SMALL AND LARGE IDEALS OF AN ASSOCIATIVE RING

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Abstract. Let R be an associative ring with identity, and let I be an (left, right, two-sided) ideal of R. Say that I is *small* if $|I| < |R|$ and *large* if $|R/I| < |R|$. In this note, we present results on small and large ideals. In particular, we study their interdependence and how they influence the structure of R. Conversely, we investigate how the ideal structure of R determines the existence of small and large ideals.

1. Introduction

All rings in this paper are associative with identity 1≠0.

Let R be a ring, and suppose that I is an (left, right, two-sided) ideal of R. Define I to be *small* provided $|I| < |R|$ and *large* if $|R/I| < |R|$. Rings R for which $|R/I|$ is finite for every nonzero two-sided ideal I of R were studied some time ago by Chew and Lawn ([2]). They call such rings *residually finite*. Many of their results were extended (in particular, to rings without identity) by Levitz and Mott in [14]. This notion was generalized in the commutative setting by Salminen and the author. To wit, let R be a commutative ring and let M be an infinite unitary R-module. Say that M is *homomorphically smaller* (HS for short) if and only if $|M/N| < |M|$ for all nonzero submodules N of M. Various structural results on HS modules were obtained in [19]. Dually, infinite modules M (over a commutative ring) for which $|N| < |M|$ for all proper submodules N of M have also received attention in the literature. Robert Gilmer and Bill Heinzer initiated their study, calling such modules *Jónsson modules*. We refer the reader to Gilmer and Heinzer ([3], [4], [5], [6]) and Oman ([10], [16], [17], [20]) for results on Jónsson modules and related structures.

The purpose of this article is two-fold: On one hand, we narrow our study by considering only ideals (instead of modules, more generally). On the other, we do not restrict our study to rings in which *all* proper ideals are small or *all* nonzero ideals are large. Instead, we study properties of small and large ideals in general. Specifically, we investigate how the existence of small ideals and large ideals affects the cardinality and ideal structure of a ring R and conversely.

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2. Fundamental Results

Following standard convention, the term 'ideal' will always denote a two-sided ideal. For one-sided results, we default to left ideals. However, all one-sided results contained in this paper remain true when 'left ideal' is replaced with 'right ideal'. If M and N are isomorphic left modules over a ring R, then we denote this fact by $M \cong_R N$. Throughout the paper, we will be careful to distinguish between ring isomorphism and module isomorphism to help mitigate any confusion (in particular, if A and B are rings, then the notation "$A \cong B$" will always mean that A and B are isomorphic *as rings*).

Before beginning our investigation, we remind the reader of the focal point of our study: A (left, right, two-sided) ideal I of a ring R is *small* if $|I| < |R|$ and *large* if $|R/I| < |R|$. Note that the zero ideal is a small ideal of any ring. Dually, R is a large ideal of any ring R. Not every ring possesses a nonzero small left ideal. Indeed, the only small left ideal of a division ring is the zero ideal. More generally, suppose the ring R does not contain (nonzero) right zero divisors. Then the only small left ideal of R is the zero ideal. To see this, suppose that I is a nonzero left ideal of R, and let $x \in I$ be an arbitrary nonzero element. Then the map $\varphi : R \to I$ defined by $\varphi(r) := rx$ is injective. Thus $|R| \leq |I| \leq |R|$, and we conclude that $|I| = |R|$. Analogously, not every ring possesses a proper large left ideal, as division rings witness. Further, it is not hard to find examples of rings which are not division rings which have neither a nonzero small left ideal nor a proper large left ideal. For instance, consider the ring $R := \mathbb{Q} \times \mathbb{Q}$. The proper nonzero ideals of R are $\{0\} \times \mathbb{Q}$ and $\mathbb{Q} \times \{0\}$. From this fact, it is clear that R does not possess a nonzero small ideal nor a proper large ideal. Lastly, we remark that if R is a finite ring, then every proper nonzero left ideal of R is both small and large.

We now commence the building of our machinery. We first recall the following fact due to Lagrange:

Fact 1 (Lagrange)**.** *Let G be a* (*possibly infinite*) *group, and let H be a subgroup of G. Then $|G| = |H| \cdot [G : H]$.*

Proof. Let H be a subgroup of the group G. Recall that the (say) left cosets of H in G partition G and each left coset of H has the same cardinality as H. The result follows. $\square$

This brings us to our first proposition.

Proposition 1. *Let R be a ring. Then R possesses a left ideal I which is both small and large if and only if R is finite and not a field.*

Proof. Let R be a ring. If R is finite and not a field, then (by Wedderburn's Theorem) there exists a nonzero, proper left ideal I of R. Since R is finite, we conclude that I is both small and large. Conversely, assume that I is a left ideal of R which is both small and large. It is patent that R is not a field. Suppose by way of contradiction that R is infinite. Fact 1 yields $|R| = |I| \cdot |R/I|$. Basic cardinal arithmetic implies that either $|I| = |R|$ or $|R/I| = |R|$. If $|I| = |R|$, then I is not small; if $|R/I| = |R|$, then I is not large. As I is both small and large, we have reached a contradiction, and the proof is complete. □

The following corollary is immediate.

