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ON MODULES M FOR WHICH $N \cong M$ FOR EVERY SUBMODULE N OF SIZE $|M|$

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ABSTRACT. Let R be a commutative ring with identity, and let M be an infinite unitary R-module. M is called a *Jónsson module* provided every submodule of M of the same cardinality as M is equal to M. Such modules have been well-studied, most notably by Gilmer and Heinzer ([**3–6**]). We generalize this notion and call M *congruent* provided every submodule of M of the same cardinality as M is isomorphic to M (note that this class of modules contains the class of Jónsson modules). These modules have been completely characterized by Scott in [**10**] when the operator domain is $\mathbf{Z}$. In [**9**], the author extended Scott's classification to modules over a Dedekind domain. In this paper, we study congruent modules over arbitrary commutative rings. We use the theory developed in this paper to prove new results about Jónsson modules as well as characterize several classes of rings.

1. Introduction and general results. In this paper, all rings are assumed commutative with identity and all modules are unitary. We begin by revisiting the definition given in the abstract.

Definition 1. Let M be an infinite module over the ring R. We call M a congruent module if and only if whenever N is a submodule of M of the same cardinality as M, then $N \cong M$.

To initiate the reader and to motivate our study, we introduce some canonical examples of congruent modules.

Example 1. Let F be an infinite field. Then F becomes a module over itself whose submodules are precisely the ideals of F. Since F has only trivial ideals, it is easy to see that F is congruent as a module over itself.

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Example 2. Consider the ring $\mathbf{Z}$ of integers as a module over itself. Any nontrivial submodule of $\mathbf{Z}$ is infinite cyclic and hence isomorphic to $\mathbf{Z}$. Thus $\mathbf{Z}$ is congruent as a module over itself.

Example 3. Let F be a field, and let $\kappa > |F|$ be an infinite cardinal. Then $\oplus_\kappa F$ is a congruent F-vector space.

Example 4. Let p be a prime number. The direct limit of the Abelian groups $\mathbf{Z}/(p^n)$ is the so called quasi-cyclic group of type p^∞, commonly denoted by $\mathbf{Z}(p^\infty)$. It is well known that every proper subgroup of $\mathbf{Z}(p^\infty)$ is finite, whence $\mathbf{Z}(p^\infty)$ is trivially a congruent module over $\mathbf{Z}$.

We conclude the list of examples by stating our classification of the congruent modules over a Dedekind domain.

Theorem 1 [**9**, Theorem 1]. *Let D be a Dedekind domain with quotient field K, and let M be an infinite D-module. Then M is congruent if and only if one of the following holds*:

(1) $M \cong \oplus_\kappa D/P$ *for some prime ideal P of D, and $\kappa > |D/P|$.*

(2) $M \cong D/P$ *where P is either a maximal ideal of D, or $P = \{0\}$ and D is a* PID.

(3) $M \cong C(P^\infty) = \{x \in K/D : P^n x = 0$ *for some $n > 0\}$, where P is a nonzero maximal ideal of D such that the residue field D/P is finite.*

We now recall the following definition.

Definition 2. Let M be an R-module. The annihilator of M, denoted by $\mathrm{Ann}\,(M)$, is the collection of all elements $r \in R$ for which $rm = 0$ for every element $m \in M$. M is said to be faithful if $\mathrm{Ann}\,(M) = 0$.

One checks immediately that $\mathrm{Ann}\,(M)$ is an ideal of R for any R-module M. Further, if I is the annihilator of M, then M has a canonical

module structure over the factor ring R/I given by $\overline{r} \cdot m := rm$. We state the following trivial but useful lemma. The proof is easy and is omitted.

Lemma 1. *Let M be an R/I-module where I is an ideal of R. Then M has a canonical R-module structure given by $r \cdot m := \overline{r}m$. With this structure, M is a congruent R-module if and only if M is a congruent R/I-module.*

We now come to the following proposition which will be indispensable in our analysis of congruent modules. This result is a generalization of [**5**, Proposition 2.5].

Proposition 1. *Let R be a ring, and suppose that M is a congruent R-module. Then the following hold*:

(1) *If $r \in R$, then either $rM \cong M$ or $rM = (0)$.*

(2) $\mathrm{Ann}\,(M)$ *is a prime ideal of R.*

Proof. Assume that M is a congruent R-module.

(1) Let $r \in R$. Define the map $\varphi : M \to M$ by $\varphi(m) = rm$. Then φ is clearly a homomorphism onto rM. Let K be the kernel. Then $rM \cong M/K$. This implies that $|K||rM| = |M|$. If $|rM| = |M|$, then since M is congruent, we see that $rM \cong M$. Otherwise $|K| = |M|$, and so $M \cong K$. Let $\psi : K \to M$ be an isomorphism. Now let $m \in M$ be arbitrary. Then $m = \psi(m')$ for some $m' \in K$. Thus $rm = r\psi(m') = \psi(rm') = \psi(0) = 0$. As m was arbitrary, we obtain $rM = (0)$.

(2) Suppose that $r, s \notin \mathrm{Ann}\,(M)$. Then by (1), we see that $rM \cong M$ and $sM \cong M$. This clearly implies that $rsM \cong M$. Hence $rs \notin \mathrm{Ann}\,(M)$ and $\mathrm{Ann}\,(M)$ is a prime ideal of R. This completes the proof. □

The following corollary follows immediately.

