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RINGS WHICH ADMIT FAITHFUL TORSION MODULES II

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All rings are assumed to have an identity and all modules are unitary.
Modules are left modules unless specified otherwise

Let R be an associative ring with identity. An (left) R-module M is said to be *torsion* if for every $m \in M$, there exists a nonzero $r \in R$ such that $rm = 0$, and *faithful* provided $rM = \{0\}$ implies $r = 0$ $(r \in R)$. We call R (left) *FT* if R admits a nontrivial (left) faithful torsion module. In this paper, we continue the study of FT rings initiated in Oman and Schwiebert [Rings which admit faithful torsion modules, to appear in *Commun. Algebra*]. After presenting several examples, we consider the FT property within several well-studied classes of rings. In particular, we examine direct products of rings, Brown–McCoy semisimple rings, serial rings, and left nonsingular rings. Finally, we close the paper with a list of open problems.

Keywords: Faithful module; torsion module; Brown–McCoy semisimple ring; nonsingular ring; serial ring; endomorphism ring; valuation ring.

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1. Introduction

The purpose of this paper is to continue our earlier work in [7]. Specifically, we continue to research the question: *Which rings admit faithful torsion modules*? This is a natural question since these modules are, in a sense, "locally annihilated" but not "globally annihilated". Recall that a module M over a ring R is *faithful* if $\operatorname{ann}(M) := \{r \in R : rM = 0\} = \{0\}$. However, it is clear that individual elements of a faithful module may be killed independently. We are interested in faithful modules where every element can be killed nontrivially, and define a module M to be *torsion* provided that for every $m \in M$, there exists some nonzero $r \in R$ such that $rm = 0$.

In [7], we defined a ring R to be (left) *FT* provided R admits a nontrivial (left) faithful torsion module. If R does not admit such a module, R is said to be *non-FT*. Moreover, we defined a rank function which played an essential role in our analysis, which we recall below.

Definition 1.1. Let R be a ring. The (left) FT rank of R, denoted $\mathrm{FT}(R)$, is defined to be the least cardinal number κ such that R admits a (left) faithful torsion module which can be generated by κ many elements. In case R is non-FT, we define the FT rank of R to be zero.

Remark 1.2. One may define the *right FT rank* of a ring R in the obvious way, and in general, the left and right FT rank of a ring need not coincide. In fact, we presented a ring which has left FT rank 0 and right FT rank 1 in [7, Example 4].

We now recall several more results from [7]. In particular, we showed that a left or right Artinian ring has finite FT rank, and a commutative Noetherian ring has countable FT rank. We completely determined the FT rank function for simple rings, semisimple Artinian rings, quasi-Frobenius rings, UFDs, commutative Noetherian domains, and commutative domains of finite Krull dimension, among other classes. It was also shown that if the FT rank of a ring is an infinite cardinal κ, then κ is regular. Moreover, if $\kappa \leq \lambda$ and κ is regular, then there exists a valuation ring V of cardinality λ such that $\mathrm{FT}(V) = \kappa$.

The outline of this paper is as follows. We begin by giving several examples of both FT and non-FT rings. We then develop some preliminary results which we utilize throughout the paper. In the next two sections, we study the FT property within the well-studied classes of serial rings and nonsingular rings, respectively. Finally, we close with some open problems.

2. Examples

In this section, we present several examples of both FT and non-FT rings to initiate the reader and to motivate our study.

Example 2.1. Let D be a division ring, and let V be a (nontrivial) vector space over D. Then the only torsion element of V is 0. It follows that D does not admit a nontrivial torsion module, and hence D is non-FT.

Example 2.2. The ring $\mathbb{Z}$ of integers is FT. Moreover, $\mathrm{FT}(\mathbb{Z}) = \aleph_0$.

Proof. Let $G := \bigoplus_{n>0} \mathbb{Z}/(n)$. It is readily checked that G is a faithful torsion module over $\mathbb{Z}$, and hence $\mathrm{FT}(\mathbb{Z}) \leq \aleph_0$. It is also easy to see that a finitely generated torsion abelian group is not faithful (as a $\mathbb{Z}$-module), whence $\mathrm{FT}\ (\mathbb{Z}) = \aleph_0$. □

Example 2.3. Let $n > 1$ be an integer. Then $\mathbb{Z}/(n)$ is non-FT.

