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On elementarily κ-homogeneous unary structures

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Abstract. Let L be a first-order language with equality and let $\mathfrak{U}$ be an L-structure of cardinality κ. If $\aleph_0 \leq \lambda \leq \kappa$, then we say that $\mathfrak{U}$ is elementarily λ-homogeneous iff any two substructures of cardinality λ are elementarily equivalent, and λ-homogeneous iff any two substructures of cardinality λ are isomorphic. In this note, we classify the elementarily λ-homogeneous structures (A, f) where $f : A \to A$ is a function and λ is a cardinal such that $\aleph_0 \leq \lambda \leq |A|$. As a corollary, we obtain a complete description of the Jónsson algebras (A, f), where $f : A \to A$.

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1 Preliminaries

We begin by defining the focal point of our study.

Definition 1.1. Let L be a first-order language with equality and let $\mathfrak{U}$ be an L-structure of cardinality κ. If $\aleph_0 \leq \lambda \leq \kappa$, then we say that $\mathfrak{U}$ is *elementarily λ-homogeneous* iff any two substructures of cardinality λ are elementarily equivalent, and *λ-homogeneous* iff any two substructures of cardinality λ are isomorphic.

Likely the first appearance of this idea is in [9]. In this paper, W. R. Scott characterizes the infinite abelian groups G which are $|G|$-homogeneous. More recently ([3]), Manfred Droste classifies the elementarily κ-homogeneous binary relational structures (A, R). We recall one of his main results.

Proposition 1.2 ([3, Theorem 1.1]). *Let A be an infinite set of cardinality κ, R a binary relation on A, and λ an infinite cardinal with $\lambda \leq \kappa$. Then (A, R) is elementarily λ-homogeneous iff $\lambda = \kappa$ and (A, R) is isomorphic to one of the following structures:*

(i) $(\kappa, \varnothing)$,

(ii) $(\kappa, =)$,

(iii) $(\kappa, \neq)$,

(iv) $(\kappa, \kappa \times \kappa)$,

(v) $(\kappa, \in)$ *(strict or inclusive),*

(vi) $(\kappa, \ni)$ *(strict or inclusive).*

Droste notes that this result shows that the structure (A, R) is elementarily λ-homogeneous iff (A, R) is λ-homogeneous.

Most recently, the author studied this concept within the context of commutative algebra. Let M be an infinite unitary module over the commutative ring R with identity. We call M *congruent* iff $N \cong M$ for any submodule N of M with $|N| = |M|$ (in other words, M is $|M|$-homogeneous). Many results were obtained in [8] and [7], including a complete characterization of the torsion-free congruent modules as well as some statements which were shown to be independent of ZFC.

There is other literature related to this subject. Neumann ([6]) shows that if G is a permutation group on a set of infinite cardinality κ and there is a cardinal λ such that either G is transitive on the collection of λ-subsets of X (where $\lambda < \kappa$) or is transitive on the set of λ-subsets with complement of size λ (where $\lambda = \kappa$), then G is k-transitive for all finite k. Thus there is no non-trivial G-invariant relational structure on X. In [5], strong structural results are obtained for a relational structure M such that, for some infinite cardinal $\mu < |M|$, there are just finitely many non-isomorphic substructures of M of size μ. There is also some model-theoretic literature on structures in a language with just a single unary relation symbol; see [10] for example.

In this paper, we study structures (X, f) where X is infinite and $f : X \to X$. For every cardinal pair (λ, κ) with $\aleph_0 \leq \lambda \leq \kappa$, we classify the structures (X, f) of cardinality κ which are elementarily λ-homogeneous. We note several differences between the unary function and the binary relation environments. In particular, for each uncountable κ, there are elementarily κ-homogeneous unary structures of size κ which are not κ-homogeneous.

The outline of the paper is as follows. In Section 2, we characterize the countably infinite structures (X, f) which are elementarily $\aleph_0$-homogeneous. In Section 3, we characterize the uncountable structures (X, f) which are elementarily $|X|$-homogeneous. In the final section, we state our main result (Corollary 4.1) and apply the results of the paper to completely characterize the Jónsson algebras (X, f), where $f : X \to X$.

