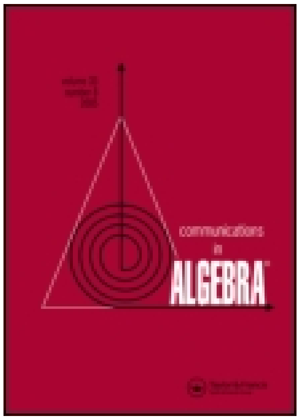

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RINGS WHICH ADMIT FAITHFUL TORSION MODULES

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Let R be a ring with identity, and let M be a unitary (left) R-module. Then M is said to be torsion provided that for every $m \in M$, there is a nonzero $r \in R$ such that $rm = 0$. In this article, we study the question of the existence (and nonexistence) of faithful torsion modules over both commutative and noncommutative rings.

Key Words: Cofinality; Faithful module; FPF ring; Quasi-Frobenius ring; Regular cardinal; Semiperfect ring; Torsion module; Valuation ring.

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1. INTRODUCTION

In this article, all rings are assumed to have an identity and all modules are left unitary modules unless stated otherwise. Further, 'ideal' means 'two-sided ideal.'

The main question we explore in this article is that of which rings admit faithful torsion modules. This is a natural question since these modules are, in a sense, 'locally annihilated' but not 'globally annihilated.' As is customary, we will say that a left module M over a ring R is *faithful* if $ann(M) := \{r \in R : rM = 0\} = \{0\}$. However, it is clear that individual elements of a faithful module may be killed independently. In fact, we are interested in faithful left modules where every element can be so killed nontrivially. Various notions in the literature refer to this property and the word 'torsion' has been used to denote several degrees of this phenomenon, typically in reference to the annihilator, $ann(m)$, of an element $m \in M$. Let $m \in M$ be arbitrary. Some authors simply require $ann(m)$ to be nonzero (Atiyah [1], Hungerford [4]), while others require $ann(m)$ to contain a regular element of R (Lam [9], Goodearl [7]). Still others require $ann(m)$ to be an essential left ideal of R (Faith [3]).

Here we settle for the most general of the conditions, and we say that a module M is *torsion* if and only if for every $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$. We caution the reader that our definition of torsion does not associate with a corresponding torsion theory. In particular, our class of torsion modules is not closed under direct sums (see Golan [6] for an introduction to torsion theories). We remark that the primary focal point of this article is local annihilation of modules

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which is not global, and the least restrictive definition of torsion is best-suited to our purposes.

The outline of the article is as follows. After developing some preliminary results, we introduce the concept of faithful torsion (FT) rank of a ring R. Recall that if M is an R-module and $X \subseteq M$, then X is a *generating set* for M provided $\langle X\rangle = M$; that is, if the submodule of M generated by X is M itself. The FT rank of R is then defined by:

$$\mathrm{FT}(R) := \min\{|X| : X \text{ is a generating set for some faithful torsion } R\text{-module}\},$$

and $FT(R) := 0$ by convention if R does not admit a faithful torsion module. Our main objective in the article is to study this rank function. Among other results, we prove that if R is (left or right) Artinian, then R has finite FT rank, and if R is commutative Noetherian, then R has countable FT rank. We also completely determine the rank function for several classes of rings, and we show that for any regular cardinal κ, there exists a ring of FT rank κ. We close the article with some open questions.

2. PRELIMINARIES

As stated in the introduction, we will be interested in determining which rings do and do not admit faithful torsion modules. Before pursuing this question in general, we answer the following (much easier) subquestions.

Question 1. Which rings admit a faithful module?

The solution is easy: Every ring admits a faithful module. Indeed, if R is any ring, then R is faithful as a module over itself since R has an identity.

Now we ask a harder question.

Question 2. Which rings admit a nonzero torsion module?

Our solution requires several lemmas. The proofs are straightforward; as such, we omit them.

Lemma 1. *Let R be a ring, M an R-module, N a submodule of M, and I a proper nonzero two-sided ideal of R. Then*:

(a) *R/I is a torsion R-module*;
(b) *If N is an essential submodule of M, then M/N is a torsion R-module.*

Lemma 2. *Let M be an R-module, and let N be a submodule of M. If N is not a direct summand of M, then N is contained in a proper essential submodule of M.*

We will need one more lemma before answering Question 2 in general. We first recall that a nonzero element x in a ring R is said to be a *right zero divisor* if and only if $ax = 0$ for some nonzero $a \in R$. Motivated by commutative terminology (and avoiding a conflict with the noncommutative definition of a regular ideal), we define a left ideal I of R to be *r-regular* if and only if I contains some element which is *not* a right zero divisor.

Lemma 3. *Let R be a ring, and let I be a left ideal of R. Then R/I is torsion if and only if I intersects every r-regular left ideal of R nontrivially.*

We now dispose of Question 2.

Theorem 1. *A ring R admits a nonzero torsion module if and only if R is not a division ring. Moreover, if R is not a division ring, then R admits a nonzero cyclic torsion module.*

Proof. Suppose first that R is a division ring. Then as every R-module is torsion-free, clearly R does not admit a nonzero torsion module. Suppose now that R is not a division ring. We will show that R admits a nonzero cyclic torsion module. We consider two cases.