Proof. Suppose by way of contradiction that $\mathbb{Z}/(n)$ admits a faithful torsion module M. Since M is unitary, we conclude that the additive order of an arbitrary element of M divides n. Let k be the lcm of the additive orders of the elements of M. We claim that $k = n$. If $k < n$, then $\overline{k}M = \{0\}$. But then M is not faithful, a contradiction. It follows from elementary abelian group theory that there exists some $m \in M$ with additive order $k = n$. But then m is not a torsion element of M, and we have another contradiction. This concludes the proof. □

Example 2.4. Let κ be a regular cardinal. There exists a valuation ring V such that $\mathrm{FT}(V) = \kappa$.

Sketch of Proof. Let $G := \bigoplus_{\kappa} \mathbb{Z}$ with the reverse lexicographic order. By the Jaffard–Ohm–Kaplanksy Theorem, there exists a field F and a valuation v on F with value group G. Let V be the associated valuation ring. Then $\mathrm{FT}(V) = \kappa$. See [7, Theorem 8] for details. □

Example 2.5. Let V be a discrete valuation ring, $J := (v)$ its maximal ideal, and let $k > 0$ be an integer. Then the ring $R := V/(v^k)$ is non-FT.

Proof. Consider the ring $R := V/(v^k)$, and let M be an arbitrary torsion R-module. Note that $I := (v^{k-1})/(v^k)$ is the minimum nonzero ideal of R. Thus for any $m \in M$, it follows that $I \subseteq \mathrm{ann}(m)$. We conclude that M cannot be faithful. □

Example 2.6. Let R be a commutative ring, and let $n > 1$ be an integer. Further, let $M_n(R)$ denote the ring of $n \times n$ matrices with entries in R. Then $\mathrm{FT}(M_n(R)) = 1$.

Proof. Let R be a commutative ring, and let $n > 1$ be an integer. It is straightforward to verify that R^n is a cyclic, faithful torsion module over $M_n(R)$, whence $\mathrm{FT}(M_n(R)) = 1$. □

Example 2.7. Suppose that G is a nontrivial torsion abelian group. Then $\mathrm{End}_{\mathbb{Z}}(G)$ is FT if and only if G is not cyclic.

Proof. Let G be a nontrivial torsion abelian group, and suppose first that $G \cong \mathbb{Z}/(n)$ for some $n > 1$. Then $\mathrm{End}_{\mathbb{Z}}(G) \cong \mathbb{Z}/(n)$ (as rings). Example 2.3 now implies that $\mathrm{End}_{\mathbb{Z}}(G)$ is non-FT.

Assume now that G is not cyclic. To show that $\mathrm{End}_{\mathbb{Z}}(G)$ is FT, it suffices to show that G is torsion over $\mathrm{End}_{\mathbb{Z}}(G)$ (since every module is faithful over its endomorphism ring).

We suppose first that there is no finite bound on the orders of the elements of G. Let $g \in G$ be arbitrary, and let n be the order of g. Then the map $f_n : G \to G$

defined by $f_n(x) := nx$ is nonzero since there are elements of G of order larger than n, yet $f_n(g) = 0$. Thus G is torsion over $\mathrm{End}_{\mathbb{Z}}(G)$ in this case.

We now assume that $nG = \{0\}$ for some positive integer n. It is well-known (see [1, Theorem 11.2], for example) that $nG = \{0\}$ implies that G is a direct sum of cyclic groups. Hence we have:

$$G = \bigoplus_{i \in I} G_i,$$

where each G_i is cyclic of prime power order $p_i^{\lambda_i}$. For each $i \in I$, let $\pi_i : G \to G$ be the projection map onto the ith coordinate. Consider an arbitrary nonzero sequence $m := (m_i : i \in I) \in G$ (each m_i is an integer taken mod $p_i^{\lambda_i}$). We must produce a nonzero $\alpha \in \mathrm{End}_{\mathbb{Z}}(G)$ such that $\alpha(m) = 0$. We consider two cases.

Case 1. I is infinite. Choose $i_0 \in I - \mathrm{Supp}(m)$. Then $\pi_{i_0}(m) = 0$, yet π_{i_0} is not the zero map.

Case 2. I is finite. Then G is a finite abelian group. Thus we may express G as:

