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More results on congruent modules

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ABSTRACT

W.R. Scott characterized the infinite abelian groups G for which $H \cong G$ for every subgroup H of G of the same cardinality as G [W.R. Scott, On infinite groups, Pacific J. Math. 5 (1955) 589–598]. In [G. Oman, On infinite modules M over a Dedekind domain for which $N \cong M$ for every submodule N of cardinality $|M|$, Rocky Mount. J. Math. 39 (1) (2009) 259–270], the author extends Scott's result to infinite modules over a Dedekind domain, calling such modules *congruent*, and in a subsequent paper [G. Oman, On modules M for which $N \cong M$ for every submodule N of size $|M|$, J. Commutative Algebra (in press)] the author obtains results on congruent modules over more general classes of rings. In this paper, we continue our study.

1. Background

Throughout this paper, a ring R is always assumed to be commutative with identity and an R-module M is assumed unitary.

In universal algebra, an *algebra* is a pair $(X, \mathbf{F})$ consisting of a set X and a collection $\mathbf{F}$ of operations on X (there is no restriction placed on the arity of these operations). In case $\mathbf{F}$ is countable and all operations have finite arity, then $(X, \mathbf{F})$ is called a *Jónsson algebra* provided each proper subalgebra of X has smaller cardinality than X. Such algebras are of particular interest to set theorists. In set theory, a cardinal κ is said to be a *Jónsson cardinal* provided there is no Jónsson algebra of cardinality κ. Many papers have been written on this topic; we refer the reader to [1] for an excellent survey of these algebras.

In the early 1980s, Robert Gilmer and William Heinzer translated these notions to the realm of commutative algebra. In [2], they define a module M over a commutative ring R with identity to be a *Jónsson module* provided every proper submodule of M has smaller cardinality than M. They applied and extended their results in several subsequent papers [3–5]. The author continued this study in [6] and [7].

Much earlier, W.R. Scott studied a more general type of algebraic structure. In particular, he classified the abelian groups G for which $G \cong H$ for every subgroup H of the same cardinality as G in [8]. Note that every Jónsson abelian group trivially possesses this property. In [9], the author extends Scott's result to infinite modules over a Dedekind domain, and defines such modules to be congruent. Note that every Jónsson module is trivially congruent. In [10], the author studies congruent modules over more general classes of rings. Formally, an infinite module M over a ring R is called *congruent* if and only if every submodule N of M of the same cardinality as M is isomorphic to M (note that every finite module is trivially congruent; this is why we define the concept only for infinite modules). We also note that this notion has received attention in model theory. In [11], Droste calls a structure S for a first-order language κ*-homogeneous* provided every two substructures of cardinality κ are isomorphic. He then characterizes the κ-homogeneous structures $(A, <)$ where A is a set, $<$ is a binary relation on A, and $\kappa \leq |A|$.

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2. Preliminaries

We begin with several examples of congruent modules to initiate the reader.

Example 1. Consider the ring $\mathbb{Z}$ of integers as a module over itself. The submodules of $\mathbb{Z}$ coincide with the subgroups of $\mathbb{Z}$. Since every nontrivial subgroup of $\mathbb{Z}$ is infinite cyclic, it is clear that $\mathbb{Z}$ is a congruent $\mathbb{Z}$-module.

Example 2. Let F be a field, and let κ be an infinite cardinal greater than $|F|$. Consider a κ-dimensional vector space V over F (thus V has cardinality κ). Any subspace W of V of cardinality κ also has dimension κ, whence $W \cong V$. It follows that V is a congruent module over F.

Example 3. The direct limit of the cyclic groups $C(p^n)$, p a prime (the so-called quasi-cyclic group of type p^∞, denoted $C(p^\infty)$) is infinite, yet all proper subgroups are finite. Thus $C(p^\infty)$ is a Jónsson (hence congruent) $\mathbb{Z}$-module.

We now recall the author's classification of the congruent modules over a Dedekind domain, extending W.R. Scott's result over $\mathbb{Z}$.

Proposition 1 ([9], Theorem 1). *Let D be a Dedekind domain with quotient field K, and let M be an infinite D-module. Then M is congruent iff one of the following holds:*
(1) $M \cong \bigoplus_\kappa D/P$ for some prime ideal P of D, and $\kappa > |D/P|$.
(2) $M \cong D/P$ where P is either a maximal ideal of D, or $P = \{0\}$ and D is a PID.
(3) $M \cong C(P^\infty) = \{x \in K/D : P^n x = 0$ for some $n > 0\}$, where P is a nonzero prime ideal of D such that the residue field D/P is finite.

In what follows, we will make use of the following lemma.

Lemma 1 ([10], Proposition 1). *Let R be a ring, and suppose that M is a congruent R-module. Then the following hold:*
(1) If $r \in R$, then either $rM \cong M$ or $rM = \{0\}$.
(2) $Ann(M)$ is a prime ideal of R.

Thus by modding out the annihilator, there is no loss of generality in restricting our study to faithful congruent modules over integral domains.

3. An important class of congruent modules

In this section, we study a restricted class of congruent modules. After some work, we will obtain a structure theorem characterizing the form of congruent modules in this class. This will ultimately allow us to obtain classification results for congruent modules over a fairly large number of domains. We begin with a lemma.

Lemma 2 ([7], Lemma 3). *Let M be an infinite R-module, and suppose $r \in R$, $n \in \mathbb{N}$. Assume that r^n annihilates M and let $M[r]$ denote the submodule of M consisting of the elements of M annihilated by r. Then $|M[r]| = |M|$.*

We now recall the following definition.