Corollary 1. *Let R be an infinite ring, and let I be a left ideal of R. If $|R/I| < |R|$, then $|I| = |R|$. Further, if $|I| < |R|$, then $|R/I| = |R|$.*

The next proposition collects some basic closure properties which will serve us well throughout the paper. For brevity, let $\mathcal{S}(R)$ denote the collection of small left ideals of a ring R, and let $\mathcal{L}(R)$ denote the collection of large left ideals of R.

Proposition 2. *Let R be an infinite ring, and let $\mathcal{I}(R)$ denote the collection of left ideals of R. Then the following hold:*

(a) If $I \in \mathcal{S}(R)$, $J \in \mathcal{I}(R)$, and $J \subseteq I$, then $J \in \mathcal{S}(R)$.
(b) If $I_1, I_2, \ldots, I_n \in \mathcal{I}(R)$, then $I_1 + I_2 + \cdots + I_n \in \mathcal{S}(R)$ if and only if $I_k \in \mathcal{S}(R)$ for all k, $1 \leq k \leq n$.
(c) If $I \in \mathcal{L}(R)$, $J \in \mathcal{I}(R)$, and $I \subseteq J$, then $J \in \mathcal{L}(R)$.
(d) If $I_1, I_2, \ldots, I_n \in \mathcal{I}(R)$, then $I_1 \cap I_2 \cap \cdots \cap I_n \in \mathcal{L}(R)$ if and only if $I_k \in \mathcal{L}(R)$ for all k, $1 \leq k \leq n$.

Proof. Assume that R is an infinite ring, and let $\mathcal{I}(R)$ be the collection of left ideals of R.

(a) Obvious.

(b) Suppose $I_1, I_2, \ldots, I_n \in \mathcal{I}(R)$, and set $I := I_1 + I_2 + \cdots + I_n$. If $I \in \mathcal{S}(R)$, then $I_k \in \mathcal{S}(R)$ for all k by (a). Conversely, suppose that $I_k \in \mathcal{S}(R)$ for all k, $1 \leq k \leq n$. Then simply note that $|I| = |I_1 + I_2 + \cdots + I_n| \leq |I_1 \times I_2 \times \cdots \times I_n| = |I_1| \cdot |I_2| \cdots |I_n| := \kappa$. If each I_k is finite, then κ is finite, and (as R is infinite) $|I| < |R|$. If some I_k is infinite, then $\kappa = \max(|I_1|, |I_2|, \ldots, |I_n|) < |R|$ since each I_k is small. This proves (b).

(c) Suppose that $I \in \mathcal{L}(R)$ and $J \in \mathcal{I}(R)$ with $I \subseteq J$. Then as $R/J \cong_R (R/I)/(J/I)$, we see that $|R/J| = |(R/I)/(J/I)| \leq |R/I| < |R|$.

(d) Suppose $I_1, I_2, \ldots, I_n \in \mathcal{I}(R)$. If $I_1 \cap I_2 \cap \cdots \cap I_n \in \mathcal{L}(R)$, then each $I_k \in \mathcal{L}(R)$ by (c). Conversely, suppose that each $I_k \in \mathcal{L}(R)$. We let $\varphi : R \to R/I_1 \times R/I_2 \times \cdots \times R/I_n$ be the natural map, and let M denote the image of R under φ. Since $R/(I_1 \cap I_2 \cap \cdots \cap I_n) \cong_R M$, we see that

$$|R/(I_1 \cap I_2 \cap \cdots \cap I_n)| = |M| \leq |R/I_1 \times R/I_2 \times \cdots \times R/I_n| = |R/I_1| \cdot |R/I_2| \cdots |R/I_n| < |R|.$$

(the final inequality follows as in the proof of (b) above) □

We now address two fundamental questions: Does the existence of a proper large left ideal imply the existence of a nonzero small left ideal? What is the status of the converse? The ring $\mathbb{Z}$ of integers shows that the answer to the first question is 'no'. On the other hand, the existence of a nonzero small left ideal does imply the existence of a proper large left ideal, as we now show.

Proposition 3. *Let R be a ring, and suppose that I is a nonzero left ideal of R. Then there exists a maximal left ideal M of R such that $|R/M| \leq |I|$. Thus if R has a nonzero small left ideal, then R also has a large maximal left ideal.*

Proof. Let R be a ring, I a nonzero left ideal of R, and let $x \in I$ be nonzero. Recall that the left annihilator of x in R is defined by $ann_R(x) := \{r \in R : rx = 0\}$. Further, $Rx \cong_R R/ann_R(x)$. Since $x \neq 0$, $ann_R(x)$ is a proper left ideal of R. Hence $ann_R(x) \subseteq M$ for some maximal left ideal M of R. It now follows easily that $|R/M| \leq |R/ann_R(x)| = |Rx| \leq |I|$. □

We use the previous proposition to classify the infinite division rings using the concept of smallness.