$$G = \bigoplus_{i=1}^{n} C(p_i^{\lambda_i})$$

where each $C(p_i^{\lambda_i})$ is the cyclic group of order $p_i^{\lambda_i}$. Suppose first that there is some m_i and some integer r such that $rm_i = 0 \bmod p_i^{\lambda_i}$, but $p_i^{\lambda_i} \nmid r$. Let $\varphi := r\pi_i$. Then note that $\varphi(m) = 0$, but as $p_i^{\lambda_i} \nmid r$, φ is not the zero map. We now suppose that for every m_i and for every integer r: if $rm_i = 0 \bmod p_i^{\lambda_i}$, then $p_i^{\lambda_i} \mid r$. But then each m_i is relatively prime to p_i. Since G is not cyclic, it follows that $p_i = p_j$ for some $i \neq j$. We may assume without loss of generality that $i = 1$, $j = 2$, and $\lambda_1 \leq \lambda_2$. Recall from above that m_1 is relatively prime to p_1 and m_2 is relatively prime to $p_2 = p_1$. Let r_1 and r_2 be such that $r_1 m_1 = 1 \bmod p_1^{\lambda_1}$ and $r_2 m_2 = 1 \bmod p_1^{\lambda_1}$. Define $\varphi : G \to G$ by $\varphi(g_1, g_2, \ldots, g_n) := (r_1 g_1 - r_2 g_2, 0, 0, 0, \ldots, 0)$ (again, each g_i is an integer taken modulo $p_i^{\lambda_i}$). Since $\lambda_1 \leq \lambda_2$, this map is well-defined and easily checked to be a homomorphism. One verifies at once that $\varphi(m) = 0$, yet φ is not the zero map. We have shown that G is torsion over $\mathrm{End}_{\mathbb{Z}}(G)$, and the proof is concluded. □

3. Preliminaries

In this section, we prove some results on direct products which we will need throughout the remainder of the paper. Before stating our first proposition, we recall the following Theorems from [7].

Lemma 3.1 ([7, Theorem 3]). *If* $\mathrm{FT}(R) = \kappa \geq \aleph_0$, *then* κ *is a regular cardinal, and there exists a strictly descending chain of nonzero ideals* $\mathcal{C} = \{I_\alpha : \alpha \in \kappa\}$ *such that* $\bigcap_{\alpha \in \kappa} I_\alpha = \{0\}$.

Lemma 3.2 ([7, Theorem 2]). *Let* $\mathcal{S} := \{I_j : j \in X\}$ *be a family of nonzero proper left ideals of the ring* R *such that the following hold*:

(1) *Either all the* I_j *are two-sided ideals and* $\mathcal{S}$ *has the finite intersection property* (*that is*, *any intersection of finitely many members of* $\mathcal{S}$ *is nonzero*), *or all the* I_j *are essential left ideals.*
(2) $\bigcap_{j \in X} I_j = \{0\}$.

Then $M := \bigoplus_{j \in X} R/I_j$ *is a faithful torsion* R*-module. Thus* $0 < \mathrm{FT}(R) \leq |X|$.

We now establish several results on the FT rank of direct products of rings.

Proposition 3.3. *Suppose* $\{R_i : i \in X\}$ *be a nonvoid collection of rings*, *and let* $R := \prod_{i \in X} R_i$.

(a) *If* $j \in X$ *and* R_j *is* FT, *then* R *is* FT *and* $\mathrm{FT}(R) \leq \mathrm{FT}(R_j)$.
(b) *Suppose that* X *is finite and* $\mathrm{FT}(R) \geq \aleph_0$. *Then there exists* $k \in X$ *such that* $\mathrm{FT}(R_k) = \mathrm{FT}(R)$.
(c) *If* X *is infinite*, *then* R *is* FT, *and* $\mathrm{FT}(R) \leq \aleph_0$.

Proof. Let $R := \prod_{i \in X} R_i$.