Definition 1. Let M be a module over the ring R, and suppose that I is an ideal of R. M is said to be I-primary provided every element of M is annihilated by some power of I.

Using the lemma, we show that all faithful I-primary congruent modules must be countable.

Proposition 2. *Let R be a domain, and suppose that M is a congruent faithful I-primary R-module for some nonzero ideal I of R. Then M is countable.*

Proof. Let R, M, and I be as in the statement of the proposition. Choose some nonzero $r \in I$. For each positive integer n, we let M_n denote the submodule of M consisting of the elements of M annihilated by r^n. Since M is I-primary, we see that M is the union of the M_n's as n ranges over the positive integers. Note that:

$$M_1 \subseteq M_2 \subseteq M_3 \ldots . \quad (1)$$

Note also from Eq. (1) that for every positive integer n:

$$M[r] = M_1 = M_n[r]. \quad (2)$$

We claim that every M_n is finite. Suppose by way of contradiction that some M_n is infinite. It then follows from Lemma 2 that $|M_n| = |M_n[r]|$. But note from Eq. (2) above that this implies $|M_n| = |M_1|$. Thus all the M_n's have the same cardinality. Since M is the union of the M_n's, it follows that $|M| = |M_1| \cdot \aleph_0 = |M_1|$. Since M is congruent, this implies that $M \cong M_1$. But then M is annihilated by r, contradicting the fact that M is faithful. Thus every M_n is finite. Since M is the union of the M_n's, it follows that M is countable. This completes the proof. □

We now recall another definition.

Definition 2. A ring R is called Laskerian iff every ideal of R admits a primary decomposition.

It is well-known that every ideal of a ring R which admits a primary decomposition admits an irredundant one in which the radicals of the primary ideals in the decomposition are all distinct. It is also well-known that the Laskerian rings properly include the class of Noetherian rings.

We are finally ready to introduce an important class of modules.

Definition 3. Suppose that M is a module over the ring R. Call M an *L-module* iff $R/Ann(m)$ is a zero-dimensional Laskerian ring for every $m \in M$.

Our next goal is to describe the congruent L-modules. In Theorem 3.1 of [2], the authors show that if M is a countable infinitely generated Jónsson module over a ring R, then M is a torsion module and M is the strictly increasing union of finite submodules each annihilated by a power of the same maximal ideal Q of R. Further, Q has finite index in R. We present the following generalization (the first half of the proof follows closely Gilmer and Heinzer's proof of Theorem 3.1 of [2]) characterizing the congruent L-modules.

Theorem 1. *Let R be a domain which is not a field and suppose M is a faithful congruent L-module over R. Then there is a maximal ideal J of R and a sequence of nonzero elements $m_1, m_2, m_3, \ldots$ of M such that the following hold:*

(i) *R/J is finite.*

(ii) *Each cyclic module (m_i) is finite and $J^i m_i = \{0\}$.*

(iii) $M = (m_1) \cup (m_2) \cup (m_3) \cup \ldots$.

(iv) *For each i, $(m_i) \subsetneq (m_{i+1})$.*

(v) $\bigcap_{n=1}^{\infty} J^n = \{0\}$.

Proof. We assume that M is a faithful congruent L-module over the domain R and that R is not a field. For each maximal ideal J of R, we let M_J denote the collection of elements $m \in M$ such that $J \subseteq \sqrt{Ann(m)}$. It is routine to verify that M_J is a submodule of M.

We now let m be an arbitrary nonzero element of M. Since M is an L-module, $R/Ann(m)$ is Laskerian. In particular, the zero submodule $\{\bar{0}\}$ of $R/Ann(m)$ admits an irredundant primary decomposition. Pulling this back to R, we see that $Ann(m)$ admits an irredundant primary decomposition. Thus:

$$Ann(m) = \bigcap_{i=1}^{n} Q_i \tag{3}$$

where the Q_i are primary ideals with distinct prime radicals. For each i, let $J_i = \sqrt{Q_i}$. Since $R/Ann(m)$ is zero-dimensional, it follows that the J_i are distinct maximal ideals of R. For each j with $1 \le j \le n$, let $I_j := \bigcap_{i \ne j} Q_i$ (if $n = 1$, we let $I_1 = R$). We claim that no maximal ideal J of R contains each I_j. This is clear if $n = 1$. Suppose that $n > 1$. If some maximal ideal J contains each I_j, then it is clear that J must contain some Q_i, Q_j with $i \ne j$ (since J is prime, if J contains a finite intersection of ideals, then it must contain one of the ideals in the intersection). But since J is prime, J must contain the distinct maximal ideals $J_i = \sqrt{Q_i}$ and $J_j = \sqrt{Q_j}$, which is clearly impossible. Since no maximal ideal contains each I_j, it follows that $R = I_1 + \cdots + I_n$. In particular, $1 = x_1 + \cdots + x_n$ with $x_j \in I_j$ for each j. Hence $m = \sum_{i=1}^{n} x_i m$. Now let j be arbitrary with $1 \le j \le n$. We claim that $Q_j x_j m = 0$. But this is clear from Eq. (3) and the definition of I_j. It now follows that $x_j m \in M_{J_j}$. Since m was arbitrary, we obtain $M = \sum_\alpha M_{J_\alpha}$, where the J_α range over the maximal ideals of R. We claim that this sum is direct. For suppose by way of contradiction that for some nonzero $m \in M$, we have $m \in M_{J_1} \cap (M_{J_2} + \cdots + M_{J_n})$ where the J_i's are distinct. We claim that

