

*Chapter 1*

# JÓNSSON MODULES OVER COMMUTATIVE RINGS

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## Abstract

Let  $M$  be an infinite unitary module over a commutative ring  $R$  with identity. Then  $M$  is called a *Jónsson module* provided every proper submodule of  $M$  has smaller cardinality than  $M$ . These modules have been studied by several algebraists, including Robert Gilmer, Bill Heinzer, and the author. In this note, we recall the major results on Jónsson modules to bring the reader up to speed on current research. Included are some applications to Artinian and uniserial modules as well as quasi-cyclic groups. There are several wide open problems in this area, some of which may be independent of the usual axioms of set theory. We close the article with a discussion of several such problems and outline some possible strategies for solving them.

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*All rings in this paper are assumed to be commutative with identity, and all modules are assumed to be unitary.*

## 1 Introduction

An old problem posed by Kurosh was to determine if there exists a group of cardinality  $\aleph_1$  in which all proper subgroups are countable. In the mid 1970's, Shelah constructed such a group ([12]). This spurred more interest from the logic community in so-called Jónsson algebras, which are algebras with countably many finitary operations in which every proper subalgebra has smaller cardinality (see [1] for an excellent survey).

These notions piqued the interest of commutative algebraists Robert Gilmer and Bill Heinzer, who translated these ideas to the context of unitary modules over a commutative ring with identity ([6]). They define an infinite module  $M$  over a ring  $R$  to be a *Jónsson*

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*module* iff every proper submodule of  $M$  has smaller cardinality than  $M$ . Using various ideal-theoretic techniques, they give a complete description of all countable Jónsson modules over a Prüfer domain, and prove several propositions about general Jónsson modules. They applied and extended these results in several subsequent papers ([3], [4], [5], [7]).

We begin by providing two canonical examples of Jónsson modules.

**Example 1.** *Let  $F$  be an infinite field, and consider  $F$  as a module over itself. The submodules of  $F$  are precisely the ideals of  $F$ . Since  $F$  has only trivial ideals, it is easy to see that  $F$  is a Jónsson module over itself.*

More generally, if  $R$  is any ring with an infinite residue field  $R/M$ , then  $R/M$  becomes a Jónsson module over  $R$ .

**Example 2.** *Let  $p$  be a prime number. The direct limit of the cyclic groups  $\mathbb{Z}/(p^n)$  is the so-called quasi-cyclic group of type  $p^\infty$ , denoted by  $C(p^\infty)$ . It is well-known that every proper subgroup of  $C(p^\infty)$  is finite, whence  $C(p^\infty)$  is a Jónsson module over  $\mathbb{Z}$ .*

It was proved by W.R. Scott that the quasi-cyclic groups are in fact the only Abelian Jónsson groups ( $\mathbb{Z}$ -modules). We close the introduction by giving a new proof of this fact. Before doing so, we recall two results from Abelian group theory. The first is exercise 2, p. 67 of [2].

**Proposition 1.** *Every Abelian group is either divisible or contains a maximal subgroup.*

**Proposition 2** (Structure Theorem for Divisible Abelian Groups). *Every divisible Abelian group is a direct sum of copies of  $\mathbb{Q}$  and copies of  $\mathbb{Z}(p^\infty)$  for various primes  $p$ .*

*Proof.* See p. 64 of [2]. □

We now provide a new and simple proof of Scott's result.

**Theorem 1** ([11], Remark 1). *The only Abelian Jónsson groups are the quasi-cyclic groups  $C(p^\infty)$ .*

*Proof.* Let  $G$  be an Abelian Jónsson group. It is easy to see that  $G$  is indecomposable: for if  $G = M \oplus N$ , then since  $G$  is infinite, either  $|M| = |G|$  or  $|N| = |G|$ . Since  $G$  is Jónsson, this forces either  $M = G$  or  $N = G$ . If  $G$  is divisible, it follows from the structure theorem for divisible Abelian groups and the fact that  $G$  is indecomposable that  $G \cong \mathbb{Q}$  or  $G \cong \mathbb{Z}(p^\infty)$  for some prime  $p$ . As  $\mathbb{Q}$  is clearly not a Jónsson group, we have  $G \cong \mathbb{Z}(p^\infty)$  and we're done. Suppose now that  $G$  has a maximal subgroup  $M$ . Since  $G$  is Jónsson,  $|M| < |G|$ . It is well-known that  $G/M \cong \mathbb{Z}/(p)$  for some prime  $p$  and thus  $M$  must be infinite. Let  $g \in G - M$ . Then  $(M, g) = G$  by maximality of  $M$ . However,  $|G| = |(M, g)| = |M| < |G|$  and this is a contradiction. This completes the proof. □

## 2 General Results on Jónsson Modules

We now review some of the most important general results from the literature. We begin with the following proposition of Gilmer and Heinzer.