(a) Assume that $j \in X$ and that $\mathrm{FT}(R_j) := \kappa > 0$. Let M_j be a κ-generated faithful torsion module over R_j. Now consider the product $\prod_{i \in X} N_i$, where $N_j = M_j$ and $N_i = R_i$ for $i \neq j$. It is straightforward to verify that $\prod_{i \in X} N_i$ is a faithful torsion module over R which can be generated by at most κ elements.
(b) Suppose now that $X = \{1, 2, \ldots, n\}$ and $\mathrm{FT}(R) := \kappa \geq \aleph_0$. Let $\pi_i : R \to R_i$ denote the canonical ring projections. Lemma 3.1 yields the existence of a strictly decreasing chain of nonzero ideals $\mathcal{C} = \{I_\alpha : \alpha \in \kappa\}$ of R with trivial intersection. Clearly for each i, $\mathcal{C}_i = \{\pi_i(I_\alpha) : \alpha \in \kappa\}$ is a decreasing chain of ideals of R_i with trivial intersection. We now claim that for some k, $\{0\} \notin \mathcal{C}_k$. If $\{0\} \in \mathcal{C}_i$ for all i, then there exists $\alpha \in \kappa$ such that $\pi_i(I_\alpha) = \{0\}$ simultaneously for all $i \in X$. But then $I_\alpha = \prod_{1 \leq i \leq n} \pi_i(I_\alpha) = \{0\}$, contradicting that every member of $\mathcal{C}$ is nonzero. We may now invoke Lemma 3.2 to conclude that $0 < \mathrm{FT}(R_k) \leq |\mathcal{C}_k| \leq \kappa = \mathrm{FT}(R)$. Part (a) implies that $\mathrm{FT}(R) \leq \mathrm{FT}(R_k)$, and hence $\mathrm{FT}(R_k) = \mathrm{FT}(R)$.
(c) First suppose that $|X| = \aleph_0$. In this case, we may assume that $X = \aleph_0$. Note that the sequence $\{I_n\}$ defined by $I_n := \prod_{i=n}^{\aleph_0} R_i$ is a chain of nonzero ideals of R, and so it has the finite intersection property. Moreover, $\bigcap_{n=1}^{\aleph_0} I_n = \{0\}$. We conclude from Lemma 3.2 that $0 < \mathrm{FT}(R) \leq \aleph_0$. Suppose now that $|X| > \aleph_0$, and let Y be a countably infinite subset of X. Note that $R \cong (\prod_{y \in Y} R_y) \times (\prod_{x \in X - Y} R_x)$. We just showed that $\prod_{y \in Y} R_y$ has nonzero, countable FT rank. Part (a) implies that R too has nonzero, countable FT rank. □

Remark 3.4. In Example 2.1, we showed that a division ring is non-FT. Combining this result with part (c) shows that the converse to part (a) fails. In particular, an infinite direct product of division rings is FT, yet every division ring is non-FT.

We now proceed to prove several consequences of the previous proposition. We will need two more lemmas.

Lemma 3.5 ([7, Proposition 3]). *Let R be a simple ring. Then* $\mathrm{FT}(R) \leq 1$. *Moreover,* $\mathrm{FT}(R) = 0$ *if and only if R is a division ring.*

Lemma 3.6 ([7, Corollary 3]). *Let R be a semisimple Artinian ring. Then* $\mathrm{FT}(R) \leq 1$. *Moreover,* $\mathrm{FT}(R) = 0$ *if and only if R is reduced* (*that is, if and only if R is a finite product of division rings*).

Corollary 3.7. *Suppose that $R_1, R_2, \ldots, R_n$ are simple rings, and let $R := R_1 \times R_2 \times \cdots \times R_n$. Then* $\mathrm{FT}(R) \leq 1$. *Moreover,* $\mathrm{FT}(R) = 0$ *if and only if each R_i is a division ring.*

Proof. Assume that each R_i is a simple ring, and let $R := R_1 \times R_2 \times \cdots \times R_n$. Suppose first that some R_i is not a division ring. In this case, Lemma 3.5 and (a) of Proposition 3.3 imply that $\mathrm{FT}(R) = 1$. Now suppose that each R_i is a division ring. Then R is a reduced semisimple Artinian ring. We conclude from Lemma 3.6 that $\mathrm{FT}(R) = 0$. □

Recall that the *Brown–McCoy radical* of a ring R is defined to be the intersection of all two-sided maximal ideals of R. One defines R to be *Brown–McCoy semisimple* provided the Brown–McCoy radical of R is trivial. We now characterize the Brown–McCoy semisimple FT rings. In what follows, $\mathrm{Max}(R)$ denotes the collection of maximal ideals of R.

Proposition 3.8. *Suppose that R is a Brown–McCoy semisimple ring. Then R is FT if and only if R is not a finite product of division rings. In any case,* $\mathrm{FT}(R) \leq |\mathrm{Max}(R)|$.

Proof. Assume that R is a Brown–McCoy semisimple ring. If R is FT, then it follows from Lemma 3.6 that R is not a finite product of division rings.

Conversely, assume that R is not a finite product of division rings. We will show that R is FT and that $\mathrm{FT}(R) \leq |\mathrm{Max}(R)|$. Suppose first that $\{0\} \in \mathrm{Max}(R)$. Then R is a simple ring which is not a division ring. Lemma 3.5 implies that $\mathrm{FT}(R) = 1 = |\mathrm{Max}(R)|$. We now assume that $\{0\} \notin \mathrm{Max}(R)$, and consider two cases.

Case 1. $\mathrm{Max}(R)$ has the finite intersection property. We invoke Lemma 3.2 to conclude that R is FT with $\mathrm{FT}(R) \leq |\mathrm{Max}(R)|$.