$$J_1 + (J_2 \cap J_3 \ldots \cap J_n) \subseteq \sqrt{Ann(m)}. \tag{4}$$

Note by definition that since $m \in M_{J_1}$, $J_1 \subseteq \sqrt{Ann(m)}$. Since $m \in M_{J_2} + \cdots + M_{J_n}$, we have $m = m_2 + m_3 + \cdots m_n$ where each $m_i \in M_{J_i}$. Hence $J_i \subseteq \sqrt{Ann(m_i)}$ for each i, $2 \le i \le n$. It follows easily that $J_2 \cap J_3 \cap \ldots \cap J_n \subseteq \sqrt{Ann(m_2 + m_3 + \cdots + m_n)} = \sqrt{Ann(m)}$, and (4) is established. Since $J_1 \subseteq \sqrt{Ann(m)}$ and $m \ne 0$, we obtain $J_1 = \sqrt{Ann(m)}$. It now follows from (4) again that $J_2 \cap J_3 \ldots \cap J_n \subseteq J_1$. Since J_1 is prime, we have $J_i \subseteq J_1$ for some i, $2 \le i \le n$. Since J_i is maximal, this forces $J_i = J_1$. This is a contradiction to the fact that the J_i's are distinct. Thus the sum is direct, and:

$$M = \bigoplus_\alpha M_{J_\alpha}. \tag{5}$$

We now show that, in fact, $M \cong M_J$ for some maximal ideal J. If there is some maximal ideal J for which $|M| = |M_J|$, then since M is congruent, $M \cong M_J$ and we are done. We assume by way of contradiction that each M_J has smaller cardinality than M. Fix some nontrivial submodule M_J. Since $|M_J| < |M|$, it follows from (5) that $|M| = |\bigoplus_{J' \neq J} M_{J'}|$. But then since M is congruent, we see that:

$$M \cong \bigoplus_{J' \neq J} M_{J'}. \tag{6}$$

Consider any nonzero element m of $\bigoplus_{J' \neq J} M_{J'}$. We may express m as $m = m_1 + m_2 + \cdots + m_n$ for some elements $m_1, m_2, \ldots m_n$ with each $m_i \in M_{J_i}$ and $J_i \neq J$. As above, it is easy to see that:

$$J_1 \cap J_2 \cap \ldots \cap J_n \subseteq \sqrt{Ann(m)}. \tag{7}$$

Now choose any nonzero $n \in M_J$. By definition, we have $J \subseteq \sqrt{Ann(n)}$. Since $n \neq 0$, it follows that $J = \sqrt{Ann(n)}$. From (6), it follows that there is some element $n' \in \bigoplus_{J' \neq J} M_{J'}$ such that $J = \sqrt{Ann(n')}$. But it now follows from (7) that $J_1 \cap J_2 \cap \ldots \cap J_n \subseteq J$. Since J is prime and each J_i is maximal, we conclude that $J_i = J$ for some i. This contradicts the fact that $J_i \neq J$ for all i. Thus we conclude that $M \cong M_J$ for some maximal ideal J of R.

We now claim that $M = M_J$. Let $\varphi : M \to M_J$ be an isomorphism, and let m be a nonzero element of M. Since $\varphi(m)$ is a nonzero element of M_J, it follows that $J = \sqrt{Ann(\varphi(m))}$. Thus if $j \in J$, then $j^i\varphi(m) = 0$ for some positive integer i. But then $\varphi(j^i m) = 0$. Since φ is injective, $j^i m = 0$ and $j \in \sqrt{Ann(m)}$. Since $j \in J$ was arbitrary, we obtain $J \subseteq \sqrt{Ann(m)}$ and so $m \in M_J$. Thus we have:

$$J = \sqrt{Ann(m)} \quad \text{for every nonzero } m \in M. \tag{8}$$

Since R is not a field, $J \neq 0$. Let j be a nonzero element of J. It is clear from (8) that M is (j)-primary. It now follows from Proposition 2 that M is countably infinite. If any cyclic module (m) is infinite, then since M is congruent, we would have $M \cong (m)$ and hence $M \cong R/I$ where I is the annihilator of (m) in R. Since M is faithful, this forces $I = \{0\}$ and thus $M \cong R$. However, for any nonzero $r \in R$, the annihilator in R of (r) is $\{0\}$ since R is a domain. But since M is an L-module, this implies that R is zero-dimensional, whence a field. This is a contradiction to our assumption that R is not a field. Thus each cyclic submodule (m) is finite, whence $R/Ann(m)$ is finite for each $m \in M$. We now claim that M is J-primary. To see this, consider an arbitrary nonzero $m \in M$. It follows from (8) that $Ann(m) \subseteq J$. Thus $J/Ann(m)$ is a proper ideal of the finite ring $R/Ann(m)$. Since $J = \sqrt{Ann(m)}$, it follows that $J/Ann(m)$ is nilpotent. In particular, there exists a positive integer n such that $J^n \subseteq Ann(m)$. Thus every element of M is annihilated by a power of J.

For each positive integer i, let M_i denoted the collection of elements of M annihilated by J^i. Clearly we have:

$$M_1 \subseteq M_2 \subseteq M_3 \ldots. \tag{9}$$

Since M is J-primary, the union of these modules is precisely M. We claim:

$$\text{Each } M_i \text{ is finite.} \tag{10}$$

If not, then since M is congruent and countable, this would imply that $M \cong M_i$ for some i. In particular, M is annihilated by J^i. But M is faithful and $J \neq \{0\}$, so this is impossible.