Corollary 1. *Let R be a ring and suppose that M is a faithful congruent R-module. Then R is an integral domain.*

Thus by modding out the annihilator, there is no loss of generality in assuming that M is faithful over a domain.

We now turn our attention toward characterizing the torsion-free congruent modules. We begin with the following technical lemma. The proof utilizes basic set-theoretic techniques and is omitted.

Lemma 2. *Suppose that R is a ring and $|R| = \kappa$. Then, for any nonzero cardinal λ, if at least one of κ, λ is infinite, then $|\oplus_\lambda R| = \max(\kappa, \lambda)$.*

The following proposition will be used shortly.

Proposition 2. *Let R be an infinite ring. Then R is congruent as a module over itself if and only if R is a principal ideal domain.*

Proof. Suppose first that R is a principal ideal domain. Suppose that I is an ideal of R of the same cardinality as R. Then $I = (x)$ for some nonzero $x \in R$. The mapping $r \mapsto rx$ is clearly an R-module isomorphism between R and I. Thus R is congruent as a module over itself. Conversely, suppose that R is congruent as a module over itself. As R is a faithful R-module (R has an identity), it follows from Corollary 1 that R must be a domain. Let I be an arbitrary nonzero ideal of R. We show that I is principal. Pick any nonzero $x \in I$. The mapping $r \mapsto rx$ is injective since R is a domain, and thus $|I| = |R|$. As R is congruent, we have $I \cong R$ as R-modules. As R is a cyclic R-module, so is I. This completes the proof. □

We state a final definition before characterizing the torsion-free congruent modules.

Definition 3. Let M be an R-module, and let S be a subset of M. S is said to be linearly independent provided that whenever $m_1, \ldots, m_k$ are distinct elements of S and $r_1, \ldots, r_k \in R$ with $r_1 m_1 + \cdots + r_k m_k = 0$, then each $r_i m_i = 0$.

It is easy to see that if S is an independent subset of an R-module M, then the submodule $\langle S\rangle$ generated by S is isomorphic to $\oplus_{s\in S}Rs$. If S is torsion-free, then $\langle S\rangle \cong \oplus_{|S|}R$. We now characterize the torsion-free congruent modules.

Theorem 2. *Let R be a ring and M an infinite torsion-free R-module. Then M is congruent if and only if $M \cong \oplus_\kappa R$ where κ is a cardinal and one of the following holds*:

(1) *$\kappa = 1$ and R is a principal ideal domain.*

(2) *κ is infinite, $\kappa > |R|$, and R is a Dedekind domain.*

Proof. That the modules in classes (1) and (2) are congruent follows from Theorem 1. Thus we let R be an arbitrary ring and M a torsion-free congruent R-module. Since M is torsion-free, it follows that R is a domain. If R is finite, then R is a field, and thus M is a vector space over R. But then $M \cong \oplus_\kappa R$ for some cardinal κ. Since M is infinite, it follows immediately that (2) holds. Thus we assume that R is an infinite domain. Suppose first that $|M| \leq |R|$. Let m be an arbitrary nonzero element of M. Since M is torsion-free, the map $r \mapsto rm$ is injective, and thus $|(m)| = |R| = |M|$. Since M is congruent, $M \cong R$ and (1) holds by Proposition 2. Now suppose that $|M| > |R|$. Let F be the quotient field of R, and let S be the set of nonzero elements of R. Then $S^{-1}M$ is an F-vector space. We have $|S^{-1}M| = |M|$ and $|F| = |R|$. Thus the dimension of $S^{-1}M$ as a vector space over F is equal to $|M|$. It follows that there exists a set X of $|M|$ elements of M which are linearly independent over R. Since M is congruent, $M \cong \langle X\rangle \cong \oplus_{|M|}R$. It remains to show that R is a Dedekind domain. If I is any nonzero ideal of R, then $|\oplus_{|M|} I| = |\oplus_{|M|} R|$, and hence as $\oplus_{|M|}R$ is congruent, $\oplus_{|M|}I \cong \oplus_{|M|}R$. Hence I is a direct summand of a free module, and is thus projective. Since every ideal of R is projective, R is Dedekind. This completes the proof. □

Now that we have this theorem in hand, we proceed to show that all congruent modules which are not torsion-free must be torsion. To begin, we note that in general, congruent modules are not closed under taking homomorphic images. However, there is a class of submodules

which preserves congruence upon factoring, and it is this class which will play the main role in establishing our next theorem.

Definition 4. Let M be an R-module and N be a submodule of M. We say that N is monomorphically invariant if and only if for any monomorphism $\varphi : M \to M$ and any $m \in M$, $\varphi(m) \in N$ if and only if $m \in N$.

Before stating our next lemma, we introduce the following notation for brevity. If $\varphi : M \to N$ is an R-module homomorphism and J is a submodule of N, we let $J^c := \varphi^{-1}(J)$.