Case 2. There exist ideals $M_1, M_2, \ldots, M_k \in \mathrm{Max}(R)$ such that $M_1 \cap M_2 \cdots \cap M_k = \{0\}$. The Chinese Remainder Theorem implies that $R \cong R/M_1 \times R/M_2 \times \cdots \times R/M_k$.

In particular, R is a finite product of simple rings, but is not a finite product of division rings (by our above assumption). It follows from Corollary 3.7 that $\mathrm{FT}(R) = 1$ and the proof is complete. $\square$

Recall that R is *Jacobson semisimple* (or *semiprimitive*) if $\mathrm{rad}(R) = \{0\}$. Under various conditions, the Brown–McCoy radical can be made to coincide with $\mathrm{rad}(R)$. A ring is called *left* (*right*) *duo* if all of its left (right) ideals are two-sided, *duo* if all its one-sided ideals are two-sided, and *left* (*right*) *quasi-duo* if its maximal left (right) ideals are two-sided. The following corollary follows immediately from Proposition 3.8.

Corollary 3.9. *If R is Jacobson semisimple and is right or left quasi-duo, then R is FT if and only if R is not a finite product of division rings. Further,* $\mathrm{FT}(R) \leq |\mathrm{Max}(R)|$.

4. Serial Rings

We begin this section by recalling some terminology. An R-module M is *uniserial* provided its submodules are linearly ordered by inclusion. M is called *serial* if and only if M is a direct sum of uniserial modules. A ring R is a *uniserial ring* if and only if R is right and left uniserial as a module over itself. R is a *serial ring* if and only if R is both right and left serial as a module over itself. A *valuation ring* is a commutative uniserial domain. We refer the reader to Puninski [8] for a treatment of serial rings and to Gilmer [2] for a thorough development of the theory of valuation rings.

If R is right or left serial, then its decomposition into uniserial modules also witnesses that R is *semiperfect*, that is, there exist idempotents $\{e_i : 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} e_i = 1$ and e_iRe_i is a local ring for every i. It is well-known that semiperfect rings are *semilocal*, that is, $R/\mathrm{rad}(R)$ is Artinian. We now arrive at our first proposition.

Proposition 4.1. *Let R be a duo, semilocal, Noetherian ring which is not Artinian. Then R is* FT. *Moreover,* $0 < \mathrm{FT}(R) \leq \aleph_0$.

Proof. Being duo and Noetherian on both sides, [4, Corollary 2] gives us that $\bigcap_{i=1}^{\infty}(\mathrm{rad}(R))^i = \{0\}$. If $\mathrm{rad}(R)$ were nilpotent, then R would be semiprimary, but in that case the Hopkins–Levitzki Theorem states that R must also be Artinian. Since we have hypothesized that this is not the case, the powers of $\mathrm{rad}(R)$ form an infinite, strictly descending chain of nonzero ideals. Since they have trivial intersection, Lemma 3.2 implies that $0 < \mathrm{FT}(R) \leq \aleph_0$. $\square$

We pause to recall a theorem ([8, Theorem 2.3]) which will be of use to us shortly.

Lemma 4.2 (Drozd–Warfield Theorem). *Every finitely presented module M over a serial ring is serial; more specifically, M is a direct sum of local uniserial (hence cyclic) modules.*

We now restate the following result (mentioned in the introduction) and prove a lemma about finite direct sums of cyclic modules over left duo rings. With these in hand, we then establish a converse to Proposition 4.1.

Lemma 4.3 ([7, Theorem 4]). *If R is a left or right Artinian ring, then R has finite* FT *rank.*

Lemma 4.4. *Let R be a left duo ring. If M is an R-module which is a finite direct sum of cyclic modules, then M is not a faithful torsion module over R.*

Proof. Suppose that R is left duo, and let $M := \bigoplus_{i=1}^{n} Rm_i$ be a direct sum of cyclic R-modules. Assume that M is torsion. We will show that M is not faithful. Since M is torsion, there exists some nonzero $r \in R$ such that $rm_1 + rm_2 + \cdots + rm_n = 0$. Since the sum is direct, we conclude that $rm_i = 0$ for each i. Hence $r \in \bigcap_{i=1}^{n} \operatorname{ann}(m_i)$. Since R is left duo, it follows that $\operatorname{ann}(m_i) = \operatorname{ann}(Rm_i)$ for each i. Thus $r \in \bigcap_{i=1}^{n} \operatorname{ann}(Rm_i)$, and we see that $rM = \{0\}$. This shows that M is not faithful. □

We now prove the main result of this section.

