ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 1, 2009

ON INFINITE MODULES M OVER A DEDEKIND DOMAIN FOR WHICH $N \cong M$ FOR EVERY SUBMODULE N OF CARDINALITY $|M|$

GREG OMAN

ABSTRACT. Let R be a commutative ring with identity, and let M be an infinite unitary R-module. Let us call M *congruent* if and only if every submodule of M of the same cardinality as M is isomorphic to M. In [**7**], Scott classified all congruent Abelian groups, i.e., congruent $\mathbf{Z}$-modules. In this paper, we extend his result to classify all congruent modules over an arbitrary Dedekind domain. As a consequence, we get a complete description of the Jónsson modules of a Dedekind domain.

1. Preliminaries. In this section, we acquaint the reader with the fundamental definitions and propositions needed to prove our classification theorem. Every ring is assumed to be commutative with identity and every module is assumed to be unitary. We begin by giving a list of the many equivalent definitions of a Dedekind domain.

Proposition 1. *Let R be a domain. The following are equivalent*:

(i) *R is a Dedekind domain.*

(ii) *Every ideal of R is projective.*

(iii) *R is Noetherian and, for every nonzero prime ideal P of R, R_P is a discrete rank-one valuation domain.*

(iv) *R is Noetherian, integrally closed, and all nonzero prime ideals are maximal.*

(v) *Every nonzero ideal of R is uniquely the finite product of prime ideals.*

We state the following two important results about submodules of free modules over a Dedekind domain:

Received by the editors on July 7, 2005, and in revised form on May 30, 2006.
DOI:10.1216/RMJ-2009-39-1-259

Proposition 2. *Suppose that D is a Dedekind domain and that F is a free D-module. Then every submodule of F is a direct sum of ideals of D.*

Proof. See [**1**, page 352] for a proof of this result. □

Proposition 3 (Kaplansky). *Let D be a Dedekind domain, and let M be a D-module which is the direct sum of an infinite number of ideals of D. Then M is free.*

Of course if M is a finite sum, then M need not be free. To see this, choose any Dedekind domain D which is not a PID (for example, $\mathbf{Z}[\sqrt{10}]$) and take an ideal I which requires two generators. Clearly I is not a free D-module.

Definition 1. Let R be a ring, P a nonzero prime ideal of R, and M an R-module. The P-component of M is defined to be the collection of all elements of M which are annihilated by some power of P.

The P-component of M is a submodule of M. Further, we recall the following well-known result:

Proposition 4. *Every torsion module over a Dedekind domain D is a direct sum of its P-components, where P ranges over the nonzero prime ideals of D.*

We now recall the definition of independence for a subset S of an R-module M:

Definition 2. Let M be an R-module. A subset S of $M - \{0\}$ is said to be independent if and only if for any distinct $m_1, \dots, m_k \in S$ and $r_1, \dots, r_k \in R$:

$$r_1 m_1 + \cdots + r_k m_k = 0$$

implies that each $r_i m_i = 0$.

Suppose that $S := \{s_i : i \in I\}$ is an independent set in M. It is easy to see that the submodule generated by S is isomorphic to $\oplus_{s \in S} Rs$. Further, if $\{r_i : i \in I\}$ is a subset of R, and for each i, $r_i s_i \neq 0$, then $\{r_i s_i : i \in I\}$ is also independent and the mapping $s_i \mapsto r_i s_i$ is injective.

Definition 3. Let R be a ring, and let M be an infinite R-module. If $|M| > |R|$, then M is said to be large (or more precisely, R-large).

We will make use of the following result of Smith and Wiegold:

Proposition 5. *Let R be a Noetherian ring, and let M be an infinite, large R-module. Then M possesses an independent subset S with $|S| = |M|$.*

Proof. This result is proved in [**8**]. □

We use this result to obtain the following corollary:

Corollary 1. *Let R be a Dedekind domain, and let M be an infinite, large R-module. Then M possesses an independent set S with $|S| = |M|$ in which all torsion elements of S have prime annihilators in R.*