We next show that $M_i = M_{i+1}$ implies that $M_{i+1} = M_{i+2}$. So suppose that $M_i = M_{i+1}$ and let $m \in M_{i+2}$. Then $Jm \subseteq M_{i+1} = M_i$. Since $Jm \subseteq M_i$, $J^iJm = \{0\}$, and hence $m \in M_{i+1}$, which was to be shown. Since each M_i is finite, but M is infinite, this proves that we cannot have $M_i = M_{i+1}$ for any i (lest $M = M_i$). Hence we see that

$$M_i \subsetneq M_{i+1} \quad \text{for each } i. \tag{11}$$

In addition, we claim:

$$M_1 \text{ is nonzero.} \tag{12}$$

In any case, M_2 must be nonzero since $M_1 \subsetneq M_2$. Choose any nonzero $m \in M_2$. If m is annihilated by J, then $m \in M_1$. Otherwise there exists some $j \in J$ with $jm \neq 0$. Then jm is annihilated by J and $jm \in M_1$.

We now let $M_0 := \{0\}$ and for every positive integer n, we let $L_n = M_n - M_{n-1}$. We form a graph as follows. Let X be a set of generators for J, and let the vertex set V be M. We draw an edge between an element m of L_i and an element m' of L_{i-1} iff $xm = m'$ for some $x \in X$. It follows from (10)–(12) that each L_i is finite and nonempty. We now claim:

$$\text{For } i > 1 \text{ and } m \in L_i, \text{ there is an } x \in X \text{ with } xm \in L_{i-1}. \tag{13}$$

To see this, let $i > 1$ and $m \in L_i$ be arbitrary. Note trivially that for any $x \in X$, $xm \in M_{i-1}$. Suppose by way of contradiction that for all $x \in X$, $xm \notin L_{i-1}$. Since $L_{i-1} = M_{i-1} - M_{i-2}$, it follows that for every $x \in X$, $xm \in M_{i-2}$. Thus by definition, for every $x \in X$, $J^{i-2}xm = 0$. Since X generates J, it follows that $J^{i-1}m = 0$. But then $m \in M_{i-1}$, contradicting that $m \in L_i = M_i - M_{i-1}$.

We may now invoke König's Lemma to obtain a sequence of elements of M: $m_1, m_2, \ldots$ such that each $m_i \in L_i$ and for each i there is some $x \in X$ with $xm_{i+1} = m_i$. Since M is congruent and countable, we may assume that M is generated by these elements. In particular, M can be expressed as the strictly increasing union:

$$(m_1) \subsetneq (m_2) \subsetneq (m_3) \ldots. \tag{14}$$

Lastly, since M is J-primary and faithful, we see that $\bigcap_{i=1}^{\infty} J^i = \{0\}$. Since (m_1) is nonzero, finite, and annihilated by the maximal ideal J, it follows that R/J is finite. This completes the proof. □

For brevity, let us call a pair (M, J) consisting of a faithful congruent module M over a domain D and a maximal ideal J of D satisfying conditions (i)–(v) of Theorem 1 *special*.

To showcase the utility of this theorem, we will give a description of the countable congruent modules over an arbitrary ring. To do this, we need a result from an earlier paper.

Lemma 3 *([10], Theorem 2). Let R be a ring, M an infinite torsion-free R-module. Then M is congruent iff $M \cong \bigoplus_{\kappa} R$ where κ is a cardinal and one of the following holds:*
(1) *$\kappa = 1$ and R is a principal ideal domain.*
(2) *κ is infinite, $\kappa > |R|$, and R is a Dedekind domain.*

Corollary 1. *Let D be a domain, and let M be a countable, faithful congruent module over D. Then one of the following holds:*
(a) *D is a principal ideal domain and $M \cong D$.*
(b) *D is a finite field and $M \cong \bigoplus_{\aleph_0} D$.*
(c) *There exists a maximal ideal J of D such that (M, J) is special.*

Proof. We assume that D is a domain and that M is a countable, faithful congruent module over D. Suppose first that there exists some $m \in M$ such that the cyclic module (m) is infinite. Since M is congruent, we have $M \cong (m)$. Since M is faithful, it follows that $M \cong D$. It now follows from Lemma 3 that D is a principal ideal domain, and (a) holds. Thus we may suppose that every cyclic submodule (m) is finite. We suppose now that D is a field. If D is a finite field, then clearly (b) holds. If D is an infinite field, then by Lemma 3, (a) holds. Thus we now assume that every cyclic submodule of M is finite and that D is not a field. Hence $D/Ann(m)$ is finite for every $m \in M$. This implies that $D/Ann(m)$ is zero-dimensional and Noetherian, hence Laskerian. It follows from Theorem 1 that (c) holds and the proof is complete. □

4. Congruent modules over Noetherian rings

As stated in the introduction, W.R. Scott classified all congruent modules over the ring $\mathbb{Z}$ of integers. In particular, the only non-free faithful congruent abelian groups are the (injective) quasi-cyclic groups $C(p^{\infty})$. The author classified all injective congruent modules over an arbitrary Noetherian domain in [10]. Recall that a domain D is an *almost DVR* provided the integral closure $\overline{D}$ of D is a discrete valuation domain which is finitely generated over D.

Proposition 3 *([10], Theorem 7). Suppose D is a Noetherian domain which is not a field, and M is an infinite (faithful) injective module. Then M is congruent iff $M \cong E(D/J)$ where J is a maximal ideal of D, D/J is finite, D_J is an almost DVR, and $E(D/J)$ is the injective hull of D/J.*

Recall from Proposition 1 that all faithful uncountable congruent modules over a Dedekind domain are free. We now apply the results of the previous section and The Generalized Continuum Hypothesis to show that for a very large class of Noetherian domains, all faithful congruent modules are free. In particular, we prove the following theorem:

Theorem 2. *The following is consistent with ZFC: Suppose that D is an uncountable Noetherian domain such that $|D|$ is not the successor of a cardinal of countable cofinality. If M is a faithful congruent module over D, then M is free and D is Dedekind.*

We begin with two new definitions.