2 The countable case

Let X be a countably infinite set and $f : X \to X$ be a function. In this section, we determine precisely when (X, f) is elementarily $\aleph_0$-homogeneous. To begin, we recall the following definition.

Definition 2.1. Let X be a set and $f : X \to X$ a function. For $x \in X$, we define the *orbit of* x by $\mathcal{O}(x) := \{f^n(x) : n \in \mathbb{N}\}$ (where $f^0(x) := x$). An orbit $\mathcal{O}(x)$ is called *a cycle* iff $f^n(x) = x$ for some $n > 0$.

Recall that a unary structure (X, f) is *connected* provided for every $x, y \in X$, there exist natural numbers m and n such that $f^m(x) = f^n(y)$. We will need the following standard result (the easy proof is left to the reader).

Lemma 2.2. *Let (X, f) be a unary structure. Define a relation $\sim$ on X by $x \sim y$ iff $f^m(x) = f^n(y)$ for some natural numbers m and n. Then $\sim$ is an equivalence relation. Further, each equivalence class is closed under f.*

For $x \in X$, the equivalence class $[x]$ of x is called a *connected component*. These components will play a vital role in our classification results. We also introduce some terminology which will be useful shortly.

Definition 2.3. Let $f : X \to X$, and suppose $x \in X$. If $y \in X$ and $f^n(y) = x$ for some $n \in \mathbb{N}$, then we say that y *lies above* x, or that x *lies below* y.

Before disposing of the countable case, we state a final lemma. The result is well known and the proof is easy, so we omit it.

Lemma 2.4. *Let X be a set and $f : X \to X$ a function. Then:*

(1) *Every finite orbit contains a cycle.*

(2) *Any two cycles are either equal or disjoint.*

We now classify the countable elementarily $\aleph_0$-homogeneous unary structures (X, f).

Theorem 2.5. *Let X be a countably infinite set and $f : X \to X$ a function. Then the structure (X, f) is elementarily $\aleph_0$-homogeneous iff one of the following holds:*

(i) $(X, f) \cong (\mathbb{N}, S)$ *where S is the successor function on $\mathbb{N}$.*

(ii) $(X, f) \cong (\mathbb{N}, P)$ *where $P(n) = n - 1$ for all $n > 0$ and $P(0)$ takes any value in $\mathbb{N}$.*

(iii) *X is the union of cycles, each of the same cardinality.*

(iv) *There exists some finite orbit $\mathcal{O}(x)$ such that for each $y \notin \mathcal{O}(x)$, $f(y) = x$.*

Proof. We first show that the structures in families (i)–(iv) are elementarily $\aleph_0$-homogeneous. Toward this end, it clearly suffices to show that for any such structure, any two infinite substructures are isomorphic. We begin with (i). Consider an arbitrary substructure Y of $\mathbb{N}$. Let m be the least element of Y. Then the mapping $n \mapsto m + n$ is clearly an isomorphism between $(\mathbb{N}, S)$ and (Y, S). Now for (ii). Let Y be a substructure of $(\mathbb{N}, P)$ where $P(n) = n - 1$ for all $n > 0$ ($P(0)$ is irrelevant). Clearly if $m \in Y$, then also $n \in Y$ for every natural number $n \leq m$. Hence if Y is infinite, then Y must coincide with $\mathbb{N}$. Thus trivially $(Y, P) \cong (\mathbb{N}, P)$. As for (iii), if X is the countably infinite union of cycles each of the same cardinality, then clearly so is any infinite substructure Y of X. It is now clear that Y and X must be isomorphic. Lastly, we examine (iv). Any infinite substructure Y of X must clearly contain infinitely many points y with $f(y) = x$. Thus Y also contains $\mathcal{O}(x)$. It is clear from these facts that X and Y are isomorphic.

Now we consider an arbitrary countably infinite (X, f) that is elementarily $\aleph_0$-homogeneous, and show that (X, f) belongs to one of the families (i)–(iv).

Suppose first that X contains an infinite orbit $\mathcal{O}(z)$. Then the following must be true in any infinite substructure Y of X since each is expressible in first-order logic and is true in $\mathcal{O}(z)$:

(a) There exists a unique element y_0 with no preimage.

