module with $|M| > |R|$. Choose an independent subset S of M of size $|M|$ by Theorem 1 of [1] quoted above and let

$s \in S$ be arbitrary. Then the submodule of M generated by $S - \{s\}$ is proper and of size $|M|$. Thus M is not a Jónsson module. □

We now easily obtain the following corollary of the results of this section.

Corollary 1. *Suppose that D is a countable Noetherian domain with quotient field K and M is an arbitrary faithful module over D. Then M is a Jónsson module over D if and only if one of the following holds*:

(1) *D is a field and $M \cong D$.*
(2) *There is a discrete valuation overring V of D with a finite residue field such that M is a homomorphic image of K/V.*

3. JÓNSSON MODULES OVER A 1-DIMENSIONAL NOETHERIAN RING

As noted in the introduction, Scott classified the Jónsson modules over the ring $\mathbb{Z}$ of integers [12]. In this section, we generalize this result by characterizing the Jónsson modules over an arbitrary one-dimensional Noetherian ring. Before doing so, we recall several important facts to be used shortly.

Lemma 5 ([11, Theorem 2.1]). *Suppose M is a Jónsson module over the ring R. Then either R is a field and $M \cong R$, or M is a torsion module.*

Lemma 6 ([11, Proposition 5]). *Suppose M is a faithful Jónsson module over the ring R. Suppose further that $r \in R$ is nonzero and that every element of M is annihilated by a power of r. Then M is countable.*

Lemma 7. *Every Jónsson module is indecomposable.*

Proof. Let M be a Jónsson module and suppose $M = H \bigoplus K$. Then by elementary cardinal arithmetic, either $|H| = |M|$ or $|K| = |M|$. Since M is Jónsson, this forces either $H = M$ or $K = M$. □

We now prove that every faithful Jónsson module over a 1-dimensional Noetherian ring is countable. Our proof follows closely the ideas of the proof of Theorem 3.1 of [8].

Theorem 3. *Suppose that M is a faithful Jónsson module over the 1-dimensional Noetherian domain R. Then M is countable.*

Proof. We assume that M is a faithful Jónsson module over the 1-dimensional Noetherian domain R. By Lemma 5, it follows that M is a torsion module. For every maximal ideal J of R, we let M_J denote the collection of elements $m \in M$ such that $J \subseteq \sqrt{Ann(m)}$. It is straightforward to check that M_J is a submodule of M for each maximal ideal J.

We now let m be an arbitrary nonzero element of M. Since M is torsion, it follows that $Ann(m)$ is a proper nonzero ideal of R. We use primary decomposition to express $Ann(m)$ as follows:

$$Ann(m) = \bigcap_{i=1}^{n} Q_i, \tag{1}$$

where the Q_i are primary ideals with distinct prime radicals. For each i, let $J_i = \sqrt{Q_i}$. Since $Ann(m)$ is nonzero, it follows that each Q_i is nonzero, and thus the J_i are distinct maximal ideals of R since R is 1-dimensional. Now, for each j with $1 \leq j \leq n$, let $I_j := \bigcap_{i \neq j} Q_i$ (if $n = 1$, we let $I_1 = R$). We claim that no maximal ideal J of R contains each I_j. This is clear if $n = 1$. Suppose that $n > 1$. If J contains each I_j, then it is clear that J must contain some Q_i, Q_j with $i \neq j$. But since J is prime, J must contain the distinct maximal ideals J_i and J_j, which is clearly impossible. Thus it follows that $R = I_1 + \cdots + I_n$. In particular, $1 = x_1 + \cdots + x_n$ with $x_j \in I_j$ for each j, and hence $m = \sum_{i=1}^{n} x_i m$. Now let j be arbitrary. We claim that $Q_j x_j m = 0$. But this is clear from Eq. (1) above and the definition of I_j. It follows from the definition of the M_J that $x_j m \in M_{J_j}$, and thus $M = \sum_\alpha M_{J_\alpha}$, where the J_α range over the maximal ideals of R. We claim that this sum is direct. For suppose that $m \in M_{J_1} \cap (M_{J_2} + \cdots + M_{J_n})$ where the J_i's are distinct. We will show that $m = 0$. Since $m \in M_{J_2} + \cdots + M_{J_n}$, $m = \alpha_2 + \cdots + \alpha_n$ where each $\alpha_i \in M_{J_i}$. By definition, we have $J_i \subseteq \sqrt{Ann(\alpha_i)}$ for each i. Thus: $J_2 \cap \cdots \cap J_n \subseteq \sqrt{Ann(\alpha_2)} \cdots \cap \sqrt{Ann(\alpha_n)} = \sqrt{Ann(\alpha_2) \cap Ann(\alpha_3) \cdots \cap Ann(\alpha_n)} \subseteq \sqrt{Ann(m)}$ (since $m = \alpha_2 + \cdots + \alpha_n$). Since also $m \in M_{J_1}$, we obtain $J_1 + (J_2 \cap \cdots \cap J_n) \subseteq \sqrt{Ann(m)}$. But the J_i are distinct maximal ideals, and so this forces $Ann(m) = R$. Thus $m = 0$ and the sum is direct. Hence we obtain:

$$M = \bigoplus_{\alpha} M_{J_\alpha}. \tag{2}$$

But since M is indecomposable (Lemma 7), $M = M_J$ for some maximal ideal J of R. Thus $J \subseteq \sqrt{Ann(m)}$ for every $m \in M$. Since R is not a field, J is nonzero. Choose any nonzero $j \in J$. Then $j \in \sqrt{Ann(m)}$ for any $m \in M$. In particular, j^n kills m for some positive integer n. It now follows from Lemma 6 that M is countable. This completes the proof. □

Using Theorem 2, we obtain a complete description of the Jónsson modules over a 1-dimensional Noetherian ring.

Corollary 2. *Suppose that D is a 1-dimensional Noetherian domain with quotient field K and M is an arbitrary faithful module over D. Then M is a Jónsson module over D if and only if there is a valuation overring V of D with a finite residue field such that M is a homomorphic image of K/V.*

Remark 2. If D is a 1-dimensional Noetherian domain, then every valuation overring V of D which is not a field is necessarily a discrete valuation ring. To see this, recall that the integral closure $\overline{D}$ of D is equal to the intersection of all valuation overrings of D (Theorem 19.8 of [4]). Thus $\overline{D} \subseteq V$ and V is a valuation overring of $\overline{D}$. However, $\overline{D}$ is 1-dimensional, Noetherian, and integrally closed, hence a Dedekind domain (Theorem 37.8 of [4]). Thus V is a valuation overring of the Dedekind domain (hence Prüfer domain) $\overline{D}$. It follows that V is a localization of $\overline{D}$, and is thus a discrete valuation ring.

4. STRONGER RESULTS WITH THE GENERALIZED CONTINUUM HYPOTHESIS

In this section, we obtain a complete description of the Jónsson modules over an arbitrary Noetherian ring by assuming the generalized continuum hypothesis (GCH), which states that for an infinite cardinal κ, there is no cardinal properly between κ and 2^κ. It is well known that GCH can neither be proved nor refuted from the usual axioms of set theory. We begin with a particular version of Krull's Intersection Theorem.

Lemma 8 (Krull's Intersection Theorem)**.** *Suppose D is a Noetherian domain and I is a proper ideal of D. Then $\bigcap_{n=1}^{\infty} I^n = \{0\}$.*

We use Krull's Intersection Theorem to prove the following. We feel it is likely that the following result is in the literature; since we could not locate the result, we include a proof.

Proposition 7. *Suppose D is a Noetherian domain and I is an ideal of D such that D/I has infinite cardinality κ. Then $\kappa \leq |D| \leq 2^\kappa$.*

Proof. Clearly $|D/I| \leq |D|$ and so $\kappa \leq |D|$. We need to show that $|D| \leq 2^\kappa$. Toward this end, consider the canonical mapping $\varphi : D \to \prod_{n=1}^{\infty} D/I^n$ defined by:

$$\varphi(d) := (\overline{d}, \overline{d}, \overline{d} \dots).$$

We claim φ is injective. For suppose that d is in the kernel of φ. Then $d \in \bigcap_{n=1}^{\infty} I^n$, and so by Krull's Intersection Theorem, $d = 0$. Thus $|D| \leq |\prod_{n=1}^{\infty} D/I^n|$. It follows from Lemma 3 that each D/I^n has cardinality κ. Let X denote the union of D/I^n as n ranges over the positive integers. Then X has cardinality κ, and trivially $|\prod_{n=1}^{\infty} D/I^n| \leq |\prod_{n=1}^{\infty} X|$. But $\prod_{n=1}^{\infty} X$ is precisely the collection of all functions $f : \mathbb{N} \to X$, which has cardinality $|X|^{|\mathbb{N}|} = \kappa^{\aleph_0} \leq \kappa^\kappa = 2^\kappa$. It now follows that $|D| \leq 2^\kappa$ and the proof is complete. □

To prove the main result of this section, we need a final lemma. This lemma is an unpublished result of Enochs, which appears as Theorem 3.1 of [9].