Case 1: R possesses a left ideal I which is not a direct summand of R. Lemma 2 implies that $I \subseteq I'$ for some proper essential left ideal I' of R. It now follows from Lemma 1 that R/I' is a torsion R-module.

Case 2: Every left ideal I of R is a direct summand of R. Then R is a semisimple Artinian ring. The Wedderburn–Artin Theorem implies that $R \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r)$ for some division rings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$. Since R is not a division ring, it follows that either $r > 1$ or $n_1 > 1$. In either case, R possesses a nonzero right zero divisor. Moreover, every element of R is either a right zero divisor or a unit. This implies that every r-regular left ideal of R contains a unit and thus coincides with R. Let $x \in R$ be a nonzero right zero divisor. Then Rx is a proper left ideal of R which intersects every r-regular left ideal of R nontrivially. Lemma 3 implies that R/Rx is torsion, and the proof is complete. □

For the remainder of the article, we focus our attention on the much harder 'intersection' of Questions 1 and 2.

Question 3. Which rings admit a faithful torsion module, and which rings do not?

It is this question which we pursue for the remainder of the article. To simplify terminology, let us agree to call a ring R which admits a faithful torsion module an *FT ring*. If R does not admit a faithful torsion module, we will call R a *non-FT ring*.

3. FUNDAMENTAL RESULTS

We begin with several examples to initiate the reader.

Example 1. If D is a division ring, then D is non-FT.

Proof. Immediate from Theorem 1. □

Example 2. Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Then R is FT.

Proof. Let R be the ring (under pointwise addition and multiplication) of real-valued continuous functions defined on $\mathbb{R}$. For every positive integer n, let

$f_n : \mathbb{R} \to \mathbb{R}$ be a continuous function with zero set $[-n, n]$. Now define $g_n := f_1 f_2 \cdots f_n$, and let $\mathcal{I} := \{(g_n) : n \in \mathbb{Z}^+\}$ (thus $\mathcal{I}$ is the collection of the principal ideals generated by the g_n). It is easily checked that $\bigoplus_{n>0} R/(g_n)$ is a faithful torsion module over R. $\square$

Example 3. Suppose D is a commutative domain which is not a field, and let K be the quotient field of D. Then K/D is a faithful torsion D-module. Thus D is an FT ring.

Proof. Let D be a commutative domain which is not a field with quotient field K. It is clear that K/D is torsion. Suppose by way of contradiction that $d \in D - \{0\}$ and d annihilates K/D. Then in particular, $d \cdot \frac{1}{d^2} \in D$, and hence d is a unit of D. But as d annihilates K/D and d is a unit of D, we see that $K \subseteq d^{-1}D \subseteq D \subseteq K$. Hence $D = K$, and D is a field, a contradiction. $\square$

We will shortly describe a large class of FT rings. First, we establish a lemma.

Lemma 4. *Let R be a ring, and let $\{M_i : i \in I\}$ be a collection of R-modules. Suppose that for each i, $N_i \leq_e M_i$. Then $\bigoplus_{i\in I} M_i/N_i$ is a torsion R-module.*

Proof. Assume that R is a ring, $\{M_i : i \in I\}$ is a collection of R-modules, and that for each i, $N_i \leq_e M_i$. Then $\bigoplus_{i\in I} N_i \leq_e \bigoplus_{i\in I} M_i$ (see p. 76 of [9], for example). It now follows from Lemma 1 that $(\bigoplus_{i\in I} M_i)/(\bigoplus_{i\in I} N_i)$ is torsion. Noting that $(\bigoplus_{i\in I} M_i)/(\bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I}(M_i/N_i)$, we obtain the desired result. $\square$

Our next result incorporates the following terminology. Say that a family $\mathcal{S} := \{I_j : j \in X\}$ of ideals of a ring R has *the finite intersection property* (FIP) if and only if every finite intersection of ideals of $\mathcal{S}$ is nonzero.

Theorem 2. *Let $\mathcal{S} := \{I_j : j \in X\}$ be a family of nonzero proper left ideals of the ring R such that the following hold*:

(1) *Either all the I_j are two-sided ideals and $\mathcal{S}$ has the finite intersection property, or all the I_j are essential left ideals*;
(2) $\bigcap_{j\in X} I_j = \{0\}$.

Then $M := \bigoplus_{j\in X} R/I_j$ is a faithful torsion R-module (*note further that conditions* (1) *and* (2) *imply that $\mathcal{S}$ is infinite*).

Proof. Assume that $\mathcal{S} := \{I_j : j \in X\}$ is a family of nonzero proper left ideals of the ring R which satisfy (1) and (2) above.

We first claim that M is faithful. For suppose that $r \in R$ and $rM = \{0\}$. Let $j \in X$ be arbitrary. Then $r \cdot 1 = 0 \pmod{I_j}$, and hence $r \in I_j$. Since j was arbitrary, we obtain $r \in \bigcap_{j\in X} I_j = \{0\}$, and M is faithful.