**Proposition 3** ([6], Proposition 2.5). *Suppose that  $M$  is a Jónsson module over the ring  $R$ . Let  $r \in R$  be arbitrary. Then:*

- (1) *Either  $rM = M$  or  $rM = 0$ ;*
- (2)  *$\text{Ann}(M) = \{s \in R : (\forall m \in M)(sm = 0)\}$  is a prime ideal of  $R$ .*

Thus by modding out the annihilator, there is no loss of generality in assuming that a Jónsson module is faithful over an integral domain.

The next result states that the nontrivial Jónsson modules are all torsion.

**Theorem 2** ([10], Theorem 2.1). *Suppose that  $M$  is a Jónsson module over the ring  $R$ . Then either  $R$  is a field and  $M \cong R$  or  $M$  is a torsion module.*

Recall the curious fact (Theorem 1) that every Jónsson module over the ring  $\mathbb{Z}$  of integers is countable. A much more general result is true if one assumes the generalized continuum hypothesis. It is an open question if the following proposition can be proved in ZFC (though the result *can* be proved in ZFC if  $R$  is Noetherian).

**Theorem 3** ([10], Corollary 4). *Assume the generalized continuum hypothesis. If  $M$  is a Jónsson module over the ring  $R$ , then  $|M| \leq |R|$ .*

We now recall the following fundamental result on countable Jónsson modules due to Robert Gilmer and Bill Heinzer. Note that this result gives quite a bit of information both about the Jónsson module  $M$  and the operator ring  $R$ .

**Theorem 4** ([6], Theorem 3.1). *Suppose that  $M$  is a countably infinite Jónsson module over the ring  $R$  and that  $M$  is not finitely generated. Then  $M$  is a torsion  $R$ -module, and there exists a maximal ideal  $Q$  of  $R$  such that the following hold:*

- (1)  *$\text{Ann}(x)$  is a  $Q$ -primary ideal of finite index for every  $x \in M - \{0\}$ ;*
- (2)  *$R/Q$  is finite;*
- (3) *The powers of  $Q$  properly descend;*
- (4)  *$\bigcap_{i=1}^{\infty} Q^i = \text{Ann}(M)$ ;*
- (5) *If  $H_i = \{x \in M : Q^i x = 0\}$ , then  $\{H_i\}_{i=1}^{\infty}$  is a strictly ascending sequence of submodules of  $M$  such that  $M = \bigcup_{i=1}^{\infty} H_i$ .*

### 3 Projective and Injective Jónsson Modules

We now turn our attention toward describing the projective and injective Jónsson modules. If  $M$  is a faithful projective Jónsson module over the domain  $R$ , then it follows from Proposition 3 and Theorem 2 that  $R$  is a field and  $M \cong R$ . The characterization of the injective Jónsson modules is much more complicated. We have solved the problem over Noetherian rings, but the general problem remains open.

**Theorem 5** ([9], Corollary 4). *Let  $D$  be a Noetherian domain, and suppose that  $M$  is an infinite injective module over  $D$ . Then  $M$  is a Jónsson module over  $D$  iff one of the following holds:*

- (1)  *$D$  is a field and  $M \cong D$ ;*
- (2)  *$M \cong E(D/J)$  for some maximal ideal  $J$  of  $D$  such that  $D/J$  is finite and  $D_J$  is an almost DVR ( $E(D/J)$  is the injective hull of  $D/J$ ).*

## 4 Which Rings Admit Faithful Jónsson Modules?

As noted in the introduction, if  $M$  is a maximal ideal of the ring  $R$  and  $R/M$  is infinite, then  $R/M$  is a Jónsson module over  $R$ . Note that if  $R$  is not a field, then  $R/M$  isn't faithful. Thus we wonder what can be said of the existence of faithful Jónsson modules over an arbitrary ring  $R$ . Must such modules exist? For many familiar rings, the answer is no. We now review most of what is currently known about this question.

**Theorem 6** ([8], Corollary 3.9.4). *Let  $F$  be a finite field, and suppose that  $\{x_i : i \in I\}$  is a set of indeterminates with  $|I| \leq 2^{\aleph_0}$ . Then both  $F[x_i : i \in I]$  and  $\mathbb{Z}[x_i : i \in I]$  admit faithful countable Jónsson modules.*

In particular, this shows that a domain admitting a faithful Jónsson module need not be Noetherian, and even in the Noetherian case, there is no finite bound on the Krull dimension of such a domain.

The situation where  $F$  is infinite is much different.

**Theorem 7** ([8], Proposition 3.9.6). *Let  $F$  be an infinite field. The domain  $F[x_1, \dots, x_n]$  does not admit a faithful Jónsson module. Suppose now that  $\{x_i : i \in I\}$  is a set of indeterminates and that  $|I| \leq |F|$ . Then it cannot be proved in ZFC that  $F[x_i : i \in I]$  admits a faithful Jónsson module.*

We close this investigation with results for power series rings and finite-dimensional valuation rings.