(b) For all y and all $n \in \mathbb{N}$, $y, f(y), f^2(y), \ldots, f^n(y)$ are all distinct (requires infinitely many sentences).

In particular, (a) and (b) are true of the entire space X. Let $x \in X$ be the unique element with no preimage. We claim that $X = \mathcal{O}(x)$. Suppose this is not the case. Then there is some $y \in X$ that does not belong to $\mathcal{O}(x)$. Consider the structure $X' := \mathcal{O}(x) \cup \mathcal{O}(y)$. Since x has no preimage in X, clearly x has no preimage in X'. But since $y \notin \mathcal{O}(x)$ and (b) holds in X', it follows that y has no preimage in X' either. Thus X' has two distinct elements with no preimage, contradicting (a). Thus $X = \mathcal{O}(x)$, and it follows that $(X, f) \cong (\mathbb{N}, S)$.

We now assume that every orbit of X is finite, and we consider the number of connected components of X.

Suppose first that there are infinitely many connected components. Label them $C_0, C_1, \ldots$. Since every orbit is finite, it follows from Lemma 2.4 that every connected component must contain a cycle. Choose a cycle $c_n \subseteq C_n$ for every n. We first claim that there are infinitely many cycles c_i which all have the same length. For suppose this is not the case. Then in particular, only finitely many cycles have the same length as c_0. Let Y be the substructure of X generated by the cycles c_n as n ranges over $\mathbb{N}$ and let Y' be the substructure of X generated by all cycles of different length than c_0. Then as Y and Y' are infinite, $Y \equiv Y'$. However, Y possesses an element of order $|c_0|$ whereas Y' does not. Since this can be expressed in first-order logic, we obtain a contradiction. Thus there exists an infinite collection of cycles, say $\{d_n : n \in \mathbb{N}\}$ each of the same length l. Let $Y := \bigcup_{n\in\mathbb{N}} d_n$. Note that Y satisfies the sentence: "For every y, $f^l(y) = y$ and $f^j(y) \neq y$ for $1 \leq j < l$." Since $X \equiv Y$, it follows that X satisfies this sentence as well, and thus X is the union of cycles, each of the same cardinality.

We now assume that every orbit in X is finite, and that there are but finitely many connected components. Thus some component C must be infinite. It follows from Lemma 2.4 that C contains a unique cycle c. Moreover, since every orbit is finite, it follows again from Lemma 2.4 that $c \subseteq \mathcal{O}(x)$ for every $x \in C$. For $x \in C$, define the *entrance number* of x, $E(x)$, to be the least natural number n for which $f^n(x) \in c$. For each natural number n, we let L_n denote the collection of all elements of C with entrance number n. Note that for $n > 0$:

$$\text{If } x \in L_n, \text{then } f(x) \in L_{n-1}. \tag{2.1}$$

Case 1: Some L_i is infinite. Let n be least such that L_n is infinite. Then $n > 0$. Since L_{n-1} is finite, it follows from (2.1) that some $x \in L_{n-1}$ has infinitely many elements of L_n lying above it. Label these elements $y_0, y_1, y_2, \ldots$. Let Y be the substructure of X generated by the y_is and let $j := |\mathcal{O}(x)|$. Note that the following sentence can be expressed in first-order logic and is true in Y:

"There exists x such that $\mathcal{O}(x)$ has order j and for all $y \notin \mathcal{O}(x)$, $f(y) = x$."

Since $X \equiv Y$, it follows that this sentence is also true in X and thus X belongs to family (iv).

Case 2: Each L_i is finite. Since C is infinite, it follows from (2.1) that every L_i is nonempty. By König's Lemma, we may pick a sequence of elements $x_1, x_2, x_3, \ldots$ such that each $x_i \in L_i$ and $f(x_{i+1}) = x_i$. Let Y be the substructure of X generated by the x_i. We claim that $Y = X$. Suppose by way of contradiction that there is some $z \in X - Y$ and consider the structure $Y' := Y \cup \mathcal{O}(z)$. Since $Y \equiv Y'$ and every member of Y has a preimage in Y, it follows that every member of Y' has a preimage in Y'. In particular, z has a preimage in Y'. Since $z \notin Y$, it follows that z is contained in a cycle. But then Y' contains two distinct cycles, whereas Y does not, and this contradicts $Y \equiv Y'$. Hence $X = Y$. It is now clear that (X, f) belongs to family (ii) and the proof is complete. □

The previous proof yields the following immediate corollary.