Proof. By the previous proposition, there exists an independent set S' with $|S'| = |M|$. We will define a function φ on S' as follows: if $s \in S'$ is not a torsion element, then we define $\varphi(s) := s$. Otherwise, the annihilator of s can be expressed as $P_1^{n_1} \cdots P_k^{n_k}$, where the P_i's are distinct nonzero prime ideals of R. Then we choose any $x \in P_1^{n_1-1} P_2^{n_2} \cdots P_k^{n_k}$ which is not in $P_1^{n_1}$ (we can clearly do this since each P_i is invertible and since nonzero prime ideals are maximal), and let $\varphi(s) = xs$. Note that P_1 clearly annihilates xs. We now let $S := \{\varphi(s) : s \in S'\}$. From the comments following Definition 2, it suffices to show that $xs \neq 0$ if s is a torsion element and x is chosen as above. If $xs = 0$, then we have that $x \in P_1^{n_1} \cdots P_k^{n_k} \subseteq P_1^{n_1}$, and thus $x \in P_1^{n_1}$, a contradiction to the way x was chosen. This completes the proof. □

Definition 4. Let D be a domain, P a prime ideal of D, and F the quotient field of D. The P-component of the D-module F/D is called the quasi-cyclic module of type P and is denoted by $C(P^\infty)$. If $P = (p)$, this module is often denoted simply by $C(p^\infty)$.

We list the following useful results about quasi-cyclic modules over PID's to be used in the next section:

Proposition 6. *Let R be a* PID*, and let p be a prime element of R. Let M be an R-module, and suppose that there exist $c_1, c_2, \dots c_n, \dots$ in M such that the c_i generate M, $c_1 \neq 0$, and $pc_1 = 0$, $pc_2 = c_1$, $pc_3 = c_2$, etc. Then $M \cong C(p^\infty)$.*

Proof. See [**2**, pages 23–25]. □

Proposition 7. *Let p be a prime of the* PID *R. For each positive integer n, let S_n be the submodule of $C(p^\infty)$ generated by $1/p^n$. We have the following*:

(i) $S_1 \subsetneq S_2 \subsetneq S_3 \subsetneq \cdots$.

(ii) *M is the union of the S_n's.*

(iii) *$S_n \cong R/(p^n)$ for each positive integer n.*

(iv) *The S_n are the only nonzero proper submodules of $C(p^\infty)$.*

Proof. See [**1**, page 265]. □

The following result is well known:

Proposition 8. *Let M be an R-module, J a maximal ideal of R, and suppose that the annihilator of every element of M is equal to some power of J (in which case M is said to be J-primary). Then M has a canonical module structure over the localization R_J given by $r/s \cdot m := (rm/s)$, where (rm/s) is equal to the unique $m' \in M$ with $sm' = rm$.*

2. The classification theorem. In this section, we provide a complete description of the infinite modules M over a Dedekind domain D with the property that every submodule of the same cardinality as M is isomorphic to M. We begin with some preliminary lemmas.

Lemma 1. *Let D be a domain, and let M be a maximal ideal of D. There is a D_M-module isomorphism from $C(M^\infty)$ into $C(MD_M^\infty)$.*

Proof. By definition, $C(M^\infty)$ is M-primary, and so has the canonical module structure over D_M as given by Proposition 8. Note trivially that the quotient field of D is the same as the quotient field of D_M. Thus, we define a mapping by $x/y \pmod{D} \mapsto x/y \pmod{D_M}$. Note that if n is a positive integer and $M^n(x/y) \subseteq D$, then $(MD_M)^n(x/y) \subseteq D_M$. Hence, the mapping is well-defined. The mapping is trivially a D_M-module homomorphism. We simply must check that the mapping is one-to-one. Hence, we suppose that $(x/y) \in D_M$ and that $M^n(x/y) \subseteq D$ for some positive integer n. We must show that $(x/y) \in D$. This is equivalent to showing that the ideal quotient $[(y) : (x)]$ is all of D. As $(x/y) \in D_M$, there is some $s \in D - M$ with $s \in [(y) : (x)]$. Further, as $M^n(x/y) \subseteq D$, we have $M^n \subseteq [(y) : (x)]$. As $(M^n, s) = D$, we see that $D = [(y) : (x)]$, which is what we needed to show. This completes the proof. □

Lemma 2. *Let D be a Dedekind domain, and suppose that P is a nonzero prime ideal of D. Then the quasi-cyclic module $C(P^\infty)$ is infinite.*