Theorem 4.5. *Let R be a duo serial ring. Then:*

(a) *If R is FT, then R is not Artinian. The converse fails.*
(b) *If R is Noetherian, then R is FT if and only if R is not Artinian. In this case, $FT(R) = \aleph_0$.*

Proof. We assume that R is a duo serial ring.

(a) Suppose by way of contradiction that R is FT and Artinian. Then Lemma 4.3 implies that $0 < \mathrm{FT}(R) < \aleph_0$. Let M be a finitely generated faithful torsion module over R. Since R is Noetherian, M is finitely presented. It follows from Lemma 4.2 that M is a finite direct sum of cyclic modules. However, this contradicts Lemma 4.4.

To see that the converse fails, we give an example of a commutative uniserial ring that is non-Artinian and non-FT. Let v be a valuation on a field F with value group $\mathbb{R}$, and let V be the associated valuation ring. Let $I := \{x \in V : v(x) > 2\}$ and $J := \{x \in V : v(x) \geq 2\}$. Clearly $I \subsetneq J$ and there are no ideals properly between I and J. It follows that J/I is the minimum ideal of the uniserial ring V/I. Thus if M is any torsion module over V/I, then $J/I \subseteq \operatorname{ann}(M)$, and it follows that M cannot be faithful. Hence V/I is non-FT. We claim that V/I is not Artinian. It suffices to show that V/I is not even Noetherian. Suppose by way of contradiction that V/I is Noetherian. Since V/I is a uniserial ring, V/I is a principal ideal ring. Hence every ideal L of V containing I can be expressed as $L = (I, x)$ for some $x \in V$. Let $K := \{x \in V : v(x) > 1\}$ (note that $I \subsetneq K$). However, there is no $y \in V$ such that $K = (I, y)$. Suppose there is such a y. Then of course, $y \notin I$. Since V is a valuation ring, we conclude that $I \subseteq Vy$. But then $K = Vy$, and so K is principal. However, clearly K is not principal (it is not even finitely generated). This contradiction shows that V/I is not Noetherian, hence not Artinian.

(b) Suppose now that R is Noetherian. Assume in addition that R is not Artinian. Then we conclude from Proposition 4.1 (recalling that a serial ring is semilocal) that $0 < \mathrm{FT}(R) \leq \aleph_0$. We have seen that Lemmas 4.4 and 4.2 preclude the existence of a finitely generated faithful torsion module, and so $\mathrm{FT}(R) = \aleph_0$. □

5. Left Nonsingular Rings

Recall that a ring R is *left nonsingular* if the *left singular ideal* $\mathrm{Sing}({}_RR) := \{r \in R : \mathrm{ann}_l(r) \leq_e {}_RR\} = \{0\}$. The left nonsingular rings encompass many well-studied and important classes of rings including von Neumann regular rings, reduced rings, left semihereditary rings, and left Rickart rings. We will show in Theorem 5.9 that "most" left nonsingular rings are FT. In fact, the only ones which are fail to be FT are the finite products of division rings. We begin with two lemmas.

Lemma 5.1. *Let $R \subseteq S$ be a unitary ring extension such that ${}_RR \subseteq_e {}_RS$ as left R-modules. If S is* FT, *then so is R.*

Proof. We assume that ${}_RR \subseteq_e {}_RS$ and that S is FT. We will show that R is FT as well. Since S is FT, S admits a faithful torsion module ${}_SM$. Clearly ${}_RM$ is faithful with the action that M inherits from R since $R \subseteq S$. We now show that ${}_RM$ is a torsion R-module. To see this, let $m \in M$ be arbitrary. Since ${}_SM$ is torsion, it follows that $\{0\} \neq \mathrm{ann}_S(m) \subseteq S$. By essentialness of ${}_RR$, we deduce that $R \cap \mathrm{ann}_S(m) \neq \{0\}$. Thus $\mathrm{ann}_R(m) \neq \{0\}$, and ${}_RM$ is torsion. □

Lemma 5.2 ([7, Proposition 1]). *Let D be a domain. Then D is* FT *if and only if D is not a division ring.*

We now make use of the *maximal left ring of quotients*, $Q^{\ell}_{\max}(R)$, of a ring R. We recall that ${}_RR$ is an essential submodule of ${}_RQ^{\ell}_{\max}(R)$, and refer the reader to [6, Chap. 5] for a thorough development of the theory of maximal rings of quotients.

Proposition 5.3. *Let R be a ring and let $Q := Q^{\ell}_{\max}(R)$ be the maximal left ring of quotients of R. Suppose that Q is a finite direct product of division rings. Then R is* FT *if and only if R is not a finite product of division rings.*

Proof. Let R and Q be as stated, and assume that Q is a finite product of division rings. If R is a finite direct product of division rings, then Lemma 3.6 implies that R is non-FT. Thus we assume that R is not a finite product of division rings.