Corollary 2.6. *Let (X, f) be a countably infinite structure. Then (X, f) is elementarily $\aleph_0$-homogeneous iff (X, f) is $\aleph_0$-homogeneous.*

3 The uncountable case

In this section, we classify the uncountable structures (X, f) which are elementarily $|X|$-homogeneous. Though there are many similarities to the countable case, there are also some differences. In particular, for every uncountable κ, there are elementarily κ-homogeneous unary structures of size κ which are not κ-homogeneous. We begin by defining a new theory T. This theory will be instrumental in proving the classification theorem of this section.

Definition 3.1. Let $\mathfrak{L}$ be the language with equality containing a single unary function symbol S. Define the theory *IST* (infinitary successor theory) to be the set of consequences of the following axioms:

(S1) S is one-to-one.

(S2) (schema) For each positive integer n: $\forall x : x, S(x), S(S(x)), \ldots, S^n(x)$ are distinct.

(S3) (schema) For each positive integer n, there exist n distinct elements which are not in the range of S.

We now define some terminology which will allow us to describe the models of IST.

Definition 3.2. Consider a unary structure (X, f). We call (X, f) an $\mathbb{N}$-*chain* iff $(X, f) \cong (\mathbb{N}, S)$ and a $\mathbb{Z}$-*chain* iff $(X, f) \cong (\mathbb{Z}, S)$ where S is the successor function on $\mathbb{N}$ and $\mathbb{Z}$, respectively.

It is easy to see that the models of IST are precisely the (disjoint) union of an infinite collection of $\mathbb{N}$-chains along with another (possibly empty) disjoint union of $\mathbb{Z}$-chains. We will ultimately need to know that this theory is complete in order to classify the elementarily κ-homogeneous unary structures for uncountable κ. As a detailed proof of the completeness of IST will take us too far afield, we sketch an outline containing the main ideas and leave the details to the reader.

Proposition 3.3. *The theory IST is complete.*

Sketch of proof. By inspection of the models of IST, the theory has countably many 1-types over the empty set. Hence, by Proposition 2 of [10], any completion of IST is ω-stable, so has a countable saturated model. Since any countable saturated model of IST consists of the disjoint union of $\aleph_0$ $\mathbb{Z}$-chains and $\aleph_0$ $\mathbb{N}$-chains, any two such models are isomorphic. The completeness of IST follows. □

Next we define a structure which will be of paramount importance in proving the classification theorem of this section.

Definition 3.4. Let (X, f) be an uncountable unary structure. Call (X, f) a *rake* provided there exists some $x \in X$ such that for all $y \in X - \mathcal{O}(x)$, $f(y) = x$.

Let us agree to call the point x in the above definition the *central point* of the rake and the elements not in $\mathcal{O}(x)$ the *initial points* of the rake. The notation $R(x)$ will denote a rake with central point x. We will make use of the following two lemmas.

Lemma 3.5. *Suppose that (X, f) is a unary structure of uncountable cardinality κ. If (X, f) is elementarily κ-homogeneous and if (X, f) possesses a rake substructure of cardinality κ, then (X, f) is itself a rake.*

Proof. Suppose that (X, f) is elementarily κ-homogeneous of uncountable cardinality κ and let $R(x)$ be a rake substructure of X of size κ. Clearly we may assume that $y \in R(x)$ for *every* $y \in X$ for which $f(y) = x$ and $y \notin \mathcal{O}(x)$ (otherwise we may extend $R(x)$ to a rake $R'(x)$ which contains all such y). In this case, we claim that in fact $R(x) = X$. Suppose by way of contradiction that this is not the case. Let $y \in X - R(x)$ and consider the structure $X' := R(x) \cup \mathcal{O}(y)$. Suppose first that $\mathcal{O}(y)$ is a cycle. Since $R(x) \equiv X'$, it follows that $R(x)$ also contains a cycle. But then X' contains two distinct cycles whereas $R(x)$ only contains one. Since this is a first-order property and $X' \equiv R(x)$, we have a contradiction. It follows that $\mathcal{O}(y)$ is not a cycle. But note that this implies that the following sentence is true in X':

"$\exists z$ with the property that $\exists! t$ such that both $f(t) = z$ and t has empty preimage."