Proof. Let p be a nonzero element of P, and let n be a positive integer. Consider the element $1/p^n \pmod{D}$ of F/D. As D is Dedekind, we have that $(p^n) = P^k Q_1 \cdots Q_r$ where $k \geq n$, $r \geq 0$, and the Q_i are nonzero prime ideals of D different from P. Choose $x \in Q_1 \cdots Q_r - P$ (in case $r = 0$, choose $x \notin P$). Consider the element $x/p^n \pmod{D}$. By our selection of x, we have that $x/p^n \pmod{D}$ is annihilated by P^k, and thus $x/p^n \pmod{D} \in C(P^\infty)$. Suppose now that $i < n$. If $x/p^n \pmod{D}$ is annihilated by the ideal P^i, then $p^i x = p^n d$ for some $d \in D$. But this implies that $x \in P$, a contradiction. Hence, for each $n \in \mathbf{N}$, there is an element $y \in C(P^\infty)$ not annihilated by P^n. Clearly this implies that $C(P^\infty)$ is infinite. □

Lemma 3. *Let R be a domain, x an element of R. Then for each positive integer n, we have $|R/(x^n)| = |R/(x)|^n$.*

Proof. We prove this by induction on $\mathbf{N}$. The case where $n = 1$ is trivial. Thus we assume the result is true for some positive integer n. Define a mapping $\varphi : R/(x^{n+1}) \to R/(x^n)$ by $(x^{n+1}) + r \mapsto (x^n) + r$. This map is clearly well-defined and surjective. The kernel K is easily seen to be $\{(x^{n+1}) + rx^n : r \in R\}$. From the fundamental theorem of ring homomorphisms, we have that $|R/(x^{n+1})| = |K||R/(x^n)|$. By the inductive hypothesis, $|R/(x^n)| = |R/(x)|^n$. We will be done if we can show that $|K| = |R/(x)|$. This is easy: the map $(x) + r \mapsto (x^{n+1}) + rx^n$ is a bijection between K and $R/(x)$. This completes the proof. □

Lemma 4. *Let M be an infinite R-module, and let $r \in R$, $n \in \mathbf{N}$. Suppose that r^n annihilates M. Let $M[r]$ denote the submodule of M consisting of the elements of M annihilated by r. Then $|M[r]| = |M|$.*

Proof. We prove this by induction on $\mathbf{N}$. The case when $n = 1$ is trivially true. Thus we assume the lemma is true for some $n \in \mathbf{N}$. Suppose that M is an infinite R-module $r \in R$, $n \in \mathbf{N}$, and suppose that r^{n+1} annihilates M. We have that $M/M[r] \cong rM$. Hence we get that $|M| = |rM||M[r]|$. As M is infinite, it follows that either $|M[r]| = |M|$ or $|rM| = |M|$. If $|M[r]| = |M|$, then we have what we want and we are done. Otherwise $|rM| = |M|$. Recall that r^{n+1} annihilates M, and therefore r^n annihilates rM. By the inductive hypothesis, we have that $|(rM)[r]| = |rM| = |M|$. Clearly $(rM)[r] \subseteq M[r]$, and thus $|M[r]| = |M|$. This completes the proof. □

Lemma 5. *Suppose that M is a J-primary D-module for some maximal ideal J of D. Then M is a JD_J-primary D_J-module, and the set of elements of M annihilated by J as a D-module is precisely the set of elements of M annihilated by JD_J as a D_J-module.*

Proof. The proof is trivial and is left to the reader. □

Our next lemma follows trivially from basic set theory. The proof is omitted.

Lemma 6. *Suppose that R is a ring and $|R| = \kappa$. Then for any nonzero cardinal λ, if at least one of κ or λ is infinite, then $|\oplus_\lambda R| = \max(\kappa, \lambda)$.*

Finally, we are in a position to prove our main theorem:

Theorem 1 (Classification of congruent modules over a Dedekind domain). *Let D be a Dedekind domain, and let M be an infinite D-module. Then M is congruent if and only if one of the following holds*:

(1) *$M \cong \oplus_\kappa D$ where $\kappa > |D|$ is an infinite cardinal.*

(2) *$M \cong D$ and D is a principal ideal domain.*

(3) *$M \cong D/P$ where P is a nonzero prime ideal of D.*

(4) *$M \cong \oplus_\kappa D/P$ where P is a nonzero prime ideal of D and $\kappa > |D/P|$ is an infinite cardinal.*

(5) *$M \cong C(P^\infty)$ where P is a nonzero prime ideal of D such that the residue field D/P is finite.*

Proof. Let D be a Dedekind domain, and suppose that M is an infinite D-module. We first show that each of the modules in families (1)–(5) is congruent.