Since R is not a finite product of division rings, $R \subsetneq Q$. We may write $Q = \prod_{i=1}^{n} e_iQe_i$, where the e_i's are mutually orthogonal central idempotents of Q such that each e_iQe_i is a division ring with identity e_i and $\Sigma_{i=1}^n e_i = 1$. Note that $R \subseteq R' := \prod_{i=1}^n e_iRe_i$ since $1 \in R'$ (if $r \in R$, then $r = r \cdot 1 = r(e_1 + e_2 + \cdots + e_n) = re_1 + re_2 + \cdots + re_n = e_1re_1 + e_2re_2 + \cdots + e_nre_n$).

We now find an FT ring S such that $R \subseteq S \subsetneq Q$. First, we claim that for some k, $e_kRe_k \subsetneq e_kQe_k$. Suppose to the contrary that for all i, $e_iRe_i = e_iQe_i$. Let $T_i := e_iRe_i \cap R$. It is clear that each e_iRe_i is a nonzero R-submodule of Q, and since ${}_RR \subseteq_e {}_RQ$, all the T_i are nonzero. Let i be arbitrary and $0 \neq e_ire_i \in T_i$. Recall that $e_iRe_i = e_iQe_i$. Thus e_iRe_i is a division ring with identity e_i. Hence e_ire_i has an inverse $e_ir'e_i$ for some $r' \in R$. Thus $e_i = (e_ir'e_i)(e_ire_i) = e_ir're_i = r'(e_ire_i) \in T_i$. But then $e_i \in T_i \subseteq R$ for every i. This implies that $e_iRe_i \subseteq R$ for every i. Since we have assumed that each $e_iRe_i = e_iQe_i$, we conclude that $R = Q$. This is a contradiction.

So, there exists k such that $e_kRe_k \subsetneq e_kQe_k$. Let $S := \prod_{i=1}^n S_i$, where $S_k = e_kRe_k$ and $S_j = e_jQe_j$ for all $j \neq k$. Since $e_kRe_k \subsetneq e_kQe_k$, it follows that $S \neq Q$. We claim that $S_k = e_kRe_k$ is a domain but not a division ring. Note that e_kRe_k is a subring of e_kQe_k. Since e_kQe_k is a division ring, it follows that e_kRe_k is a domain. Suppose by way of contradiction that e_kRe_k is a division ring. Then S is a finite product of division rings. Moreover, S is left self-injective and contains R as an essential submodule. But then by [6, Proposition 13.39, p. 378], $S = Q$, and we have reached a contradiction. Thus e_kRe_k is a domain which is not a division ring. We invoke Lemma 5.2 to conclude that e_kRe_k is FT. Proposition 3.3 now implies that S is FT. Since ${}_RR \subseteq_e {}_RS$, it follows from Lemma 5.1 that R is FT. □

We now recall some terminology and establish several lemmas.

Definition 5.4. A ring R is a left full linear ring if and only if R is of the form $\mathrm{End}({}_DV)$ for some vector space V over a division ring D.

Lemma 5.5. *A left full linear ring R is* FT *if and only if R is not a division ring.*

Proof. Let R be a left full linear ring. If R is a division ring, then as we have noted several times, R is non-FT. Conversely, assume that R is not a division ring. By definition, $R = \mathrm{End}({}_DV)$ for some vector space V over a division ring D. Note that V is faithful over R; to complete the proof, it suffices to show that V is torsion over R. Let $0 \neq v \in V$ be arbitrary. We will find a nonzero $\overline{\varphi} \in R$ such that $\overline{\varphi}(v) = 0$. Since R is not a division ring, $\mathrm{Dim}({}_DV) > 1$. Extend $\{v\}$ to a basis $\beta := \{v\} \cup \{w_i : i \in I\}$ for V over D. Now define φ on β by $\varphi(v) = 0$ and $\varphi(w_i) = w_i$ for each $i \in I$. Extend φ (by linearity) to an endomorphism $\overline{\varphi} : V \to V$. Note that by construction, $\overline{\varphi}(v) = 0$, yet $\overline{\varphi}$ is not the zero map. This completes the proof. □

Lemma 5.6. *A ring R is a left full linear ring if and only if R is a prime, von Neumann regular, left self-injective ring with* $\mathrm{soc}({}_RR) \neq \{0\}$. *Thus a prime,* *von Neumann regular, left self-injective ring R is* FT *if and only if R is not a division ring.*

Proof. The first statement is a fact found in many sources; one such source is [6, p. 380]. Suppose now that R is prime, von Neumann regular, and left-self injective but is not a division ring. We will show that R is FT. To do this, we distinguish two cases.