To see this, let $z = f(y)$ and note that t must be y. However, it is clear upon reflection that this sentence is not true in $R(x)$ (note in particular that there are *multiple* x' for which $f(x') = x$ and x' has empty preimage), and this is a contradiction. Hence $R(x) = X$ and the proof is complete. □

To simplify notation, let us agree to call a rake of cardinality κ a κ*-rake*.

Lemma 3.6. *Suppose that (X, f) is connected of uncountable cardinality κ and suppose κ is regular. Then X possesses a κ-rake substructure.*

Proof. We assume that (X, f) is connected of uncountable regular cardinality κ. Fix an arbitrary $x \in X$. For every nonnegative integer n, let X_n denote the collection of all elements of X which lie above $f^n(x)$. Since X is connected, X is the union of the X_ns. Since κ is uncountable and regular, some X_n has cardinality κ. We may assume without loss of generality that X_0 has cardinality κ. For each nonnegative n, we define Y_n by $Y_n := \{y \in X : f^n(y) = x\}$. Note that X_0 is the union of the Y_ns as n ranges over the nonnegative integers. Again, by regularity it follows that some Y_n has cardinality κ. Choose the least such n. Note that $n > 0$, and thus Y_{n-1} has cardinality smaller than κ. Trivially, f maps Y_n into Y_{n-1}. Yet again, since κ is regular, there exists some $y \in Y_{n-1}$ whose preimage in Y_n has cardinality κ. Thus we obtain our κ-rake by taking the substructure generated by all elements in the preimage of y not lying in $\mathcal{O}(y)$. □

We prove one final result before presenting our classification theorem.

Proposition 3.7. *Suppose that (X, f) is connected of uncountable cardinality κ and that (X, f) is elementarily κ-homogeneous. Then (X, f) is a rake.*

Proof. Suppose by way of contradiction that (X, f) is connected of uncountable cardinality κ and elementarily κ-homogeneous, yet (X, f) is not a rake. It follows from Lemma 3.5 that X does not possess a κ-rake substructure. It now follows from Lemma 3.6 that κ is singular of cofinality $\lambda < \kappa$. Thus there is a strictly increasing sequence $(\lambda_i : i \in \lambda)$ of uncountable regular cardinals each larger than λ which is cofinal in κ. Applying Lemma 3.6 and transfinite recursion, we conclude that there exist λ_i-rake substructures $R_i(x_i)$ of X for each $i \in \lambda$ such that $x_i \neq x_j$ for $i \neq j$. This implies the following:

(i) The set of initial points of the rake $R_j(x_j)$ is disjoint from the set of initial points of $R_0(x_0)$ for $j \neq 0$.

To see this, note that if y is an initial point of both $R_j(x_j)$ and $R_0(x_0)$, then $f(y) = x_j = x_0$. We claim:

(ii) For $j \neq 0$, the rake $R_j(x_j)$ contains *at most one* of the initial points of $R_0(x_0)$.

Indeed, suppose by way of contradiction that y and z are distinct initial points of $R_0(x_0)$ which belong to $R_j(x_j)$. It follows from (i) above that y and z are not initial points of $R_j(x_j)$, and hence y and z belong to $\mathcal{O}(x_j)$. Thus without loss of generality, $f^i(y) = z$ for some positive integer i. However, recall that y and z are initial points of $R_0(x_0)$, and so this is impossible.