Suppose first that $M \cong \oplus_\kappa D$ where $\kappa > |D|$ is an infinite cardinal. By Lemma 6, we see that $|M| = \kappa$. Suppose that N is a submodule of cardinality κ. By Proposition 2, N is a direct sum of ideals of D, say $N \cong \oplus_{i\in I} J_i$. As D is a domain, $|J_i| = |D|$ for each i, and thus $|N| = |\oplus_{i\in I} J_i| = |\oplus_{i\in I} D| = \max(|D|, |I|) = \kappa$. As $\kappa > |D|$, we must have $|I| = \kappa$. In particular, N is isomorphic to an infinite direct sum of ideals of D. By Proposition 3, N is free and is thus isomorphic to M.

Next assume that $M \cong D$ and D is a PID. We must show that D is a congruent D-module. This is trivial. If I is any ideal of D with $|I| = |D|$, then I is principal and generated by a nonzero element $x \in D$. The mapping $d \mapsto dx$ is clearly an isomorphism between D and I.

Suppose that $M \cong D/P$ where P is a nonzero prime ideal of D. Then D/P is a field, and so is trivially congruent.

Now suppose that $M \cong \oplus_\kappa D/P$ where P is a nonzero prime ideal of D and $\kappa > |D/P|$ is an infinite cardinal. It clearly suffices to show that $\oplus_\kappa D/P$ is a congruent D/P-module. But this is easy. D/P is a field, and $\oplus_\kappa D/P$ is a vector space of infinite dimension $\kappa > |D/P|$ over D/P. It follows from Lemma 6 that $\oplus_\kappa D/P$ has cardinality κ. If N is any submodule of size κ, then N is free. It follows again from Lemma 6 that N has dimension κ and is thus isomorphic to $\oplus_\kappa D/P$.

Lastly, suppose that $M \cong C(P^\infty)$ where P is a nonzero prime ideal of D such that D/P is finite. Let $R = D_P$. Then P extends to a principal ideal in R; denote it by (p). It is well known that D_P/PD_P is isomorphic (as a ring) to the quotient field of D/P, which is D/P since P is maximal (see [**3**, page 57] for example). In particular, we get that $R/(p)$ is finite. Proposition 7 parts (iii) and (iv) along with Lemma 3 implies that every proper submodule of the R-module $C(p^\infty)$ is finite. By Lemma 1, there is an R-module isomorphism from $C(P^\infty)$ into $C(p^\infty)$. As $C(P^\infty)$ is infinite (Lemma 2), we see that this isomorphism is surjective, and thus as R-modules, we get $C(P^\infty) \cong C(p^\infty)$. In particular, all proper R-submodules of $C(P^\infty)$ are finite. But the R-submodules of $C(P^\infty)$ are the same as the D-submodules of $C(P^\infty)$. Thus, $C(P^\infty)$ is congruent and the proof that the modules in (1)–(5) are congruent is complete.

Conversely, suppose that M is an infinite congruent D-module. We will show that M belongs to one of the above families. We distinguish two cases:

Case 1. M is a large D-module. Let $|M| = \kappa > |D|$. By Corollary 1, M possesses an independent set S with $|S| = |M|$ in which all torsion elements have prime annihilators in R. Let F be the submodule of M generated by the nontorsion elements of S, and for each nonzero prime ideal P of D, let M_P be the submodule of M generated by the elements of S with annihilator P. Then as M is congruent, we see that $M \cong F \oplus [\oplus_P M_P]$. If $|F| = \kappa$, then we have that $M \cong F$. In particular, F is a free D-module of cardinality $\kappa > |D|$. Hence $M \cong \oplus_\kappa D$, and M belongs to family (1). Otherwise $\oplus_P M_P$ has cardinality κ and we get $M \cong \oplus_P M_P$. Fix any nonzero prime ideal P_0 with $M_{P_0} \neq 0$. We claim that M_{P_0} has cardinality κ. If not, then $\oplus_{P \neq P_0} M_P$ has cardinality κ and thus $M \cong \oplus_{P \neq P_0} M_P$, which is absurd. Hence, we obtain that $M \cong M_P$ for some nonzero prime ideal P. By definition of M_P, we have that every element of M_P is annihilated by P. Thus M_P has a canonical vector space structure over the field D/P. As M_P has cardinality $\kappa > |D| \geq |D/P|$, it follows that $M_P \cong \oplus_\kappa D/P$ as a D/P-vector space, and hence as a D-module, and M belongs to family (4).