We now show that M is torsion. Suppose first that the ideals are all two sided and $\mathcal{S}$ has the finite intersection property. Consider an arbitrary nonzero element $m \in M$. Let $\{j_1, j_2, \ldots, j_k\}$ be the support of m. By assumption, $J_{j_1} \cap J_{j_2} \cdots \cap J_{j_k} \neq \{0\}$. It is readily checked that any element of this intersection kills m, and so M is torsion in this case. Now suppose that the ideals are all left essential. We invoke Lemma 4 to conclude that M is torsion, and the proof is complete. □

We will use the previous theorem to characterize the FT domains. Recall that the *left socle* of a ring R is the sum of all minimal left ideals of R. We first prove a lemma yielding a sufficient (though not necessary) condition for a ring R to be FT.

Lemma 5. *Let R be a ring. If the left socle of R is trivial, then R is FT.*

Proof. Let R be a ring, and suppose the left socle of R is trivial. Since $soc(R) = \bigcap\{I : I \leq_e R\}$ (see [9, p. 242]), it follows that the intersection of the essential left ideals of R is trivial. Theorem 2 implies that R is FT. □

Proposition 1. *Let D be a domain. Then D is FT if and only if D is not a division ring.*

Proof. We have already seen that a division ring is not FT. Conversely, suppose that D is a domain which is not a division ring. By the previous lemma, it suffices to show that the left socle of D is trivial. Suppose by way of contradiction that I is a minimal left ideal of D. Let $x \in I$ be nonzero. By minimality of I and the assumption that D is a domain, it follows that $Dx = Dx^2 = I$. Hence $x = dx^2$ for some $d \in D$. Since D is a domain, we obtain $dx = 1$. But then $I = D$, and D is a division ring, a contradiction. □

We now introduce an important definition which will assume the central role throughout the remainder of the article.

Definition 1. Let R be a ring. Define the FT rank of R, $FT(R)$, to be the least cardinal number κ such that R admits a faithful torsion module which can be generated by κ many elements. In case R is non-FT, we define $FT(R) := 0$.

Remark 1. We caution the reader that when making statements such as 'R has finite FT rank' or 'R has countable FT rank,' we do not preclude the possibility that $FT(R) = 0$.

Using the notion of FT rank, we proceed to prove a sort of converse to Theorem 2. Moreover, our next result gives an important relationship between the FT rank of a ring R and the ideal structure of R (we remark that the notion of cofinality of a cardinal appearing in the following theorem is defined more generally in Section 6).

Theorem 3. *Let R be a ring, and suppose that R has infinite FT rank κ. Then the following hold:*

(a) *R admits a faithful torsion module which is a direct sum of κ cyclic modules. Further, κ is a regular cardinal.*
(b) *There exists a collection $\mathcal{S} := \{J_i : i \in \kappa\}$ of two-sided ideals of R indexed by κ and satisfying:*

(i) $J_j \subsetneq J_i$ *for* $i < j$;
(ii) $\bigcap_{i\in\kappa} J_i = \{0\}$.

(c) $\kappa \leq |R|$.

Proof. Let R be a ring, and assume that R has infinite FT rank κ. Let M be a faithful torsion module over R with generating set $\{m_i : i < \kappa\}$. For each $j < \kappa$, let M_j denote the submodule of M generated by $\{m_i : i \leq j\}$.

(a) Let γ be the cofinality of κ, and let $(\alpha_i : i < \gamma)$ be a strictly increasing sequence of cardinals cofinal in κ. Define $\mathcal{S}_\gamma := \{ann(M_{\alpha_i}) : i < \gamma\}$. Note trivially that $\mathcal{S}_\gamma$ is a chain of nonzero two-sided ideals of R. Since M is faithful and $(\alpha_i : i < \gamma)$ is cofinal in κ, it is easy to see that $\bigcap \mathcal{S}_\gamma = \{0\}$. Theorem 2 implies that $\bigoplus_{i<\gamma} R/ann(M_{\alpha_i})$ is a faithful torsion module over R. This fact along with the minimality of κ implies that $\gamma = \kappa$. Hence κ is a regular cardinal.

(b) Simply take $\mathcal{S} := \{ann(M_i) : i < \kappa\}$. Note that it is possible for $ann(M_i) = ann(M_j)$ for $i \neq j$. Upon enumerating this set (i.e., discarding redundancies), an analogous argument to the proof of (a) shows that $|\mathcal{S}| = \kappa$. After reindexing, the result follows.

(c) Let $\{J_i : i < \kappa\}$ be an (inclusion reversing) enumeration of $\mathcal{S}$. For each $i < \kappa$, pick (by the axiom of choice) some element $x_i \in J_i - J_{i+1}$. This defines an injection from κ into R. □

4. FINITE FT RANK

In this section, we investigate some classes of rings with finite FT rank and open with the following theorem.

Theorem 4. *If R is a left or right Artinian ring, then R has finite FT rank.*

Proof. Let R be a ring, and suppose that R has infinite FT rank. Then Theorem 3 implies that R cannot be left or right Artinian. □

We now find necessary and sufficient conditions for a ring to have FT rank 1. Recall that if I is a left ideal of a ring R, then the *core* of I is the largest two-sided ideal of R contained in I. It is well-known (and easy to see) that if I is any left ideal of R, then $ann(R/I)$ is equal to the core of I.