Now recall that each λ_i is larger than λ. In particular, $\lambda_0 > \lambda$. It follows from (ii) that there are λ_0 initial points of the rake $R_0(x_0)$ with empty preimage in $Y := \bigcup_{i\in\lambda} R_i(x_i)$. Let Y' denote the substructure of Y obtained by deleting all but one of the initial points of the rake $R_0(x_0)$ which have empty preimage in Y. Note that $|Y| = |Y'| = \kappa$ and hence $Y \equiv Y'$. But as in the proof of Lemma 3.5, it follows that Y' satisfies the sentence:

"$\exists z$ with the property that $\exists! t$ such that both $f(t) = z$ and t has empty preimage"

whereas Y does not satisfy it. This is a contradiction, and the proof is complete. □

We now present our classification result for uncountable X.

Theorem 3.8. *Suppose that (X, f) is a unary structure and $|X| = \kappa$ is uncountable. Then (X, f) is elementarily κ-homogeneous iff one of the following holds:*

(i) *(X, f) is a disjoint union of κ $\mathbb{N}$-chains and λ $\mathbb{Z}$-chains, and $\lambda < \kappa$.*

(ii) *(X, f) is a union of cycles, each of the same cardinality.*

(iii) *(X, f) is a rake.*

Proof. We first show that the structures in the families (i)–(iii) are elementarily κ-homogeneous. Consider (i). Suppose that Y is a substructure of X of the same cardinality as X. It is not hard to see that Y and X are both models of the theory IST. It follows from Proposition 3.3 that $Y \equiv X$. Now suppose that X falls into family (ii) or (iii). It is easy to see that any substructure of X of the same cardinality as X is actually isomorphic to X, hence must be elementarily equivalent to X.

We now suppose that (X, f) is an arbitrary elementarily κ-homogeneous structure of uncountable cardinality κ and show that (X, f) belongs to one of the above three families. To do this, we consider the number of connected components of X.

Suppose first that X has κ connected components; say $\{C_i : i \in \kappa\}$ is the collection of connected components of X. Pick an element $x_i \in C_i$ for each i and let $S := \{x_i : i \in \kappa\}$. Let Y be the union of all infinite orbits $\mathcal{O}(x_i)$ for $x_i \in S$ and let Z be the union of all finite orbits $\mathcal{O}(x_j)$ for $x_j \in S$. Note that either Y or Z has cardinality κ. We suppose first that Y has size κ. Note that Y is a model of IST. Since $X \equiv Y$, it follows that X is also a model of IST. In particular, X is a disjoint union of $\mathbb{N}$-chains and $\mathbb{Z}$-chains. We claim that X has fewer $\mathbb{Z}$-chains than $\mathbb{N}$-chains. Suppose by way of contradiction that there are at least as many $\mathbb{Z}$-chains as $\mathbb{N}$-chains. Then there are κ $\mathbb{Z}$-chains. Let X' be the union of all the $\mathbb{Z}$-chains and let T be a transversal of the set of $\mathbb{Z}$-chains. Now define $X'' := \{\mathcal{O}(x) : x \in T\}$. Since both X' and X'' have size κ, it follows that $X' \equiv X''$. However, X'' has elements with empty preimage whereas X' does not. Since this is a first-order property, we have a contradiction. We have shown that (X, f) belongs to family (i) in this case. We now suppose that Z has cardinality κ. It follows from (1) of Lemma 2.4 that for each $x \in Z$, $\mathcal{O}(x)$ contains a cycle. Hence X is elementarily equivalent to a union of κ cycles. The proof that each cycle has the same cardinality proceeds just as in the countable case and is omitted. Hence (X, f) belongs to family (ii).

We now suppose that X has λ connected components and $\lambda < \kappa$. If any component has cardinality κ, then it follows from Lemma 3.5 and Proposition 3.7 that X is a rake and (X, f) belongs to family (iii). Hence we now assume that X has λ connected components, $\lambda < \kappa$, and each connected component C_i has cardinality less than κ. Thus κ is a singular cardinal. We now derive a contradiction to finish the proof. Let $\gamma := \text{cf}(\kappa)$ and let $(\gamma_i : i \in \gamma)$ be a strictly increasing sequence of regular uncountable cardinals cofinal in κ. For each $i \in \gamma$, choose a component C_i such that $\gamma_i \leq |C_i|$ and such that $C_i \neq C_j$ for $i \neq j$. Now choose a γ_i-rake substructure $R(x_i)$ of C_i for every $i \in \gamma$ (guaranteed by Lemma 3.6). Let X' be the substructure generated by the rakes $R(x_i)$. Now choose an arbitrary rake and delete all but one of its initial points. Call the new structure X''. Since X' and X'' have cardinality κ, it follows that $X' \equiv X''$. But as before, the sentence:

"$\exists z$ with the property that $\exists! t$ such that both $f(t) = z$ and t has empty preimage"

is true in X'' but false in X'. This is a contradiction, and the theorem is proved. □

4 Some consequences

Using the results of the previous section, we establish our main result.

Corollary 4.1. *Let (X, f) be a unary structure of cardinality κ, and suppose that $\aleph_0 \leq \lambda \leq \kappa$. Then the following hold:*

(1) *If $\kappa = \aleph_0$, then (X, f) is elementarily λ-homogeneous iff (X, f) belongs to one of the families* (i)–(iv) *of Theorem 2.5.*

(2) *If $\kappa > \aleph_0$ and $\lambda = \aleph_0$, then (X, f) is elementarily λ-homogeneous iff (X, f) is a disjoint union of $\mathbb{N}$-chains, a union of cycles each of the same cardinality, or a κ-rake whose central point has finite orbit.*

(3) *If $\lambda > \aleph_0$, then (X, f) is elementarily λ-homogeneous iff (X, f) is the disjoint union of $\mathbb{N}$-chains and $\mathbb{Z}$-chains and the number of $\mathbb{Z}$-chains is strictly less than λ, a union of cycles each of the same cardinality, or a κ-rake.*

Proof. We assume that (X, f) is a unary structure of cardinality κ and that $\aleph_0 \leq \lambda \leq \kappa$.

(1) This is clear.

(2) Suppose that $\kappa > \aleph_0$ and $\lambda = \aleph_0$, and assume that (X, f) is elementarily λ-homogeneous. Note trivially that if Y is any substructure of size $\lambda = \aleph_0$, then Y is elementarily $|Y|$-homogeneous. It follows easily from Theorem 2.5 that there

exists a *unique* family $\mathcal{F}$ from (i)–(iv) of Theorem 2.5 such that every countably infinite substructure of X belongs to $\mathcal{F}$. It is now easy to see that (X, f) is a disjoint union of $\mathbb{N}$-chains, a union of cycles each of the same cardinality, or a κ-rake whose central point has finite orbit. Conversely, each such structure is easily seen to be elementarily λ-homogeneous.

(3) The verification proceeds analogously to (2) and is omitted. □

Our final result is an application to Jónsson algebras. We recall that an infinite algebra $(X, \mathbf{F})$ is said to be a *Jónsson algebra* provided $\mathbf{F}$ is countable, every function $f \in \mathbf{F}$ has finite arity, and every proper subalgebra of X has smaller cardinality than X. Such algebras are of considerable interest to set theorists. A cardinal κ is said to be a *Jónsson cardinal* provided there is no Jónsson algebra of cardinality κ. We refer the reader to [2] for an excellent survey of Jónsson algebras.

It is not hard to see that the structure $(\mathbb{N}, P)$ is a Jónsson algebra, where $P(n) = n - 1$ for $n > 0$ and $P(0) = 0$ (note that $P(0)$ can be defined arbitrarily and still $(\mathbb{N}, P)$ is a Jónsson algebra). Using the results of this paper, we can easily prove that this is the only Jónsson algebra with a single unary operation.

Corollary 4.2. *Let X be an infinite set and let $f : X \to X$. The algebra (X, f) is a Jónsson algebra iff $(X, f) \cong (\mathbb{N}, P)$ where $P(n) = n - 1$ for all $n > 0$.*

Proof. Let X be an infinite set and let $f : X \to X$ be a function. Suppose that (X, f) is a Jónsson algebra. Clearly this implies that (X, f) is elementarily $|X|$-homogeneous. Using Theorem 2.5 and Theorem 3.8, it is easy to check that $(X, f) \cong (\mathbb{N}, P)$, where $P(n) = n - 1$ for all $n > 0$. □

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