**Theorem 8** ([8], Proposition 3.9.7). *Let  $F$  be a field. Then  $F[[x_1, \dots, x_n]]$  admits a faithful Jónsson module iff  $F$  is finite and  $n = 1$ . In this case the Jónsson module is unique and is isomorphic to  $K/F[[x]]$  where  $K$  is the quotient field of  $F[[x]]$ .*

**Theorem 9** ([8], Corollary 3.9.18). *Let  $V$  be a valuation domain of positive dimension.  $V$  admits a faithful Jónsson module iff  $V$  is a DVR with a finite residue field. In this case the Jónsson module is also unique and is isomorphic to  $K/V$  (again,  $K$  is the quotient field of  $V$ ).*

## 5 Applications

We begin by recalling that an infinite group  $G$  is said to be a *Jónsson group* provided every proper subgroup of  $G$  has smaller cardinality than  $G$ . An infinite semigroup  $S$  is a *Jónsson semigroup* if every proper subsemigroup of  $S$  has smaller cardinality than  $S$ .

Our first two results impose bounds on the cardinality of uniserial and Artinian modules.

**Theorem 10** ([8], Proposition 5.1.2). *Let  $M$  be an infinite uniserial module over the ring  $R$ . Then  $|M| \leq |R|$ .*

**Theorem 11** ([8], Proposition 5.1.4). *Let  $M$  be an infinite Artinian module over the ring  $R$ . Then  $|M| \leq |R|$ .*

The next result shows that a nonabelian Jónsson group is, in some sense, highly non-abelian.

**Theorem 12** ([8], Corollary 5.2.5). *Suppose  $G$  is a Jónsson group with derived subgroup  $G'$ . Then  $G' = \{e\}$  or  $G' = G$ .*

We now consider an infinite semigroup  $S$ . Let  $S^*$  be the subsemigroup of  $S$  generated by all commutators of  $S$  (if  $S$  has no commutators, then we put  $S^* := \emptyset$ ). We obtain the following new characterization of the quasi-cyclic groups:

**Theorem 13** ([8], Theorem 5.2.4). *Assume the generalized continuum hypothesis, and let  $S$  be an infinite semigroup. Then  $S \cong \mathbb{Z}(p^\infty)$  for some prime number  $p$  iff  $S$  satisfies:*

- (i)  $S$  is a Jónsson semigroup;
- (ii)  $S^* \neq S$ .

Lastly we state an interesting theorem due to Robert Gilmer and Bill Heinzer whose proof utilizes Jónsson modules.

**Theorem 14** ([3], Corollary 2). *If  $R$  admits a proper subring, and if every proper subring of a ring  $R$  is Artinian, then  $R$  is Artinian.*

## 6 Conclusion

We close this paper with a list of three open problems we feel are interesting. As noted in the introduction, if  $R$  has an infinite residue field  $R/M$ , then  $R/M$  is a Jónsson module over  $R$ . It is also easy to show that every finitely generated Jónsson module over a ring  $R$  has the form  $R/M$  for some maximal ideal  $M$  of  $R$ . All known examples of infinitely generated Jónsson modules are countable. Thus we ask:

**Question 1.** *Does there exist an infinitely generated uncountable Jónsson module?*

We have obtained a characterization of countable Jónsson modules below. Before stating the result, we recall that a module  $M$  is said to be *almost Noetherian* provided  $M$  is not finitely generated, but every proper submodule of  $M$  is finitely generated.

**Theorem 15** ([10], Theorem 4.1). *Suppose that  $M$  is an infinitely generated faithful Jónsson module over the domain  $D$ . The following are equivalent:*

- (a)  $M$  is countable;
- (b)  $M$  is Artinian;
- (c)  $M$  is almost Noetherian.

Considering that the Jónsson modules over  $\mathbb{Z}$  are all of the form  $\mathbb{Q}/\mathbb{Z}_{(p)}$  for some prime  $p$ , it is natural to consider modules of the form  $D/V$  where  $V$  is a valuation domain and  $D$  is a domain containing  $V$  as a subring when trying to find examples of uncountable Jónsson modules. As the following proposition shows, the only possible such Jónsson modules are of the form  $K/V$  where  $K$  is the quotient field of  $V$ .

**Proposition 4** ([8], Proposition 3.9.20). *Suppose that  $V$  is a valuation domain which is not a field, and  $D$  is a domain containing  $V$  as a subring. If  $D/V$  is a faithful Jónsson module over  $V$ , then  $D = K$ , the quotient field of  $V$ .*

Earlier in this paper, we saw that many familiar rings did not admit faithful Jónsson modules. We would like to know if it is possible to isolate a ring-theoretic property which is both necessary and sufficient to guarantee the existence of faithful Jónsson modules.

**Question 2.** *Which rings  $R$  admit faithful Jónsson modules?*

We end this paper with what could be the most interesting unsolved question on Jónsson modules. Recall from Theorem 3 that if one assume the generalize continuum hypothesis, then all Jónsson modules over a ring  $R$  cannot be larger than  $R$ . Call a module  $M$  over a ring  $R$  large if  $|M| > |R|$ .

**Question 3.** *Can the nonexistence of large Jónsson modules be proved in ZFC?*

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