Proposition 2. *Let R be a ring. Then $FT(R) = 1$ if and only if R possesses a left ideal I satisfying:*

(i) *The core of I is trivial*;
(ii) *I intersects every r-regular left ideal of R nontrivially.*

Proof. Suppose R is a ring and I is a left ideal of R satisfying (i) and (ii) above. Then Lemma 3 implies that R/I is torsion. It follows from (i) above and the preceding comments that $ann(R/I) = \{0\}$, and hence R/I is faithful. Conversely, if I is a left ideal of R such R/I is faithful torsion, then (i) and (ii) follow. □

Corollary 1. *Let R be a ring. Then*:

(a) *If R is left duo (that is, every left ideal of R is two-sided), then $FT(R) \neq 1$.*
(b) *If R is left Artinian, then $FT(R) = 1$ if and only if R possesses a minimal left ideal which is not two-sided.*

Proof. Let R be a ring. Suppose first that R is left duo. Then the core of any left ideal I is equal to I. Hence (i) and (ii) of Proposition 2 cannot possibly be satisfied. Now suppose that R is left Artinian. If I is a minimal left ideal of R which is not two-sided, then I has trivial core. Since R is left Artinian, every element of R which is not a right zero divisor is a unit (to see this, note that if $x \in R$ is not a right zero divisor, then consider the descending chain $\cdots Rx^n \subseteq Rx^{n-1} \subseteq \cdots Rx$. The fact that this chain stabilizes implies that x is left-invertible. As Artinian rings are Dedekind finite (one-sided inverses are two-sided), it follows that x is a unit). Hence I intersects every r-regular left ideal of R nontrivially. Proposition 2 implies that $FT(R) = 1$. Conversely, suppose by way of contradiction that $FT(R) = 1$, but every minimal left ideal of R is two-sided. Let I be a left ideal satisfying (i) and (ii) of Proposition 2. As R is left Artinian, I contains a minimal left ideal J. Since J is two-sided, it follows that the core of I is nontrivial, a contradiction. □

We easily determine the FT rank of a simple ring.

Proposition 3. *Let R be a simple ring. Then $FT(R) \leq 1$. Moreover, $FT(R) = 0$ if and only if R is a division ring.*

Proof. Suppose that R is a simple ring. If R is a division ring, then Theorem 1 shows that $FT(R) = 0$. Suppose now that R is not a division ring. It follows from Theorem 1 that R admits a nonzero cyclic torsion module M. The simplicity of R implies that $ann(M) = \{0\}$, and hence M is faithful. The proof is complete. □

We now study the FT rank of several subclasses of semiperfect rings. Recall that a ring R is *semiperfect* if and only if 1 can be decomposed into $e_1 + e_2 + \cdots + e_n$, where the e_is are mutually orthogonal local idempotents. A semiperfect ring R is *self-basic* if and only if $R/rad(R)$ is a finite direct product of division rings if and only if 1 is a basic idempotent of R. It is well-known that left and right Artinian rings are semiperfect. We proceed to determine the FT rank of a non-self-basic left Artinian ring.

Proposition 4. *Suppose that R is a left Artinian ring which is not self-basic. Then $FT(R) = 1$.*

Proof. Assume that R is a left Artinian ring which is not self-basic, and let e be a basic idempotent of R. It is easy to see that Re is faithful. To complete the proof, it suffices to show that Re is torsion. Since R is not self-basic, it follows from the above comments that 1 is not a basic idempotent of R. Hence e is not a unit. Since R is Dedekind finite and e is not a unit, we see that $Re \neq R$. As noted in the proof of Corollary 1, every element of R which is not a right zero divisor is a unit. Thus every proper left ideal of R consists entirely of right zero divisors. Since $Re \neq R$, it follows that Re is torsion. This completes the proof. □

We now consider FPF rings. Recall that a ring R is *finitely pseudo-Frobenius* (FPF) provided every finitely generated faithful (left) R-module is a generator for Mod R. We record the following facts and then prove a result on the FT rank of a semiperfect FPF ring.

Fact 1 (Page [11, Theorem 1.1]). *Let R be a semiperfect FPF ring and M be a finitely generated faithful (left) R-module. If e is any basic idempotent of R, then $M \cong Re \oplus X$ for some (left) R-module X.*

Fact 2 ([11, Corollary 1.2]). *If R is a self-basic semiperfect ring, then R is (left) FPF if and only if every faithful finitely generated module M is of the form $R \oplus X$ for some R-module X.*

Proposition 5. *Let R be a semiperfect FPF ring. Then $FT(R) = 0$, $FT(R) = 1$, or $FT(R) \geq \omega$.*

Proof. Let R be a semiperfect FPF ring. We assume that $0 < FT(R) := n < \omega$ and show that $n = 1$. Let M be an n-generated faithful torsion module over R, and let e be a basic idempotent of R. By Fact 1, $M \cong Re \oplus X$ for some left R-module X. As noted in the proof of Proposition 4, Re is faithful. But since $Re \oplus X$ is torsion, it follows that Re is also torsion. Hence Re is a faithful torsion R-module, and so $FT(R) = 1$. $\square$