Definition 4. Let R be an infinite ring, and let I be an ideal of R. Call I large provided $|R/I| < |R|$.

Definition 5. Let R be a domain. Call R an LPM domain iff R is Noetherian and every large prime ideal of R is maximal.

It turns out that the class of LPM domains has several nice closure properties. For example, this class is closed under factor rings, quotient rings, and finite integral extensions. As we will not make explicit use of these facts, we do not present a proof of this assertion. An important question which will be explored soon is the question of which Noetherian domains are LPM domains. We will have a considerable amount to say about this shortly. We first prove a theorem analogous to Corollary 1 of the previous section. We will need the following lemma to complete the proof. We recall that an infinite R-module M is *large* if it has cardinality larger than that of R.

Lemma 4 *([10], Theorem 6). Let D be a Noetherian domain and suppose that M is a large faithful congruent module over D. Then D is a Dedekind domain and $M \cong \bigoplus_{|M|} D$.*

Proposition 4. *Let D be an LPM domain, and suppose that M is a faithful congruent module over D. Then one of the following holds:*
(a) *D is a principal ideal domain and $M \cong D$.*
(b) *D is a Dedekind domain and $M \cong \bigoplus_{\kappa} D$ for some infinite cardinal $\kappa > |D|$.*
(c) *(M, J) is special for some maximal ideal J of D.*

Proof. We suppose that D is an LPM domain and that M is a faithful congruent module over D. If M has cardinality larger than that of D, then we see that (b) holds from Lemma 4. Thus we assume that $|M| \leq |D|$. If D is a field, then it follows from Lemma 3 that (a) or (b) holds. Thus we assume that D is not a field. Suppose there exists an $m \in M$ with $|(m)| = |M|$. Since M is congruent, $M \cong (m)$. Since M is faithful, $M \cong D$. It follows from Lemma 3 that D is a PID and thus (a) holds. Hence we now suppose that $|(m)| < |M|$ for every $m \in M$. Consider an arbitrary nonzero element $m \in M$. Then we have:

$$|D/Ann(m)| = |(m)| < |M| \leq |D|. \tag{15}$$

Since D is Noetherian, $D/Ann(m)$ Laskerian. We claim that $D/Ann(m)$ is zero-dimensional. Let P be any prime ideal of D containing $Ann(m)$. We must show that P is maximal. Simply note that $|D/P| \leq |D/Ann(m)|$ since $Ann(m) \subseteq P$. It now follows from (15) that $|D/P| < |D|$. Thus P is a large prime ideal of D, whence is maximal since D is an LPM domain. Thus M is an L-module, and it follows that (c) holds from Theorem 1. This completes the proof. □

It is now of interest to know how abundant LPM domains are. It is easy to see that every countable Noetherian domain D is an LPM domain: for if P is a large prime of D, then D/P is finite, whence a field. Thus P is maximal. It is also easy to see that not every Noetherian domain is an LPM domain. In fact, there exist Noetherian domains of arbitrarily large cardinality which are not LPM domains. Recall that the *cofinality* of an infinite cardinal κ, cfκ, is the least cardinal α such that κ can be expressed as the union of α smaller cardinals. It is well-known that if κ is any infinite cardinal, then $\kappa < \kappa^{\mathrm{cf}\kappa}$ (see Theorem 3.11, p. 33 of [12]). Let a be an arbitrary ordinal. Clearly $\aleph_{a+\omega}$ has cofinality ω. Let F be a field of cardinality $\aleph_{a+\omega}$. The ground set of the power series ring $F[[x]]$ consists of all functions $f : \omega \to F$. Thus $|F[[x]]| = |F|^{\aleph_0} = (\aleph_{a+\omega})^{\aleph_0} > \aleph_{a+\omega}$. Note that $F[[x]]/(x) \cong F$ and thus $|F[[x]]/(x)| = \aleph_{a+\omega}$. Let $R = F[[x]]$, let $P = (x)$, and consider the Noetherian domain $R[y]$. Note that $|R[y]| = |R| = |F[[x]]|$. Now observe that $R[y]/P[y] \cong (R/P)[y]$. Hence $|R[y]/P[y]| = |(R/P)[y]| = |F[y]| = |F| < |F[[x]]| = |R[y]|$. Thus $P[y]$ is a large prime of $R[y]$ which is not maximal since $R[y]/P[y]$ is not a field. We would like to thank Alan Loper and Keith Kearnes for their ideas which contributed to the construction of this example.

We will show, assuming The Generalized Continuum Hypothesis, that for many cardinals κ, all Noetherian domains of cardinality κ are LPM domains. We recall that The Generalized Continuum Hypothesis (GCH) is the assertion that for every infinite cardinal κ, there is no cardinal properly between κ and 2^κ. It is well-known that GCH is neither provable nor refutable from the standard axioms of ZFC. One consequence of GCH is that cardinal exponentiation becomes trivial. The following result is Theorem 5.15 of [12]. Note that κ^+ denotes the least cardinal greater than κ.

Lemma 5. *Suppose GCH holds. Let κ and λ be infinite cardinals. Then:*
(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$*
(ii) *If cf$\kappa \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$*
(iii) *If $\lambda <$ cfκ, then $\kappa^\lambda = \kappa$.*

We will need the following result.

Lemma 6. *Let R be a Noetherian domain. If I is any proper ideal of R, then $|R| \leq |R/I|^{\aleph_0}$.*

Proof. The proof of this lemma is contained in the proof of Proposition 7 from [6]. □

We now prove the following.