Lemma 9. *Let R be a Noetherian ring, and suppose that M is an R-module which is not finitely generated. Then M has a factor module which is countably generated but not finitely generated.*

We are now in position to prove the main result of this section. Gilmer and Heinzer have shown in Theorem 2.4 of [8] that a Noetherian ring does not admit an infinitely generated Jónsson module M of uncountable cofinality. Assuming GCH, we obtain a much stronger result.

Theorem 4. *Assume the generalized continuum hypothesis. Suppose that D is a Noetherian domain which is not a field and that M is a faithful Jónsson module over D. Then M is countable.*

Proof. Assume GCH and suppose D is a Noetherian domain which is not a field and M is a faithful Jónsson module over D. Suppose for the purpose of contradiction that M is uncountable of cardinality κ. We claim that M is not finitely generated. For suppose by way of contradiction that $M = Rm_1 + \cdots + Rm_k$. It follows from elementary cardinal arithmetic that $|Rm_i| = |M|$ for some i and since M is a Jónsson module, it follows that M is cyclic. Since M is faithful, this implies that $M \cong D$. But now D is a Jónsson module over itself. By Proposition 2.2 of [8], D must be a field. This contradicts our assumption that D is not a field. Thus M is not finitely generated. By Lemma 9, M has a factor module $N = M/K$ which is countably generated but not finitely generated. Since K is a proper submodule of M and M is Jónnson, $|K| < |M|$. This clearly implies that $|N| = |M|$, and hence N is uncountable. Further, N is also a Jónsson module by Lemma 1. Let $N = (n_1, n_2, n_3, \ldots)$. Note that not every cyclic module (n_i) can be finite, lest N be countable. Thus some (n_{i_1}) is infinite; say $|(n_{i_1})| = \alpha$. Suppose by way of contradiction that every other (n_k) has cardinality $\leq \alpha$. Then it is easy to see that N itself has cardinality α. Since N is a Jónsson module and $|(n_{i_1})| = \alpha = |N|$, it follows that $N = (n_{i_1})$. But then N is a cyclic module, contradicting the fact that N is not finitely generated. Thus there exists some n_{i_2} with $|(n_{i_1})| < |(n_{i_2})|$. Continuing inductively, we obtain a sequence of generators $n_{i_1}, n_{i_2}, \ldots$ such that:

$$|(n_{i_1})| < |(n_{i_2})| < \cdots \tag{3}$$

Let I be the annihilator of n_{i_1} in D. Then $|D/I| = \alpha = |(n_{i_1})|$. It now follows from Proposition 7 that $\alpha \leq |D| \leq 2^{\alpha}$. Since we are assuming GCH, it follows that either $|D| = \alpha$ or $D = 2^{\alpha}$. Suppose that $|D| = \alpha$. Then by (3), we see that $|(n_{i_2})| > |D|$, and hence $|N| > |D|$. This contradicts Lemma 4. Thus we are forced to conclude that $|D| = 2^{\alpha}$. But since GCH holds, it follows again from (3) that $|(n_{i_3})| > 2^{\alpha}$, and so again $|N| > |D|$, a contradiction as above. This contradiction completes the proof. □

Thus by applying Theorem 2, we obtain a complete description of the faithful Jónsson modules over an arbitrary Noetherian domain if we assume GCH. We also have the following corollary.

Corollary 3. *Let D be a Noetherian domain of cardinality greater than $2^{\aleph_0}$ which is not a field. Then it cannot be proved in ZFC that D admits a faithful Jónsson module.*

Proof. Suppose by way of contradiction that $|D| > 2^{\aleph_0}$ and such a Jónsson module M can be proved to exist in ZFC. Then M is countable by Theorem 4 (if M were uncountable, then one would have a disproof of GCH, which is well known not to exist within ZFC), and thus D admits a faithful countable module M. It follows from Remark 1 that $|D| \leq 2^{\aleph_0}$, and this is a contradiction. □

5. OPEN QUESTIONS

We conclude this paper with the following open problems.

Question 1. Can the nonexistence of non-cyclic uncountable Jónsson modules over Noetherian domains be proved in ZFC?

As noted in Lemma 4, there does not exist a Jónsson module M over a Noetherian ring R with $|M| > |R|$. For arbitrary R, it was shown in [11, Proposition 3], that if the cofinality of $|M|$ exceeds $|R|$, then M is not a Jónsson module. If one assumes GCH, it also follows that if $|M| > |R|$, then M is not a Jónsson module [11, Corollary 4]. Hence we ask:

Question 2. Can it be shown in ZFC that if R is a ring, there does not exist a Jónsson module M over R with $|M| > |R|$?

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