Using these results, we can completely determine the FT rank of a quasi-Frobenius ring. We recall that a ring R is *quasi-Frobenius* if and only if R is left (right) Artinian and left (right) self-injective. The following implication is well known (we refer the reader to Lam [8, 9] for details):

$$\text{quasi-Frobenius} \Rightarrow \text{(left and right) FPF and semiperfect.} \tag{4.1}$$

Corollary 2. *Let R be a quasi-Frobenius ring. Then $FT(R) \leq 1$. Moreover, $FT(R) = 0$ if and only if R is self-basic.*

Proof. Assume R is quasi-Frobenius. Then by definition, R is left Artinian. Theorem 4 implies that R has finite FT rank, and Proposition 5 along with implication (4.1) implies that $FT(R) = 0$ or $FT(R) = 1$. If R is not self-basic, then Proposition 4 yields that $FT(R) = 1$. Now suppose that R is self-basic. By implication (4.1), it follows that R is a self-basic semiperfect FPF ring. Let M be a finitely generated faithful module over R. Fact 2 implies that R is a direct summand of M, whence M cannot be torsion. Hence $FT(R) = 0$, as claimed. $\square$

Using this result, we easily determine the rank function for semisimple Artinian rings.

Corollary 3. *Let R be a semisimple Artinian ring. Then $FT(R) \leq 1$. Moreover, $FT(R) = 0$ if and only if R is reduced (that is, R has no nonzero nilpotent elements).*

Proof. Let R be a semisimple Artinian ring. Then R is quasi-Frobenius, and Corollary 2 implies that $FT(R) \leq 1$. Suppose first that R is reduced. Then R is a finite direct product of division rings. It follows that R is left duo, and Corollary 1 along with the fact that $FT(R) \leq 1$ implies that $FT(R) = 0$. Now suppose that R is not reduced. Then any basic subring of R is isomorphic to a finite direct product of division rings (see [8, p. 373]). Since a finite product of division rings is reduced, it follows that R cannot be self-basic. Corollary 2 implies that $FT(R) = 1$. □

We now give some further examples of non-FT rings.

Corollary 4. *A commutative quasi-Frobenius ring has FT rank* 0. *Thus a finite direct product of commutative subdirectly irreducible rings which satisfy ascending chain condition* (*ACC*) *on annihilators has FT rank* 0.

Proof. Let R be a commutative quasi-Frobenius ring. Since R is a commutative Artinian ring, $R/radR$ is a finite product of fields. Hence R is self-basic. Corollary 2 implies that $FT(R) = 0$. It is well-known (see Pan [12], for example) that the commutative quasi-Frobenius rings are exactly the finite direct products of subdirectly irreducible rings which satisfy ACC on annihilators. □

Corollary 5. *Suppose that R is a commutative Artinian principal ideal ring. Then $FT(R) = 0$.*

Proof. Let R be a commutative Artinian principal ideal ring. Then R decomposes as $R = R_1 \oplus R_2 \oplus \cdots \oplus R_k$, where each R_i is an Artinian local ring, which clearly must also be a principal ideal ring. Cohen's structure theorem for local rings implies that all (commutative) local Artinian principal ideal rings are proper homomorphic images of complete discrete valuation rings (see McLean [10] for details). Thus each R_i is chained with only finitely many ideals. It follows that each R_i is subdirectly irreducible. We may now invoke Corollary 4 to complete the proof. □

We have only considered left modules so far in this article. A natural question is whether FT rank is a sided condition; that is, whether there exists a ring R which has different FT rank if we consider right modules instead of left modules. Toward this end, we define the *right FT rank* of a ring R to be the minimum cardinality of a generating set of a faithful torsion right R-module. To be clear, we define *left FT rank* analogously (which is simply our earlier definition of FT rank). We will shortly present an example of a ring with different FT ranks on the right and left, and begin with a lemma.

Lemma 6. *Let K be a field with endomorphism σ such that, for $L = \sigma(K)$, $[K : L] := n > 1$ (for example, we could take $K = \mathbb{Q}(x)$ and $\sigma : \mathbb{Q}(x) \to \mathbb{Q}(x^2)$ to be the natural map). Let $K[x; \sigma]$ be the skew polynomial ring with multiplication rule $xa = \sigma(a)x$ for $a \in K$. Let (x^2) be the (two-sided) ideal generated by x^2, and let*

$$R := K[x; \sigma]/(x^2) = K \oplus K\overline{x}$$

Let $\{a_1, a_2, \ldots, a_n\}$ be a basis of K over L. Then:

(a) *R is an Artinian ring;*
(b) *$J := K\overline{x} = \bigoplus a_i L\overline{x} = \bigoplus a_i\sigma(K)\overline{x} = \bigoplus a_i\overline{x}K$;*

(c) *The left ideals of R are exactly* $\{0\}$, J, *and* R;
(d) *Each* $a_i\overline{x}K$ *is a minimal right ideal of* R.