Case 1. $\mathrm{soc}({}_R R) = \{0\}$. Then R is FT by Lemma 3.2 (since the left socle is the intersection of all essential left ideals of R).

Case 2. $\mathrm{soc}({}_R R) \neq \{0\}$. Then R is a left full linear ring. Lemma 5.5 implies that R is FT, and the proof is complete. □

Let $B(R)$ denote the set of central idempotents of a ring R. It is well-known that for a left self-injective ring R, $B(R)$ forms a complete Boolean algebra which is a lattice under the partial ordering defined by $e \leq f \Leftrightarrow Re \subseteq Rf$. We remind the reader that an *atom* of a lattice $\mathcal{L}$ is an element that is minimal among the set of nonzero elements. The lattice $\mathcal{L}$ is *atomic* provided that for every $x > 0$, there exists some atom a such that $x \geq a > 0$. The next lemma relates these concepts to regular, self-injective rings. The proof of part (a) can be found in [3, Proposition 9.9], and part (b) in [3, Corollary 9.11].

Lemma 5.7. *Suppose that R is a regular, left self-injective ring. Then the following hold*:

(a) *$B(R)$ is a complete Boolean algebra. Moreover, for any $\{e_i\} \subseteq B(R)$, we have $R(\inf\{e_i\}) = \bigcap(Re_i)$.*
(b) *R is a direct product of prime rings if and only if $B(R)$ is an atomic lattice.*

We establish a final proposition and then prove the main result of this section.

Proposition 5.8. *If R is a regular, left self-injective ring, then R is FT if and only if R is not a finite product of division rings.*

Proof. As usual, a finite product of division rings is non-FT, thus we assume that the regular, left self-injective ring R is not such a product. Again, we distinguish two cases.

Case 1. $B(R)$ is atomic. Then Lemma 5.7 implies that R factors into a product of prime, regular, left self-injective rings. If this product is infinite, then R is FT by (c) of Proposition 3.3. If the product is finite, at least one factor is not a division ring. By Lemma 5.6, this factor ring is FT, and hence R is FT by (a) of Proposition 3.3.

Case 2. $B(R)$ is not atomic. Then there must be $0 \neq e_0 \in B(R)$ such that there is no atom $a \in B(R)$ with $a \leq e_0$. Invoking The Hausdorff Maximal Principle, select a maximal linearly ordered subset L of $B(R)$ containing e_0. It is clear that $0 \in L$, and we now consider $L' := L \backslash \{0\}$. We claim that $\bigcap\{Re : e \in L'\} = \{0\}$.

Suppose by way of contradiction that $\bigcap\{Re : e \in L'\} \neq \{0\}$. By (a) of Lemma 5.7, there exists some $0 \neq f \in B(R)$ such that $Rf = \bigcap\{Re : e \in L'\}$. In particular, $f \leq e$ for all $e \in L'$. Since $e_0 \in L'$ and no atom lies below e_0, we conclude that f is not an atom. Hence there exists some $f' \in B(R)$ such that $0 < f' < f$. But then $L \subsetneq L \cup \{f'\}$ is linearly ordered, and this contradicts the maximality of L. We conclude that $\{Re : e \in L'\}$ is a chain of nonzero ideals of R with trivial intersection (the ideals are two-sided since each $e \in L'$ is central). Lemma 3.2 establishes that R is FT. □

Finally, the main result follows.

Theorem 5.9. *A left nonsingular ring R is* FT *if and only if R is not a finite product of division rings.*

Proof. We assume R is left nonsingular which is not a finite product of division rings, and we will show that R is FT. Let $Q = Q^{\ell}_{\max}(R)$. Johnson's Theorem (see [6, p. 376]) tells us that $Q^{\ell}_{\max}(R)$ is a left self-injective von Neumann regular ring. If Q is not a finite product of division rings, then Proposition 5.8 implies that Q is FT. Since R is essential in Q, we infer from Lemma 5.1 that R is FT. If Q is a finite product of division rings, then Proposition 5.3 implies that R is FT, and the proof is complete. □

6. Open Problems

We end the paper with several questions which we feel are interesting.

Question 6.1. Can one classify the Jacobson semisimple FT rings?

Question 6.2. Is there a Noetherian ring R with $\mathrm{FT}(R) > \aleph_0$?

Question 6.3. Are there rings of finite FT rank $n > 1$? If so, can a theory be developed for rings of finite FT rank?

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