Proposition 5. *Assume GCH holds. Let R be an uncountable Noetherian domain such that $|R|$ is not the successor of a cardinal of countable cofinality. If I is any proper ideal of R, then $|R| = |R/I|$.*

Proof. Assume GCH, and suppose that R is an uncountable Noetherian domain such that $|R|$ is not the successor of a cardinal of countable cofinality. Let I be a proper ideal of R. From Lemma 6, we see that

$$|R| \leq |R/I|^{\aleph_0}. \tag{16}$$

We claim that R/I cannot be finite. For if this is the case, then it follows from (16) that $|R| \leq 2^{\aleph_0} = \aleph_1$. Since R is uncountable, this forces $|R| = \aleph_1$. But this is impossible since $|R|$ is not the successor of a cardinal of countable cofinality. Thus R/I is infinite. Let $|R/I| = \kappa$. Since GCH holds, it follows from (16) and (i) of Lemma 5 that:

$$\kappa \leq |R| \leq \kappa^{\aleph_0} \leq \kappa^\kappa = \kappa^+. \tag{17}$$

Thus $|R| = \kappa$ or $|R| = \kappa^+$. Suppose by way of contradiction that $|R| = \kappa^+$. It then follows from (16) that $\kappa^+ \leq \kappa^{\aleph_0}$. Recall by assumption that $|R|$ is not the successor of a cardinal of countable cofinality. Since $|R| = \kappa^+$, it follows that κ has uncountable cofinality. Thus by (iii) of Lemma 5, $\kappa^{\aleph_0} = \kappa$. This contradicts the fact that $\kappa^+ \leq \kappa^{\aleph_0}$. We conclude that $|R| = \kappa = |R/I|$ and the proof is complete. □

It follows from this proposition that if R is an uncountable Noetherian domain constructible in ZFC such that $|R/I| < |R|$ for some proper ideal I of R, then the cardinality of R must be the successor of a cardinal of countable cofinality. Moreover, we can apply Proposition 5 to obtain a proof of Theorem 2.

Proof of Theorem 2. Assume GCH. We suppose that D is an uncountable Noetherian domain and $|D|$ is not the successor of a cardinal of countable cofinality. It follows from Proposition 5 that there are no large primes, and so vacuously every large prime ideal of D is maximal. Since $|D/M| = |D|$ for every maximal ideal M of D, D cannot possess a maximal ideal of finite index. Hence by Proposition 4, M is free and D is Dedekind. □

We further exploit Proposition 4 to establish a nonexistence result (not dependent on GCH) for two well-studied classes of Noetherian domains.

Proposition 6. *Let F be an infinite field and $n > 1$ an integer. Then the rings $F[x_1, x_2, \ldots, x_n]$ and $F[[x_1, x_2, \ldots, x_n]]$ do not admit faithful congruent modules.*

Proof. Let F be an infinite field, and consider first the polynomial ring $R := F[x_1, x_2, \ldots, x_n]$ where $n > 1$. If I is any proper ideal of R, then the map $f \mapsto I + f$ is clearly a one-to-one map from F into R/I. Since F is infinite, it follows that $|R/I| = |F| = |R|$. Thus R is vacuously an LPM domain. Since R has dimension $n > 1$, R is not Dedekind, and R does not possess a maximal ideal of finite index. It follows from Proposition 4 that R does not admit a faithful congruent module.

Now consider the power series ring $D := F[[x_1, x_2, \ldots, x_n]]$ where $n > 1$. We will also show that D is an LPM domain. Suppose that P is a large prime ideal of D and fix an arbitrary i with $1 \le i \le n$. It is easy to see that:

$$|F[[x_i]]| = |D|. \tag{18}$$

Since P is a large prime of D, it follows from (18) that the map $\varphi : F[[x_i]] \to D/P$ given by $\varphi(f) = P + f$ cannot be injective. Thus there exist distinct elements $f, g \in F[[x_i]]$ such that $f - g \in P$. But then $f - g$ is a nonzero element of $P \cap F[[x_i]]$. It is well-known that $F[[x_i]]$ is a DVR with maximal ideal (x_i), and thus as $P \cap F[[x_i]]$ is a nonzero prime ideal of $F[[x_i]]$, we obtain $P \cap F[[x_i]] = (x_i)$. In particular, $x_i \in P$. Since i was arbitrary, it follows that $(x_1, x_2, \ldots, x_n) \subseteq P$. As $(x_1, x_2, \ldots, x_n)$ is maximal in D, $P = (x_1, x_2, \ldots, x_n)$. We have shown that P is maximal, and hence D is an LPM domain. Since D does not have a finite residue field and is not Dedekind, it follows from Proposition 4 that D does not admit a faithful congruent module. This completes the proof. □

We now show that the assumption in the previous proposition that F is a field cannot be dispensed with. We remind the reader that an infinite module M over a ring R is a Jónsson module provided every proper submodule of M has smaller cardinality than M. Recall that every Jónsson module is trivially congruent.

Proposition 7. *Let I be a set with $|I| \le 2^{\aleph_0}$ and let $\{x_i : i \in I\}$ be a set of indeterminates. Then $\mathbb{Z}[\{x_i : i \in I\}]$ admits a faithful congruent module.*

Proof. Let p be a prime, and consider the quasi-cyclic group $C(p^\infty)$. It is well-known that every proper subgroup of $C(p^\infty)$ is finite, and thus (as noted in Example 3) $C(p^\infty)$ is a faithful Jónsson module over $\mathbb{Z}$. It is also well-known that the endomorphism ring of $C(p^\infty)$ is the ring J_p of p-adic integers. As every module is faithful over its endomorphism ring, we see that $C(p^\infty)$ is a faithful Jónsson module over J_p. As J_p has characteristic 0 and $|J_p| = 2^{\aleph_0}$, it follows that there exists a subset $X \subseteq J_p$ of elements which are algebraically independent over $\mathbb{Z}$ with $|X| = 2^{\aleph_0}$. Clearly $C(p^\infty)$ is a faithful Jónsson module over the ring $\mathbb{Z}[X]$. This completes the proof. □

We end this section with an undecidability result.