Proof. These claims appear explicitly in [9, Ex. 16.2, p. 319]. □

Example 4. The ring R in Lemma 6 has left FT rank 0 and right FT rank 1.

Proof. We first claim that the left FT rank of R is 0. Indeed, suppose that M is a left torsion module over R, and let $m \in M$ be arbitrary. Since M is torsion, it follows from (c) above that $J \subseteq ann(m)$. Since m was arbitrary, we conclude that $J \subseteq ann(M)$, and hence M is not faithful. It follows that R has left FT rank 0.

We now claim that R has right FT rank 1. By (d), $a_1\overline{x}K$ is a minimal right ideal of R. It follows from (b) and (c) above and the fact that $n > 1$ that $a_1\overline{x}K$ is not two-sided. Since R is Artinian (by (a)), we may invoke Corollary 1 to conclude that the right *FT* rank of R is 1. □

Remark 2. At this point, we revert to our original definition of rank. For the remainder of the article, 'FT rank' will always mean 'left FT rank.'

Noticeably absent from this section are examples of rings of finite FT rank $n > 1$. We do not have any examples of such rings, but we conjecture that they do exist.

5. COUNTABLE FT RANK

We begin this section by obtaining (analogous to the bound on the FT rank of an Artinian ring) a bound on the FT rank of a commutative Noetherian ring. We first recall the following result found in Bass [2].

Fact 3. *Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Every descending chain of submodules of M is countable.*

Theorem 5. *Let R be a commutative Noetherian ring. Then* $FT(R) \leq \omega$.

Proof. Immediate from the previous fact (taking $M = R$) and Theorem 3. □

Remark 3. There is a simple proof of Theorem 5 when R is a domain. We may of course assume that R is not a field. Let I be a proper nonzero ideal of R. Krull's Intersection Theorem implies that $\bigcap_{n>0} I^n = \{0\}$. It now follows from Theorem 2 that $\bigoplus_{n>0} R/I^n$ is faithful torsion. Hence $FT(R) \leq \omega$.

We turn our attention toward identifying various classes of rings with countably infinite FT rank. Toward this end, we investigate the FT rank of rings R which have the property that the set $\mathscr{S}$ of *all* nonzero (two-sided) ideals of R has the finite intersection property. Note that a large number of rings belong to this class. In particular, left and right uniform rings as well as prime rings have this property (hence, so do domains and simple rings). We prove the following theorem on the FT rank of such rings.

Theorem 6. *Let R be a ring for which the collection $\mathcal{S}$ of nonzero ideals of R has FIP. Then the following hold*:

(a) $FT(R) = 0$, $FT(R) = 1$, *or* $FT(R) \geq \omega$;
(b) *If R is commutative, then $FT(R) = 1$ is impossible, but $FT(R) = 1$ can hold if R is noncommutative* (*we can even choose R to be a domain*).
(c) *If $FT(R) = 0$, then R possesses a minimum ideal. The converse holds if R is commutative, but fails in general.*

Proof. Assume that the set of nonzero ideals of R has FIP.

(a) Suppose by way of contradiction that $FT(R) = k$ and $1 < k < \omega$. Let $M = (m_1, \ldots, m_k)$ be a faithful torsion module over R. For each j with $1 \leq j \leq k$, let $I_j := ann(Rm_j)$. Note that if $I_j = \{0\}$, then Rm_j is a faithful torsion module over R, and this contradicts $FT(R) > 1$. Hence $I_j \neq \{0\}$. Since each I_j is a nontrivial two-sided ideal of R, and since R has FIP, it follows that $I_1 \cap I_2 \cdots \cap I_k \neq \{0\}$. But this implies that M is not faithful, and we have a contradiction.

(b) If R is commutative, then $FT(R) \neq 1$ by Corollary 1. To see that $FT(R) = 1$ is possible in general, let k be a field with an automorphism σ of infinite order. Then $R := k[x, x^{-1}, \sigma]$ is a non-Artinian simple domain (this appears as Corollary 3.19 in [8]). It follows from Proposition 3 that $FT(R) = 1$.

(c) Assume that R does not possess a minimum ideal. Then the intersection of all nonzero two-sided ideals of R is trivial. Theorem 2 implies that R is FT. Suppose further that R is commutative, and let I be the minimum ideal of R. Suppose that M is a torsion module over R. It follows from the minimality of I that $I \subseteq ann(Rm)$ for any $m \in M$. Hence M cannot be faithful. To show that the converse fails in general, consider an arbitrary simple ring R which is not a division ring. Clearly, R has FIP on nonzero ideals and R itself is a mimimum ideal, yet R is FT by Proposition 3. □

Corollary 6. *Let D be a commutative domain which is not a field. Then $FT(D) \geq \omega$.*

Proof. Immediate from Proposition 1 and Theorem 6. □

We proceed to prove the following useful result. We recall that an *overring* of a commutative domain D is a ring R lying between D and its quotient field.