Lemma 3. *Let M be a congruent R-module, and suppose that N is a monomorphically invariant submodule of M. If M/N is infinite, then M/N is a congruent R-module.*

Proof. Let us assume that M is a congruent R-module and that N is a monomorphically invariant submodule of M. Suppose that M/N is infinite, and let J be a submodule of M/N of the same cardinality as M/N. We must show that $J \cong M/N$. If N has the same cardinality as M, then as $N \subseteq J^c$ (J^c is taken with respect to the natural map from M onto M/N), we see that J^c has the same cardinality as M. Otherwise $|N| < |M|$, in which case $|M/N| = |M| = |J|$, and thus in this case also $|J^c| = |M|$. As M is congruent, $J^c \cong M$. Let $\varphi : M \to J^c$ be an isomorphism. We define a mapping $\overline{\varphi} : M/N \to J$ by $\overline{\varphi}(\overline{m}) := \overline{\varphi(m)}$. We first must show that this mapping is well-defined. Thus suppose that $\overline{m}_1 = \overline{m}_2$. Then $m_1 - m_2 \in N$. Since N is monomorphically invariant, $\varphi(m_1 - m_2) \in N$, and thus clearly $\overline{\varphi(m_1)} = \overline{\varphi(m_2)}$. Hence $\overline{\varphi}$ is well-defined. $\overline{\varphi}$ is clearly a surjective homomorphism. We now show that $\overline{\varphi}$ is injective. Suppose that $\overline{\varphi}(\overline{m}) = \overline{0}$. Then $\overline{\varphi(m)} = \overline{0}$, and thus $\varphi(m) \in N$. As N is monomorphically invariant, $m \in N$, and thus $\overline{m} = \overline{0}$. This shows that $\overline{\varphi}$ is injective and thus $M/N \cong J$ and the proof is complete. □

We also need the following simple lemma.

Lemma 4. *Let M be a module over the domain D, and let T be the torsion submodule of M. Then T is monomorphically invariant.*

Proof. Let M, D and T be as stated, and let $\varphi : M \to M$ be a monomorphism. Let $m \in M$ be arbitrary. We must show that $m \in T$ if and only if $\varphi(m) \in T$. Suppose first that $m \in T$. By definition, there exists a nonzero $r \in D$ with $rm = 0$. Applying φ to both sides yields $r\varphi(m) = 0$. Hence $\varphi(m) \in T$. Conversely, suppose $\varphi(m) \in T$. Then $r\varphi(m) = 0$ for some nonzero $r \in D$ and thus $\varphi(rm) = 0$. As φ is injective, $rm = 0$ and $m \in T$. This completes the proof. □

We now state our result.

Theorem 3. *Let M be a congruent module over the ring R. Then M is either a torsion module or a torsion-free module.*

Proof. We first prove this in a special case and then generalize. We suppose that M is a congruent module over a domain D. Assume first that $|M| \leq |D|$. If M is not a torsion module, then let $m \in M$ be a nontorsion element. Then the cyclic module (m) has cardinality $|D|$. Since $|M| \leq |D|$, it follows that $|(m)| = |M| = |D|$. As M is congruent, $M \cong (m)$. Since m is not a torsion element, we have $(m) \cong D$. Thus $M \cong D$ and M is torsion-free since D is a domain. We now suppose that $|M| > |D|$. Let T be the torsion submodule of M. If $|T| = |M|$, then since M is congruent, we see that $M \cong T$ and M is a torsion module. Hence we suppose $|T| < |M|$. In particular, $|M/T| = |M| > |D|$. By Lemma 3 and Lemma 4, M/T is a congruent D-module. Since M/T is torsion-free, it follows from Theorem 2 that $M/T \cong \oplus_\kappa D$ for some cardinal κ. Recall that $|M/T| = |M| > |D|$. Hence $\kappa = |M|$. There exist κ elements $\{\overline{m}_i : i \in \kappa\}$ of M/T which are linearly independent over D. Clearly this implies that $\{m_i : i \in \kappa\}$ is linearly independent over D. Since $\kappa = |M|$ and M is congruent, $M \cong \oplus_\kappa D$ and so M is torsion-free.

Now for the general case. Suppose that M is an arbitrary congruent R-module. Let P be the annihilator of M in R. Then M is a congruent R/P-module. Since R/P is a domain (Proposition 1), it follows from what we just proved that M is a torsion or a torsion-free congruent

R/P-module. If M is a torsion R/P-module, it is clear that M is a torsion R-module. Suppose that M is a torsion-free R/P-module. If M is a torsion-free R-module, then we are done. Thus suppose that $rm = 0$ for some nonzero $m \in M$ and nonzero $r \in R$. Since M is a torsion-free R/P-module, we see that $\overline{r} = \overline{0}$, and so $r \in P$. But then r annihilates every element of M. In particular, M is a torsion R-module. This completes the proof. □

Recall that an infinite R-module M is a Jónsson module provided every proper submodule of M has smaller cardinality than M. In [**5**, Theorem 3.1], Gilmer and Heinzer prove (among other results) that every infinitely generated countable Jónsson module is torsion. We presently provide a significant generalization. First we prove a simple lemma (which is also discussed in [**5**]).

Lemma 5. *Suppose M is a Jónsson module over the ring R. Then M is indecomposable.*

Proof. If $M = N \oplus P$, then by elementary cardinal arithmetic, $|N| = |M|$ or $|P| = |M|$. As M is a Jónsson module, this forces either $N = M$ or $P = M$. Thus M is indecomposable. □

We now present our generalization of a portion of Theorem 3.1 from [**5**].