Proposition 8. *Let κ be an uncountable cardinal satisfying the following:*

(a) *κ is not the successor of a cardinal of countable cofinality.*

(b) *It is consistent with ZFC that $\kappa \le 2^{\aleph_0}$.*

Then it is undecidable in ZFC whether there exists a non-Dedekind Noetherian domain D of cardinality κ which admits a faithful congruent module.

Proof. Let κ be an uncountable cardinal satisfying (a) and (b) above. Suppose that $\kappa \le 2^{\aleph_0}$. We will show that there is a non-Dedekind Noetherian domain D of cardinality κ which admits a faithful congruent module. Let p be a prime and consider again the DVR J_p of p-adic integers. As noted in Proposition 7, $C(p^\infty)$ is a faithful Jónsson module over J_p. Since $C(p^\infty)$ is an abelian Jónsson group, it follows immediately that $C(p^\infty)$ is a faithful Jónsson module over *any* subring R of J_p. Let $\{t_i : i \in \kappa\}$ be a set of κ indeterminates. Since $|J_p| = 2^{\aleph_0}$ and $\kappa \le 2^{\aleph_0}$, it follows that (up to isomorphism) $\mathbb{Z}[\{t_i : i \in \kappa\}, x]$ is a subring of J_p. Let K be the quotient field of $\mathbb{Z}[\{t_i : i \in \kappa\}]$. Then K has cardinality κ and $K \cap J_p$ is a DVR on K. Now $(K \cap J_p)[x]$ is a Noetherian two-dimensional (hence not Dedekind) subring of J_p of cardinality κ that admits the faithful Jónsson module $C(p^\infty)$.

Since (a) holds, it follows from Theorem 2 that it is consistent that there does not exist a non-Dedekind Noetherian domain D of cardinality κ that admits a faithful congruent module. This completes the proof. □

To give a specific example, it is undecidable in ZFC whether there exists a non-Dedekind Noetherian domain D of cardinality $\aleph_2$ which admits a faithful congruent module. It is consistent that CH fails. In this case, $\aleph_2 \le 2^{\aleph_0}$. Recall that every cardinal of the form $\aleph_{a+1}$ is regular (this is a theorem of ZFC). Of course, $\aleph_2$ is the successor of the regular cardinal $\aleph_1$, whence $\aleph_2$ is not the successor of a cardinal of countable cofinality.

5. Congruent modules over Prüfer domains

Our objective in this section is to characterize the faithful congruent modules over a Prüfer domain D which are I-primary for some nonzero ideal I of D.

We first comment on terminology. If D is a domain and P is a prime ideal, we denote the localization of D at P by D_P. If I is an ideal of D contained in P, then we denote the extension of I to D_P by ID_P. If I and J are ideals of D, then we denote the ideal quotient of I by J by $[I : J] := \{x \in D : xJ \subseteq I\}$.

We begin with two lemmas (both of which are well-known).

Lemma 7. *Let V be a valuation domain which is not a field and let J be the maximal ideal of V. If $\bigcap_{n=1}^{\infty} J^n = \{0\}$, then V is a discrete valuation ring.*

Proof. Every prime ideal properly contained in J is contained in $\bigcap_{n=1}^{\infty} J^n = \{0\}$ (see Theorem 17.1 of [13], p. 187). Thus $\{0\}$ is the only prime ideal properly contained in J and hence V is one-dimensional. Since $\bigcap_{n=1}^{\infty} J^n = \{0\}$, J is not idempotent. Since J is the unique nonzero prime ideal of V, it follows that V is discrete. It now follows from Theorem 17.5 of [13] that V is a DVR. This completes the proof. □

Lemma 8. *Let D be a domain and let J be a maximal ideal of D. If $\bigcap_{n=1}^{\infty} J^n = \{0\}$, then also $\bigcap_{n=1}^{\infty} (JD_J)^n = \{0\}$.*

Proof. We assume $\bigcap_{n=1}^{\infty} J^n = \{0\}$. Suppose that $\frac{x}{s} \in \bigcap_{n=1}^{\infty} (JD_J)^n$. Let $n > 0$ be arbitrary. Then $\frac{x}{s} \in (JD_J)^n = J^n D_J$. It is clear that $x \in J^n D_J$ and thus $x = \frac{a}{s'}$ for some $a \in J^n$ and $s' \in D - J$, whence $s'x = a \in J^n$. But this implies that $s' \in [J^n : (x)]$. Clearly $J^n \subseteq [J^n : (x)]$. Thus we have $(J^n, s') \subseteq [J^n : (x)]$. However, since $s' \notin J$, it follows that $(J^n, s') = D$, whence $[J^n : (x)] = D$. It follows that $x \in J^n$. Since n was arbitrary, we have $x \in \bigcap_{n=1}^{\infty} J^n = \{0\}$, and thus $x = 0$. This completes the proof. □

We now characterize the I-primary congruent modules over Prüfer domains.