Lemma 7. *Let D be a commutative domain with quotient field $\mathcal{F}$, and suppose that $R \neq \mathcal{F}$ is an overring of D. Then $FT(D) \leq FT(R)$.*

Proof. Assume that D and R are as stated. Since R is not a field, Corollary 6 implies that $FT(R) := \kappa \geq \omega$. By Theorem 3, there exists a collection of nonzero ideals $\{J_i : i \in \kappa\}$ of R indexed by κ and satisfying the following:

(i) $J_j \subsetneq J_i$ for $i < j$;

(ii) $\bigcap_{i \in \kappa} J_i = \{0\}$.

For each i, define $K_i := J_i \cap D$. It is easy to see that $\{K_i : i \in \kappa\}$ is a chain of nonzero ideals of D (it is possible for $K_i = K_j$ for $i \neq j$, but these redundancies cause no problems). Further, (ii) above implies that $\bigcap_{i\in\kappa} K_i = \{0\}$. Finally, it follows from Theorem 2 that $FT(D) \leq \kappa = FT(R)$. □

We will shortly catalog a wide variety of rings with countably infinite FT rank. We first prove two lemmas. The first lemma is likely well known, but since we could not locate a source, we sketch its proof.

Lemma 8. *Let D be a domain with a minimal nonzero prime ideal P. Then D possesses a rank one valuation overring V (i.e., V has Krull dimension one).*

Proof. Let D and P be as stated. It suffices to prove the claim for the one-dimensional quasi-local ring D_P. Let V be a valuation overring of D_P which is not a field. We claim that V possesses a minimal nonzero prime ideal. Suppose by way of contradiction that this is not the case. Let $\{P_i : i \in I\}$ be the collection of nonzero prime ideals of V. Since V does not possess a minimal nonzero prime ideal, it follows that $\bigcap_{i\in I} P_i = \{0\}$. For each i, set $Q_i := P_i \cap D_P$. Then each Q_i is a nonzero prime ideal of D_P. As $\bigcap_{i\in I} P_i = \{0\}$ and $Q_i \subseteq P_i$, it follows that $\bigcap_{i\in I} Q_i = \{0\}$. Since $\{Q_i : i \in I\}$ is a chain of nonzero prime ideals of D_P with trivial intersection, it follows that the set $\{Q_i : i \in I\}$ is infinite. This contradicts the fact that PD_P is the unique nonzero prime ideal of D_P. Hence V possesses a minimal nonzero prime ideal P^*. It follows that V_{P^*} is a rank one valuation overring of D_P, and the proof is complete. □

Lemma 9. *Let V be a rank one valuation ring. Then $FT(V) = \omega$.*

Proof. Assume V is a rank one valuation ring, and let P be the unique nonzero prime ideal of V. Let $x \in P$ be nonzero. Then it is well known (see Theorem 17.1 of Gilmer [5], for example) that $\bigcap_{n>0}(x^n)$ is a prime ideal of V. Since P is minimal, it follows that $\bigcap_{n>0}(x^n) = \{0\}$. Theorem 2 and Corollary 6 imply that $FT(V) = \omega$. □

We now prove the following theorem.

Theorem 7. *Let D be a commutative domain with a minimal nonzero prime ideal P. Then $FT(D) = \omega$.*

Proof. We assume that D is a commutative domain and that P is a minimal nonzero prime ideal of D. Corollary 6 implies that $FT(D) \geq \omega$. By Lemma 8, there exists a rank one valuation overring V of D. Lemma 9 implies that $FT(V) = \omega$, and Lemma 7 yields $FT(D) \leq FT(V) = \omega$. It follows that $FT(D) = \omega$, and the proof is complete. □

We can now classify a wide collection of rings of countably infinite FT rank.

Corollary 7. *Each of the following commutative rings has FT rank ω:*

(i) *Domains (which are not fields) which satisfy the descending chain condition (DCC) on prime ideals;*
(ii) *Unique factorization domains (which are not fields);*

(iii) $D[x]$, *where* D *is a domain*;
(iv) $D[[x]]$, *where* D *is a domain*;
(v) *Domains of finite Krull dimension* $d > 0$.

Proof. (i) Immediate from Theorem 7. (ii) Let D be a unique factorization domain which is not a field, and let $p \in D$ be a prime element. Then (p) is a minimal (nonzero) prime ideal, hence Theorem 7 implies that $FT(D) = \omega$. Showing that the rings in (iii) and (iv) have FT rank ω proceed similarly by noticing that in both cases, (x) is a (nonzero) minimal prime ideal. (v) If D has finite Krull dimension $d > 0$, then D possesses minimal prime ideals. Again, we are done by Theorem 7. $\square$

6. UNCOUNTABLE FT RANK

In this section, we give examples of rings of FT rank κ for any regular cardinal κ. We do this via valuation theory (we refer the reader to [5] for a thorough development of valuation theory). In particular, we relate the FT rank of a valuation ring to a property of its value group. Let $(A, <)$ be a partially ordered set, and let B be a subset of A. Recall that B is *cofinal* in A provided that for every $a \in A$, there exists some $b \in B$ such that $a \leq b$. The *cofinality* of $(A, <)$, cf A, is the smallest cardinality of a cofinal subset of A. To illustrate, consider the structure $(\mathbb{R}, <)$, where $<$ is the usual order on the set of real numbers. The set $\mathbb{Z}^+$ of positive integers is cofinal in $\mathbb{R}$, and clearly every cofinal subset of $\mathbb{R}$ is infinite. Thus the cofinality of the structure $(\mathbb{R}, <)$ is ω. We relate this concept to FT rank.