Proposition 3. *Let M be a faithful Jónsson module over the domain D. Then either M is torsion or D is a field and $M \cong D$.*

Proof. Suppose that M is a faithful Jónsson module over the domain D. If M is torsion we are done. Thus suppose M is not torsion. By Theorem 3, it follows that M is torsion-free. As M is indecomposable, it follows from Theorem 2 that $M \cong D$ and D is a principal ideal domain. It remains to show that D is a field. Let d be an arbitrary nonzero element of D. Since D is a domain, $|(d)| = |D|$. As D is a Jónsson module, we have $(d) = D$ and thus d is a unit. As $d \in D - \{0\}$ was arbitrary, this shows that D is a field. □

2. Large congruent modules. In this section, we study congruent modules which are 'large' relative to the operator ring R. The precise definition follows.

Definition 5. Let M be an infinite R-module. M is said to be large provided $|M| > |R|$.

Before proceeding, we recall several definitions from set theory.

Definition 6. Let κ be an infinite cardinal. The cofinality $\mathrm{cf}(\kappa)$ of κ is the least cardinal λ such that κ is the sum of λ many cardinals, each smaller than κ. The cardinal κ is called regular if $\mathrm{cf}(\kappa) = \kappa$ and singular if $\mathrm{cf}(\kappa) < \kappa$.

The regular cardinals include $\aleph_0$ as well as every successor cardinal; that is, every cardinal of the form κ^+ for some cardinal κ (κ^+ is simply the least cardinal larger than κ). It is well known that $\mathrm{cf}(\kappa)$ is regular for every infinite cardinal κ.

Recall from Theorem 1 that every large faithful congruent module over a Dedekind domain is free. Using Theorems 2 and 3, we obtain the following.

Theorem 4. *Suppose that D is a domain, and suppose that M is a large faithful congruent module over D. If $|D| < cf(|M|)$, then D is a Dedekind domain, and M is isomorphic to a direct sum of copies of D.*

Proof. Assume that $|D| < \mathrm{cf}(|M|)$. By Theorem 2, it suffices to show that M is torsion-free. Suppose by way of contradiction that this is not the case. Then by Theorem 3, it follows that M is a torsion module. For each nonzero $d \in D$, we let $M[d] := \{m \in M : dm = 0\}$. Clearly $M[d]$ is a submodule of M for each d. Since M is torsion, we have $M = \cup_{d \in D - \{0\}} M[d]$. If each $M[d]$ has smaller cardinality than M, then M is expressed as the union of at most $|D|$ many subsets each of smaller cardinality than M. This contradicts the fact that $|D| < \mathrm{cf}(|M|)$. Hence some $M[d]$ has the same cardinality as M. Since M is congruent, we obtain $M \cong M[d]$. But, by definition, $M[d]$ is

annihilated by d, and thus so is M. This contradicts the fact that M is faithful and the proof is complete. □

We immediately obtain the following corollary.

Corollary 2. *Let D be a domain, and suppose that M is a large faithful congruent module of regular cardinality. Then D is a Dedekind domain and M is free.*

We now show that a somewhat weaker result holds for large congruent modules of any cardinality if one assumes the GCH. We recall a final definition from set theory.

Definition 7. Let κ be an infinite cardinal. Then κ is called a strong limit cardinal provided that for every $\lambda < \kappa$, one has $2^\lambda < \kappa$.

(*) Note that if κ is a strong limit, and $\alpha, \beta < \kappa$, then $\alpha^\beta \leq (2^\alpha)^\beta = 2^{\alpha \cdot \beta} < \kappa$.

We utilize the following result of Ecker in the proof of our next theorem.

Proposition 4 [**2**, Proposition 3.2]. *Let R be an infinite ring and I a maximal independent set in an R-module M. Then we have the following*:

(1) *If $|I| = 0$, then $M = \{0\}$.*

(2) *If $|I| = 1$, then $|M| \leq 2^{|R|}$.*

(3) *If $|I| > 1$, then $|M| \leq |I|^{|R|}$.*

Theorem 5. *Assume that every singular cardinal is a strong limit. Then every large faithful congruent module M over a domain D is a direct sum of cyclic modules.*

Proof. Suppose every singular cardinal is a strong limit, and suppose M is a large faithful congruent module over the domain D. If M has regular cardinality, then M is free by Corollary 2, and we are clearly

done. Similarly if D is finite, then D is a field and M is free in this case as well. Thus we assume that M has singular cardinality and D is infinite. An easy application of Zorn's Lemma shows that M has a maximal independent subset I. It now follows easily from $(*)$ and (2) and (3) above that $|I| = |M|$. Since M is congruent, we obtain $M \cong \oplus_{i \in I} Ri$. This completes the proof. □

Recall that Generalized Continuum Hypothesis is the statement that for every infinite cardinal κ, there are no cardinals properly between κ and 2^{κ}. It is well known that GCH is independent of the usual axioms for set theory. It is also easy to see that GCH implies that every singular cardinal is a strong limit. Thus we obtain the following corollary.

Corollary 3. *Assume* GCH. *Then every large faithful congruent module M is a direct sum of cyclic modules.*

We apply these results to obtain nonexistence results for large Jónsson modules.

Proposition 5. *Let R be a ring. Then*:

(1) *There are no large Jónsson modules over R of regular cardinality.*

(2) *Assuming* GCH, *there are no large Jónsson modules.*

Proof. This follows immediately from Corollary 2, Corollary 3 and the fact that Jónsson modules are indecomposable. □

We now proceed to show that the characterization of large congruent modules can be done without GCH if the operator domain is Noetherian. We state two lemmas.