Theorem 3. *Let D be a Prüfer domain which is not a field with quotient field K and suppose M is a faithful I-primary congruent module over D for some nonzero ideal I of D. Then there is a discrete valuation overring V of D with finite residue field such that $M \cong K/V$. Conversely, if V is a DVR overring of D with finite residue field, then K/V is a congruent D-module.*

Proof. We suppose D is a Prüfer domain (which is not a field) with quotient field K and that M is a faithful I-primary congruent module over D for some nonzero ideal I of D. It follows from Proposition 2 that M is countable, and from Corollary 1 that there is a maximal ideal J of D such that (M, J) is special. Since J is maximal and M is J-primary, M has a canonical module structure over the local ring D_J. As D is Prüfer, D_J is a valuation domain. It now follows from Lemma 7 and Lemma 8 that D_J is a DVR. As $D_J/JD_J \cong D/J$, it follows that D_J has a finite residue field. Let $V := D_J$ and let (j) be the maximal ideal of V. Then as M is a faithful congruent V-module, it follows from the proof of Theorem 1 that M is generated by elements $m_1, m_2, m_3, \ldots$ such that for each i, $(m_i) \subsetneq (m_{i+1})$ and $jm_{i+1} = m_i$. It is also well-known (and easy to show) that K/V is generated over V by the elements $\frac{1}{j}, \frac{1}{j^2}, \frac{1}{j^3}, \ldots \pmod V$, $(j_i) \subsetneq (j_{i+1})$, and trivially for each i, $j \cdot \frac{1}{j^{i+1}} = \frac{1}{j^i} \pmod V$. It is easy to verify that the map $m_i \mapsto \frac{1}{j^i}$ defines a V-module isomorphism between M and K/V. Conversely, suppose V is a DVR overring of D with finite residue field. By Theorem 2 of [6], K/V is a faithful Jonsson module over D, whence K/V is a faithful congruent module over D. This completes the proof. □

We end this section with a final nonexistence result. We first recall a fact about valuation rings which we will need shortly as well as an earlier result from the literature due to the author.

Lemma 9. *Let V be a valuation ring which is not a field. If A is a proper ideal of V, then $\sqrt{A}$ is a prime ideal of V.*

Proof. The proof of this assertion is contained in the proof of Theorem 17.1 of [13]. □

Lemma 10 (*[10], Theorem 3*). *If M is a congruent module over the ring R, then M is either torsion or torsion-free.*

Proposition 9. *Let V be a finite-dimensional valuation domain of dimension $d > 1$. Then V does not admit a faithful congruent module.*

Proof. We assume by way of contradiction that V admits a faithful congruent module M. By Lemma 10, M is either torsion or torsion-free. If M is torsion-free, it follows from Lemma 3 that V is Dedekind. But then V has dimension at most one, and we have a contradiction. Thus M is torsion. In particular, if m is any nonzero element of M, then $Ann(m)$ is a nonzero proper ideal of V. It follows from Lemma 9 that $\sqrt{Ann(m)}$ is a nonzero prime ideal of V. Since V has positive finite dimension, V possesses a (unique) minimal nonzero prime ideal P. It now follows that:

$$P \subseteq \sqrt{Ann(m)} \text{ for every nonzero element } m \in M. \tag{19}$$

Fix any nonzero element $p \in P$. Then (19) implies that M is (p)-primary. It follows from Proposition 2 that M is countable, and from Corollary 1 that if J is the maximal ideal of V, then (M, J) is special. In particular, $\bigcap_{n=1}^{\infty} J^n = \{0\}$. But now Lemma 7 implies that V is one-dimensional, and we have a contradiction. This completes the proof. □

6. Open questions

We end this paper with a list of open questions which we feel are important and/or interesting.

Question 1. *Is it possible to classify the faithful congruent modules over an arbitrary Noetherian domain in ZFC?*

Question 2. *Is there a faithful uncountable congruent module over a domain D which is not free?*

Question 3. *Suppose that M is a faithful congruent module over the domain D and (M, J) is special for some maximal ideal J of D. Is M injective? Must M be a Jónsson module?*

References

[1] Eoin Coleman, Jonsson groups, rings, and algebras, Irish Math. Soc. Bull. (36) (1996) 34–45.
[2] R. Gilmer, W. Heinzer, On Jónsson modules over a commutative ring, Acta Sci. Math. 46 (1983) 3–15.
[3] R. Gilmer, W. Heinzer, An application of Jónsson modules to some questions concerning proper subrings, Math. Scand. 70 (1) (1992) 34–42.
[4] R. Gilmer, W. Heinzer, Cardinality of generating sets for modules over a commutative ring, Math. Scand. 52 (1983) 41–57.
[5] R. Gilmer, W. Heinzer, On Jónsson algebras over a commutative ring, J. Pure Appl. Algebra 49 (1-2) (1987) 133–159.
[6] G. Oman, Jónsson modules over Noetherian rings, Comm. Algebra (in press).
[7] G. Oman, Some results on Jónsson modules over a commutative ring, Houston J. Math. 35 (1) (2009) 1–12.
[8] W.R. Scott, On infinite groups, Pacific J. Math. 5 (1955) 589–598.
[9] G. Oman, On infinite modules M over a Dedekind domain for which $N \cong M$ for every submodule N of cardinality $|M|$, Rocky Mount. J. Math. 39 (1) (2009) 259–270.
[10] G. Oman, On modules M for which $N \cong M$ for every submodule N of size $|M|$, J. Commutative Algebra (in press).
[11] M. Droste, k-homogeneous relations and tournaments, Quart. J. Math. (Oxford Second Series) 40 (1989) 1–11.
[12] T. Jech, Set Theory, third millennium edition, in: Springer Monographs in Mathematics, New York, 2002.
[13] R. Gilmer, Multiplicative Ideal Theory, in: Queen's Papers in Pure and Applied Mathematics, Kingston, 1992.