Proposition 6. *Let v be a valuation on the field K, let G be its value group, and let V be the associated valuation ring (we assume, of course, that V is not a field). Then $FT(V) =cf\ G$.*

Proof. Let $FT(V) := \kappa$. Note that κ is infinite (Corollary 6). We first show that there exists a cofinal subset of G with cardinality at most κ. By Theorem 3, there exists a collection $\{J_i : i < \kappa\}$ of nonzero ideals of V with trivial intersection. For each i, pick some nonzero $x_i \in J_i$, and consider the set $X := \{v(x_i) : i < \kappa\}$. We claim that X is cofinal in G. Suppose by way of contradiction that this is not the case. Then there is some $g \geq 0$ in G such that $v(x_i) \leq g$ for all $i < \kappa$. Let $x^* \in V$ be such that $v(x^*) = g$. Hence for each i, $v(x_i) \leq v(x^*)$. It follows that $x_i | x^*$ for each i, and thus x^* is a nonzero element of $\bigcap_{i \in I} J_i$, a contradiction. Thus X is cofinal in G, and so cf $G \leq FT(V)$. Suppose by way of contradiction that cf $G < FT(V)$, and let $A := \{g_i : i \in I\}$, $|I| < FT(V)$ be a cofinal subset of G (we may assume without loss of generality that each $g_i \geq 0$). For each i, let $x_i \in V$ be such that $v(x_i) = g_i$. Then it follows as above that $\bigcap_{i \in I}(x_i) = \{0\}$. But then $\bigoplus_{i \in I} V/(x_i)$ is faithful torsion, and we have a contradiction to $|I| < FT(V)$. This completes the proof. $\square$

We now show that there exist valuation rings of arbitrary regular FT rank (recall that the FT rank of a ring is always a regular cardinal by Theorem 3).

Theorem 8. *Let κ be a (infinite) regular cardinal. There exists a valuation ring V such that $FT(V) = \kappa$.*

Proof. Consider the abelian group $G := \bigoplus_{i\in\kappa}\mathbb{Z}$ (all finitely nonzero sequences in $\mathbb{Z}$ indexed by the cardinal κ). Recall that the reverse lexicographic order on G is defined as follows: A nonzero element $g := (g_i : i \in \kappa)$ is defined to be positive if and only if $g_j > 0$, where j is the largest element of the support of g (note that '$g_j > 0$' is an assertion about the usual integer ordering and 'j is the largest element of the support of g' is an assertion about the usual ordinal ordering). It is then straightforward to verify that this defines a partition of G and that the set P of positive elements of G is closed under addition. Hence G becomes a totally ordered abelian group via this ordering. By The Jaffard–Ohm–Kaplansky Theorem, there is a field F and a valuation v on F with value group G. Let V be the associated valuation ring. It suffices by Proposition 6 to show that cf $G = \kappa$. Define the function $\pi : G - \{0\} \to \kappa$ by $\pi((g_i : i \in \kappa)) := i_0$, where i_0 is the largest element of the support of $(g_i : i \in \kappa)$. Let $A \subseteq G^+$ with $|A| < \kappa$. We will show that A is not cofinal in G. Note trivially that as $|A| < \kappa$, also $|\pi[A]| < \kappa$. Since κ is a regular cardinal, $\pi[A]$ is not cofinal in κ. Let $j \in \kappa$ be such that $\pi[A] < j$ (that is, every element of $\pi[A]$ is less than j). Let $g \in G$ be the sequence with a 1 in the jth coordinate and zeros everywhere else. Then it is clear that $g > h$ for every $h \in A$, and hence A is not cofinal in G. This completes the proof. □

Corollary 8. *Suppose $\kappa \leq \lambda$ are cardinals with κ regular. Then there is a valuation ring V of cardinality λ such that $FT(V) = \kappa$.*

Proof. Let G be an ordered abelian group of size κ which also has cofinality κ (such as the group G defined in Theorem 8), and let H be any ordered abelian group of size λ. Then $G \oplus H$ becomes an ordered abelian group under the usual lexicographic order. It is straighforward to check that $G \oplus H$ has cofinality κ (in particular, $\{(g_i, h_i) : i \in I\}$ is cofinal in $G \oplus H$ if and only if $\{g_i : i \in I\}$ is cofinal in G). By Jaffard–Ohm–Kaplansky, there is a valuation v on a field K with value group $G \oplus H$. K can be chosen to have size $|G \oplus H| = \lambda$. Hence also the associated valuation ring V has size λ. The result now follows by Proposition 6. □

7. OPEN QUESTIONS

We close the article with two questions which we feel are interesting.

Question 4. Is there a bound on the FT rank of noncommutative Noetherian rings?

Question 5. Are there examples of rings of finite FT rank $n > 1$? In particular, does the ring of continuous functions on $\mathbb{R}$ have finite FT rank?

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