Lemma 6. *Let M be an R-module, and let $m \in M$. Suppose that* $\operatorname{Ann}(m) = P$ *is a prime ideal of R. If $r \in R$ and $rm \neq 0$, then* $\operatorname{Ann}(rm) = P$.

Proof. We suppose that $r \in R$ and $rm \neq 0$. Clearly $P \subseteq \operatorname{Ann}(rm)$. Thus suppose that $x \in \operatorname{Ann}(rm)$. Then $xrm = 0$, and thus $xr \in P$.

Since P is prime, either $x \in P$ or $r \in P$. As $rm \neq 0$, this forces $x \in P$. This completes the proof. □

Lemma 7. *Let M be an R-module, and suppose that $m_1, \ldots, m_k$ are distinct elements of M which are linearly independent over R. Let J_i be the annihilator of m_i. Then the annihilator of $m_1 + \cdots + m_k$ is $J_1 \cap J_2 \cdots \cap J_k$.*

Proof. Let $r \in R$. Simply observe that $r(m_1 + \cdots + m_k) = 0$ if and only if $rm_i = 0$ for each i (since $\{m_1, \ldots, m_k\}$ is independent) if and only if $r \in J_1 \cap J_2 \cdots \cap J_k$. □

We prove a proposition which will serve as a cornerstone of our next two classification theorems.

Proposition 6. *Let D be a Noetherian domain and suppose M is a faithful congruent D-module. Suppose $M = \oplus_{i \in I} Dm_i$ for some set $\{m_i : i \in I\}$ of elements of M and for each i, $\mathrm{Ann}\,(m_i)$ is a prime ideal of D. Then D is a Dedekind domain and M is a direct sum of copies of D.*

Proof. Let $S = \{m_i : i \in I\}$, and for each prime ideal P of D, we let $M[P]$ be the submodule of M generated by $\{x \in S : \mathrm{Ann}\,(x) = P\}$. It is clear that $M = \oplus_{P \mathrm{prime}} M[P]$. We will show that each nontrivial $M[P]$ has the same cardinality as M. Suppose by way of contradiction that some nontrivial $M[P_0]$ has smaller cardinality than M. It follows that $\oplus_{P \neq P_0} M[P]$ has the same cardinality as M. Since M is congruent, we obtain $M \cong \oplus_{P \neq P_0} M[P]$. Since $M[P_0]$ is nonzero, there exists an element of M with annihilator P_0. Since $M \cong \oplus_{P \neq P_0} M[P]$, it follows that there exists an element m of $\oplus_{P \neq P_0} M[P]$ with annihilator P_0. We may express this element in the form $m = m_1 + \cdots + m_k$ where each $m_i \in M[P_i]$ and $P_i \neq P_0$. It follows from Lemmas 6 and 7 that each m_i has annihilator P_i. It follows from Lemma 7 that m has annihilator $P_1 \cap P_2 \cdots \cap P_k$. But recall by assumption that m has annihilator P_0. Hence $P_0 = P_1 \cap P_2 \cdots P_k$. But since each P_i is prime, this forces $P_0 = P_i$ for some i with $1 \leq i \leq k$. This is a contradiction. Hence each nontrivial $M[P]$ has the same cardinality as M. Since M is congruent,

we obtain $M \cong M[P]$ for some (any) nontrivial $M[P]$. Clearly this implies that $M \cong \oplus_\lambda D/P$ for some cardinal λ. But M is faithful and P annihilates every element of M. It follows that $P = 0$, and $M \cong \oplus_\lambda D$. That D is a Dedekind domain now follows from Theorem 2. This completes the proof. □

We use this proposition to characterize large congruent modules over Noetherian rings. First we need two more lemmas.

Lemma 8. *Let R be a Noetherian ring, and let I be a proper ideal of R. There exists an $x \in R$ such that $[I : x]$ is a (proper) prime ideal of R.*

Proof. Assume that I is a proper ideal of R. Let $\mathcal{S}$ denote the collection of all ideals of the form $[I : x]$ where $x \notin I$. Note in particular that $\mathcal{S}$ consists of proper ideals of R. Since R is Noetherian, $\mathcal{S}$ possesses a maximal element, say $[I : x]$. We claim that $[I : x]$ is a prime ideal. Suppose not. Then there exist elements $a, b \in R$ such that $ab \in [I : x]$ but $a \notin [I : x]$ and $b \notin [I : x]$. Then $ax \notin I$. In particular, this means $[I : ax] \in \mathcal{S}$. Clearly $[I : x] \subseteq [I : ax]$. Note that by our assumptions, $b \in [I : ax] - [I : x]$. This contradicts the maximality of $[I : x]$ and completes the proof. □

The proof of our next lemma is contained in the proof of Theorem 1 of [**1**].

Lemma 9 [**1**]. *Let R be a Noetherian ring, and suppose that M is an infinite R-module with $|M| > |R|$. Then M possesses an independent subset S of the same cardinality as M.*

We are now ready to characterize the large congruent modules over a Noetherian ring.