Case 2. $|M| \leq |D|$. Suppose first that M is not a torsion module. Let $m \in M$ have annihilator (0). Then the mapping $d \mapsto dm$ is injective, and we see that $|M| = |D|$ and by the condition on M, $M \cong (m) \cong D$. Thus $M \cong D$, and D is a congruent D-module. If I is any nonzero ideal of D, then $|I| = |D|$, and so $I \cong D$. In particular, I is principal and so D is a principal ideal domain. Thus, M belongs to family (2).

Thus, we assume that M is a torsion module. By Proposition 4, M is a direct sum of its P components as P ranges over the nonzero prime ideals of D. The same reasoning applied in Case 1 can be applied here, and we conclude that M is P-primary for some nonzero prime ideal P of D. Let $M[P]$ denote the submodule of elements of M which are annihilated by P. We now distinguish two subcases:

Subcase 1. $|M[P]| = |M|$. The condition on M then implies that $M \cong M[P]$, and so as in Case 1, we see that $M \cong \oplus_\kappa D/P$ for some cardinal κ. If κ is finite, then it is clear that the congruence of M implies that $\kappa = 1$, and M belongs to family (3). Otherwise, κ is infinite, and we have that $|M| = |\oplus_\kappa D/P| = \max(\kappa, |D/P|)$. If $\kappa \leq |D/P|$, then $|M| = |D/P|$, and the condition on M implies that $M \cong D/P$, which is absurd. Thus, we are forced to conclude that $\kappa > |D/P|$, and so M belongs to family (4).

Subcase 2. $|M[P]| < |M|$. Now let $R = D_P$, and let $(p) = PD_P$. Lemma 5 says that the set $M[P]$ (viewing M as a D-module) is the same as the set $M[p]$ (viewing M as an R-module), and M is a p-primary R-module. Thus, we view M now as a module over the PID R, and we have that $|M[p]| < |M|$. For each positive integer n, we let M_n be the collection of elements of M annihilated by p^n. Clearly, $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$, and M is the union of the M_n's as n ranges over the positive integers. We claim that $M[p] = M_1$ is finite. Suppose by way of contradiction that this is not the case. We claim that this forces some M_n to have greater cardinality than $M[p]$. Otherwise, by elementary set theory, we have that $|M| = |\cup_n M_n| \leq \aleph_0 \cdot |M[p]| = |M[p]|$, contradicting that $|M[p]| < |M|$. So we conclude that some M_n has greater cardinality than $M[p]$. Recall that our assumption (by way of contradiction) is that $M[p]$ is infinite. Hence, M_n is also infinite. Let $|M_n| = \kappa > |M[p]|$. By Lemma 4, there are κ elements of M annihilated by p, i.e., $|M[p]| \geq \kappa$, contradicting that $\kappa > |M[p]|$. Thus, finally, we conclude that $M[p]$ is finite. It follows again by Lemma 4 that each M_n must also be finite. As M is infinite, it follows that each M_n is finite and nonempty. It is easy to see that $M_1 \subsetneq M_2 \subsetneq M_3 \dots$.

Next we form a graph with the vertex set V equal to the elements of M. We draw an edge from v to w if and only if $pw = v$ and $w \neq 0$. As each M_n is finite and nonempty and M is infinite, it is easy to see that this defines an infinite, finitely branching tree (with 0 as the base node). By König's Lemma, there exists an infinite path $(0, v_1, v_2, \dots)$. Thus, by definition, we obtain $v_1 \neq 0$ and $pv_1 = 0$, $pv_2 = v_1$, etc. Let M' be the R-submodule of M generated by the v_i. By Proposition 6, we see that $M' \cong C(p^\infty)$. As M is countable, the congruence of M implies that $M \cong C(p^\infty)$ as an R-module. Recall that $M[p]$ is finite, and thus by Proposition 7 and Lemma 3 (along with the simple fact that $M[p] = S_1$), we see that every proper submodule of $C(p^\infty)$ is finite. In particular, $R/(p)$ is finite. By Lemma 1, we have an R-module isomorphism of $C(P^\infty)$ into $C(p^\infty)$. Since $C(P^\infty)$ is infinite (Lemma 2), we see that this isomorphism is onto $C(p^\infty)$, and thus $C(P^\infty) \cong C(p^\infty)$ as R-modules. Recall that $M \cong C(p^\infty)$ as an R-module. Hence, $M \cong C(P^\infty)$ as an R-module, and hence also as a D-module. As $D/P \cong R/(p)$, the residue field D/P is finite, and we see that M belongs to family (5). The proof is complete. □