Theorem 6. *Let D be a Noetherian domain, and suppose that M is a large faithful congruent module over D. Then D is a Dedekind domain and $M \cong \oplus_{|M|} D$.*

Proof. Let M be a large faithful congruent module over the Noetherian domain D. By Lemma 9, M possesses an independent subset S with $|S| = |M|$. Of course we may assume that each element of S is nonzero. Let $x \in S$ be arbitrary and consider $\mathrm{Ann}\,(x)$. By Lemma 8, there is an element $d \in D$ such that $[\mathrm{Ann}\,(x) : d] = P$ is a prime ideal of D. We claim that $\mathrm{Ann}\,(dx) = P$. Indeed, $y \in \mathrm{Ann}\,(dx)$ if and only if $ydx = 0$ if and only if $yd \in \mathrm{Ann}\,(x)$ if and only if $y \in [\mathrm{Ann}\,(x) : d] = P$. Hence for every $x \in S$, we may choose an element $d_x \in D$ with $\mathrm{Ann}\,(d_x x)$ a prime ideal of D. Since S is independent, so is $\{d_x x : x \in S\}$. Since M is congruent and $|S| = |M|$, we have $M \cong \oplus_{x \in S} D d_x x$. The proof is now completed by invoking Proposition 6. □

3. Injective congruent modules. Most of the faithful congruent modules we've studied to this point have all ended up being free. We now look at the congruent injective modules over a Noetherian ring. Shortly we will provide a complete description of these modules. We begin by recalling a few definitions.

Definition 8. Let M be an R-module. The set of prime ideals of R associated to M, denoted by $\mathrm{Ass}_R(M)$, is the set $\{P | P$ is a prime ideal of R and $P = \mathrm{Ann}\,(y)$ for some $y \in M\}$.

Definition 9. A domain D is called an almost DVR if and only if D is a local Noetherian domain of Krull dimension 1 and is such that the integral closure of D is both finitely generated over D and a DVR.

Definition 10. A module M over a ring R is said to be almost finitely generated if and only if M is not finitely generated, but every proper submodule of M is finitely generated.

We now state the following result from Weakley in [**11**] on almost finitely generated modules and then recall some standard results from Matlis about injective modules over Noetherian rings. The second proposition is a collection of several results of Matlis. For proofs, see [**8**].

Proposition 7 [**11**, Proposition 3.3]. *Let D be a domain, and let M be a maximal ideal of D. Let $E(D/M)$ be the injective hull of D/M. Then the following are equivalent*:

(1) *$E(D/M)$ is Artinian and almost finitely generated.*

(2) *D_M is an almost* DVR.

Proposition 8 [**8**]. *Let R be a Noetherian ring. Then*:

(1) *An arbitrary direct sum of R-modules is injective if and only if each summand is injective.*

(2) *If E is an injective module over R, then $E \cong \oplus_i E(R/P_i)$, where each P_i is a prime ideal of R and $E(R/P_i)$ is the injective hull of R/P_i.*

(3) *For a prime ideal P, $E := E(R/P)$ is P-primary and* $\mathrm{Ass}_R(E) = \{P\}$.

(4) *For each prime ideal P, $E(R/P)$ is indecomposable.*

We are almost ready to classify the injective congruent modules over a Noetherian ring. We first state a theorem of Gilmer and Heinzer on countable Jónsson modules and then prove two lemmas.

Proposition 9 [**5**, Theorem 3.1]. *Suppose that M is a countably infinite Jónsson module over the ring R and that M is not finitely generated. Then M is a torsion R-module, and there exists a maximal ideal Q of R such that the following hold*:

(1) *$Ann(x)$ is a Q-primary ideal of finite index for every $x \in M - \{0\}$.*

(2) *R/Q is finite.*

(3) *The powers of Q properly descend.*

(4) *$\cap_{i=1}^{\infty} Q^i = Ann(M)$.*

(5) *If $H_i = \{x \in M : Q^i x = 0\}$, then $\{H_i\}_{i=1}^{\infty}$ is a strictly ascending sequence of submodules of M such that $M = \cup_{i=1}^{\infty} H_i$.*

Lemma 10. *Let R be a ring and I a finitely generated ideal of R. If R/I is finite, then R/I^n is finite for every positive integer n.*

Proof. Induct on n, the case $n = 1$ being obvious. By the second isomorphism theorem, there is a surjective module homomorphism from R/I^n to R/I with kernel I/I^n, whence $|R/I^n| = |I/I^n||R/I|$. Thus it suffices to show that I/I^n is finite. Simply note that I/I^n is a finitely generated module over the ring R/I^{n-1}, which by our inductive hypothesis is finite. This completes the proof. □

Lemma 11. *Let M be an infinite R-module, and let $r \in R$, $n \in \mathbf{N}$. Suppose that r^n annihilates M. Let $M[r]$ denote the submodule of M consisting of the elements of M annihilated by r. Then $|M[r]| = |M|$.*

Proof. We prove this by induction on $n \in \mathbf{N}$. The case when $n = 1$ is trivially true. Thus we assume the lemma is true for some $n \in \mathbf{N}$. Suppose that M is an infinite R-module, $r \in R$, $n \in \mathbf{N}$, and suppose that r^{n+1} annihilates M. It is clear that $M/M[r] \cong rM$. Hence we get that $|M| = |rM||M[r]|$. As M is infinite, it follows that either $|M[r]| = |M|$ or $|rM| = |M|$. If $|M[r]| = |M|$, then we have what we want and we are done. Otherwise $|rM| = |M|$. Recall that r^{n+1} annihilates M, and therefore r^n annihilates rM. By the inductive hypothesis, we have $|(rM)[r]| = |rM| = |M|$. Clearly $(rM)[r] \subseteq M[r]$, and thus $|M[r]| = |M|$. This completes the proof. □

Finally, we are ready to characterize the congruent injective modules over a Noetherian ring.

Theorem 7. *Let D be a Noetherian domain, and let M be an infinite module over D. Then M is congruent and injective if and only if one of the following holds*:

(1) *D is an infinite field and $M \cong D$.*

(2) *D is a field, and $M \cong \oplus_\kappa D$, where $\kappa > |D|$ is an infinite cardinal.*

(3) *$M \cong E(D/J)$ where J is a nonzero maximal ideal of D, D/J is finite, and D_J is an almost* DVR (here $E(D/J)$ is the injective hull of D/J).

Proof. We first show that each of the modules in (1)–(3) is congruent and injective. It follows from Theorem 2 that the modules in (1) and

(2) are congruent, and from Proposition 8 that they are injective. Now consider $M = E(D/J)$ where J is a nonzero maximal ideal of D, D/J is finite, and D_J is an almost DVR. By Proposition 7, we see that M is Artinian and almost finitely generated. From (3) of Proposition 8, it follows that M is J-primary. Consider any nonzero element $m \in M$. Then $J^i m = 0$ for some positive integer i. Let I be the annihilator in D of m. Then $J^i \subseteq I$, and hence $|D/I| \leq |D/J^i|$. Since D/J is finite and D is Noetherian, it follows from Lemma 10 that D/I is finite. Thus for each $m \in M$, the cyclic submodule (m) is finite. Let N be a proper submodule of M. Since M is almost finitely generated, N is finitely generated over D. But since each cyclic submodule is finite, this clearly implies that N itself is finite. Thus we have shown that M is a Jónsson module and is consequently congruent.

Conversely, let M be any infinite module over D, and suppose that M is congruent and injective. We will show that M belongs to family (1), (2), or (3). From (2) of Proposition 8, we have:

$$M \cong \bigoplus_{i \in I} E(D/P_i)$$

for some set $\{P_i\}$ of prime ideals of D (with repetitions allowed). We first dispose of the case where some $P_i = \{0\}$. In this case, note that D embeds into M, and thus M is not a torsion module. It follows from Theorem 3 that M is torsion-free, and thus M is isomorphic to a direct sum of copies of D. It follows from (1) of Proposition 8 that D itself must be injective. But then D is divisible, hence a field. Thus M belongs to family (1) or (2). Hence we assume that all of the prime ideals P_i are nonzero. We suppose first that each $E(D/P_i)$ has smaller cardinality than M. Fix an arbitrary $E(D/P_{i_0})$, and select a nonzero element $m \in E(D/P_{i_0})$. Let I be the annihilator of m in D. By Lemma 8 there exists an $x \in D$ such that $[I : x]$ is a prime ideal of D. Consider the element xm, and let $d \in D$ be arbitrary. Then note that $d(xm) = 0$ if and only if $dx \in I$ if and only if $d \in [I : x]$. Hence the element xm has prime annihilator in D. Now, since $E(D/P_{i_0})$ has smaller cardinality than M, we see that $\oplus_{i \neq i_0} E(D/P_i)$ must have the same cardinality as M, and thus $|(\oplus_{i \neq i_0} E(D/P_i)) \oplus (xm)| = |M|$. Since M is congruent, this implies that $(\oplus_{i \neq i_0} E(D/P_i)) \oplus (xm)$ is injective. It now follows from (1) of Proposition 8 that each summand is injective. In particular, (xm) is injective. But recall that xm is

a nonzero element of $E(D/P_{i_0})$. Since (xm) is injective, it follows that $(xm) \oplus N = E(D/P_{i_0})$ for some module N. As $E(D/P_{i_0})$ is indecomposable by (4) of Proposition 8, we are forced to conclude that $N = 0$ and thus $E(D/P_{i_0})$ is a cyclic module with prime annihilator in D. Since i_0 was arbitrary, M is a direct sum of injective cyclic modules, each with prime annihilator. Proposition 6 now implies that D is a Dedekind domain and M is isomorphic to a direct sum of copies of D. As above, since M is injective, D must be a field, and thus M belongs to family (1) or (2). Thus, finally, we may assume that some $E(D/P)$ has the same cardinality as M and $P \neq \{0\}$. Since M is congruent, $M \cong E(D/P)$. By (3) of Proposition 8, we see that M is P-primary, and that P is the unique associated prime ideal of M. We now show that this forces M to be countable. Choose an arbitrary nonzero $r \in P$. Since M is P-primary, every element of M is killed by some power of r. For each positive integer n, we let M_n be the collection of elements of M annihilated by r^n. Clearly $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$, and M is the union of the M_ns as n ranges over the positive integers. We claim that $M[r] = M_1$ is finite. Suppose by way of contradiction that M_1 is infinite. Since $M_1 \subseteq M_2 \subset \cdots$, it follows from Lemma 11 that $|M_n| = |M_1|$ for every positive integer n. But, since M is the union of the M_n's, it is clear that $|M| = |M_1|$. As M is congruent, $M \cong M_1 = M[r]$, whence every element of M is killed by r. Recall that as M is an injective module over a Noetherian domain, M is divisible, thus faithful. This is a contradiction. Thus M_1 is finite. It follows from Lemma 11 that M_n is finite for every positive integer n, and hence M is countable. We now show that M is a Jónsson module. Let N be a proper submodule of M. We must show that N is finite. Suppose by way of contradiction that N is infinite. Then since M is congruent, $M \cong N$. But then N is injective, and so must be a direct summand of M. This contradicts (4) of Proposition 8. We have shown that $M \cong E(D/P)$ and that M is a Jónsson module. By (2) and (5) of Proposition 9, there is an associated maximal ideal J of M with D/J finite. By (3) of Proposition 8, we must have $J = P$. Lastly, we must show that D_J is an almost DVR. To prove this, it suffices by Proposition 7 to show that $E(D/J)$ is Artinian and almost finitely generated. But this follows immediately from the fact that $E(D/J)$ is a countable Jónsson module. This completes the proof. □