3. Some consequences.

Definition 5. Let M be an infinite module over the ring R. M is called a Jónsson module provided every proper submodule of M has smaller cardinality than M.

In [**4**], Gilmer and Heinzer classify the countable Jónsson modules over an arbitrary Prüfer domain. Noting that a Jónsson module is clearly congruent, we are able to obtain a classification of all Jónsson modules over an arbitrary Dedekind domain:

Corollary 2. *Let D be a Dedekind domain, and let M be an infinite D-module. Then M is a Jónsson module if and only if one of the following holds*:

(1) *$M \cong D$ and D is a field.*

(2) *$M \cong D/P$ for some nonzero prime ideal P of D.*

(3) *$M \cong C(P^\infty)$ for some nonzero prime ideal P of D and D/P is finite.*

Proof. As noted, M is trivially congruent. A quick glance at our classification theorem reveals that M must belong to family (2), (3), or (5). As we've seen, the modules in families (3) and (5) are Jónsson modules. Thus, suppose that $M \cong D$ and D is a PID. Note that if x is any nonzero element of D, then $|(x)| = |D|$. By the condition on D, this forces $(x) = D$, and so D is a field. This completes the proof. □

We immediately obtain the following corollary:

Corollary 3. *Suppose that D is a Dedekind domain that is not a field but which contains an infinite field F. Then the Jónsson modules over D are precisely the modules D/P where P is a nonzero prime ideal of D.*

Proof. Clearly it suffices to show that D/P is infinite for every nonzero prime ideal P of D. Let P be such an ideal. The mapping $\varphi : F \to D/P$ defined by $\varphi(x) := P + x$ is clearly injective. Thus, D/P is infinite and the proof is complete. □

For example, if F is an infinite field, then the Jónsson modules over $F[X]$ are precisely the $F[X]$-modules $F[X]/(p(X))$, where $p(X)$ is an irreducible polynomial in $F[X]$.

As a final application, we show that an uncountable Jónsson group (the definition is the obvious one) G has the property that its derived subgroup G' coincides with G. This shows that G is in some sense highly nonabelian.

Proposition 9 *Let G be an uncountable Jónsson group with derived subgroup G'. Then $G' = G$.*

Proof. Let G' be the derived subgroup of G. Suppose $G \neq G'$. Then G' has smaller cardinality than G. It follows that G/G' is Abelian and of the same cardinality as G. Let $\varphi : G \to G/G'$ be the canonical map. If H is a proper subgroup of G/G', then $\varphi^{-1}[H]$ is a proper subgroup of G, and thus H has smaller cardinality than $|G| = |G/G'|$. Thus, G/G' is an uncountable Jónsson $\mathbf{Z}$-module. This contradicts Corollary 2. Thus, we are forced to conclude that $G = G'$. This completes the proof. □

REFERENCES

1. M. Auslander and D. Buchsbaum, *Groups, rings, modules*, Harper and Row, New York, 1974.

2. L. Fuchs, *Abelian groups*, Pergamon Press, Oxford, 1960.

3. R. Gilmer, *Multiplicative ideal theory*, Queens Papers Pure Appl. Math., Kingston, 1992.

4. R. Gilmer and W. Heinzer, *On Jónsson modules over a commutative ring*, Acta Sci. Math. **46** (1983), 3–15.

5. Thomas W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.

6. I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. **72**, (1952), 327–340.

7. W.R. Scott, *On infinite groups*, Pacific J. Math. **5** (1955), 589–598.

8. H. Smith and J. Wiegold, *On the number of subgroups of an uncountable soluble group*, Arch. Math. (Basel) **42** (1984), 289–295.

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210
Email address: ggoman@gmail.com