We immediately obtain the following corollary.

Corollary 4. *Let D be a Noetherian domain, and let M be an infinite module over D. Then M is an injective Jónsson module over D if and only if D is a field and $M \cong D$ or $M \cong E(D/J)$ for some nonzero maximal ideal J of D such that D/J is finite and D_J is an almost* DVR.

4. Ring characterizations. Using the theory we have developed, we give characterizations of fields, principal ideal domains, and Dedekind domains using the notion of a congruent module.

Proposition 10. *Let R be an infinite ring. The following are equivalent*:

(a) *R is a field.*

(b) *Every R-module may be embedded into a congruent R-module.*

(c) *Every large R-module is congruent.*

Proof. (a) $\Rightarrow$ (b). This follows immediately from Theorem 1 by taking a sufficiently large direct sum of copies of R.

(b) $\Rightarrow$ (a). Suppose that every R module is embeddable in a congruent R-module. Let x be a nonzero element of R. Suppose by way of contradiction that x is not invertible. Then the R-module $M := R \oplus R/(x)$ is neither torsion nor torsion-free. However, M embeds into a congruent R-module M'. By Theorem 3, M' is either torsion or torsion-free, hence so is M. This contradiction completes the proof of this implication.

(a) $\Rightarrow$ (c). This also follows immediately from Theorem 1.

(c) $\Rightarrow$ (a). Suppose every large R-module is congruent. Let $\kappa > |R|$. Then $\oplus_\kappa R$ is congruent. Since R has an identity, $\oplus_\kappa R$ cannot be torsion. By Theorem 3, $\oplus_\kappa R$ must be torsion-free. In particular, R is a domain. Now let $x \in R$ be nonzero. Suppose by way of contradiction that x is not invertible. Then $(x) \neq R$. Thus also $(x^2) \neq R$. Consider the R-module $\oplus_\kappa R/(x^2)$. By assumption, this module is congruent. The annihilator is clearly (x^2). By Proposition 1, (x^2) is a prime ideal.

But then $x \in (x^2)$. Since R is a domain, this clearly implies that x is invertible, a contradiction. This completes the proof. □

Proposition 11. *Let R be an infinite ring. The following are equivalent*:

(a) *R is a principal ideal domain.*

(b) *R is congruent as a module over itself.*

(c) *R possesses a congruent ideal I which is not contained in the set of zero-divisors of R.*

Proof. We already have (a) $\Leftrightarrow$ (b) from Proposition 2. Now (b) clearly implies (c), and thus we need only show that (c) $\Rightarrow$ (a). Suppose I is an ideal of R which is congruent as an R-module and suppose I is not contained in the zero-divisors of R. Then by Theorem 3 and Theorem 2, we see that R must be a domain. Let J be an arbitrary nonzero ideal of R, and let $x \in I$ be nonzero. Then $|Jx| = |I|$ and $Jx \subseteq I$. Since I is congruent, we have $Jx \cong I$. But also $J \cong Jx$, whence $J \cong I$. As J was an arbitrary nonzero ideal of R, it follows that R is congruent as a module over itself. By Proposition 2, it follows that R is a principal ideal domain. □

Proposition 12. *Let R be a ring. The following are equivalent*:

(a) *R is a Dedekind domain.*

(b) *R admits a torsion-free congruent R-module.*

(c) *R admits a free congruent R-module.*

(d) *R admits a faithful projective congruent R-module.*

(e) *Every R-module is the homomorphic image of a congruent R-module.*

Proof. (a) $\Rightarrow$ (b). This follows immediately from Theorem 1.

(b) $\Rightarrow$ (c). If R admits a torsion-free congruent R-module, then R is a Dedekind domain by Theorem 2. Now by Theorem 1, R admits a free congruent R-module.

(c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (a). Suppose that R admits a faithful projective congruent R-module M. Since M is faithful, it follows from Corollary 1 that R is a domain. But then M must be torsion-free, whence R is a Dedekind domain by Theorem 2.

(a) $\Rightarrow$ (e). This follows immediately from Theorem 1 and the fact that every module is the homomorphic image of a free module.

(e) $\Rightarrow$ (a). Suppose that every R-module is the homomorphic image of a congruent R-module. In particular, R itself is the homomorphic image of a congruent R-module M. If M is torsion, then clearly so is R, and this is impossible since R has an identity. Therefore by Theorem 3, M must be torsion-free. By Theorem 2, R is a Dedekind domain. □

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