Proposition 4. *Let R be an infinite ring. Then R is a division ring if and only if every proper left ideal of R is small.*

Proof. Let R be an infinite ring. If R is a division ring, then the zero ideal is the only proper left ideal of R, whence every proper left ideal of R is certainly small. Conversely, suppose that every proper left ideal of R is small. It suffices to show that the only left ideals of R are $\{0\}$ and R. Suppose by way of contradiction that I is a proper nonzero left ideal of R. Then I is small by assumption. By Proposition 3, R possesses a proper large left ideal J. But then J is both small and large, contradicting Proposition 1. □

Corollary 2. *Let R be an infinite left Noetherian ring which is not a division ring. Then there exists a nonunit $x \in R$ such that $|Rx| = |R|$.*

Proof. We assume that R is an infinite left Noetherian ring which is not a division ring. By the previous proposition, there exists a proper left ideal I of R such that $|I| = |R|$. As R is left Noetherian, $I = \langle x_1, x_2, \ldots, x_n \rangle$ for some $x_1, x_2, \ldots, x_n \in R$. Since R is infinite, it follows from basic cardinal arithmetic that $|Rx_i| = |R|$ for some i, and the proof is complete. □

One may naturally inquire about a dual result: Is there a nice characterization of the infinite rings R for which every nonzero left ideal of R is large? It appears such a characterization is currently untenable. As evidence, the residually finite rings have not been classified. Moreover, for every uncountable cardinal κ, there exist non-Noetherian valuation rings V of cardinality κ for which every nonzero left ideal of V is large (see Theorem 2.8 of [19]).

3. Sizes of Ideal Products

Let R be an infinite ring, and suppose that I and J are large left ideals of R. We proved in (d) of Proposition 2 that $I \cap J$ is large. A natural question is whether IJ must also be large[1]. The following example shows that the answer is 'no' in general.

Example 1. *Let F be a finite field, and let $R := F[x_1, x_2, x_3, \ldots]$ be the polynomial ring in (countably) infinitely many variables over F. Now set $I := \langle x_1, x_2, x_3, \ldots \rangle$. Then I is large, but I^2 is not.*

Proof. Let R and F be as stated above. Clearly, $|R| = \aleph_0$. Further, $R/I \cong F$ (as rings). Since F is finite, we see that I is a large ideal of R. It remains to show that I^2 is not large. Toward this end, it suffices to show that $I^2 + x_i \neq I^2 + x_j$ for $i \neq j$. To wit, it is easy to see that every nonzero element of I^2 has degree at least 2. Thus $x_i - x_j \notin I^2$, and the proof is complete. □

Note that in the previous example, the ideal I was not finitely generated. We will shortly prove that if we impose some finiteness conditions, the product of two large left ideals is again large. We begin with two lemmas.

Lemma 1. *Let R be a ring, and suppose that M is a (unitary) bimodule over R which can be generated (as an R-bimodule) by κ elements. If R and κ are finite, then M is finite; if R is infinite or κ is infinite, then $|M| \leq |R| \cdot \kappa$.*

Proof. Let R be a ring, and assume M is a unitary bimodule over R which can be generated by κ elements (as a bimodule). Let $\{m_i : i < \kappa\}$ be a generating set for M. Fix $i < \kappa$, and let $X_i := \{rm_is : r, s \in R\}$. We remind the reader that Rm_iR, the (bi)submodule of M generated by m_i, is the set of all finite sums of elements of X_i. Now, clearly $|X_i| \leq |R \times R|$. It follows that $|Rm_iR| \leq |(R \times R)^{<\omega}|$, the latter cardinal being the cardinality of the set of finite subsets of $R \times R$. It follows from this observation (and basic cardinal arithmetic) that

$$\text{If } R \text{ is finite, then so is } Rm_iR, \text{ and} \tag{3.1}$$

$$\text{If } R \text{ is infinite, then } |Rm_iR| \leq |R|. \tag{3.2}$$

It is easy to see that

$$|M| = \left|\sum_{i<\kappa} Rm_iR\right| \leq \left|\bigoplus_{i<\kappa} Rm_iR\right|. \tag{3.3}$$

[1]It is trivial that the product of two small left ideals is small.

We conclude from (3.1) and (3.3) that if R and κ are finite, then so is M. Thus assume that either R or κ is infinite. Then (3.1)-(3.3) imply that $\bigoplus_{i<\kappa} Rm_iR$ has cardinality at most $|R| \cdot \kappa$. We conclude that $|M| \leq |R| \cdot \kappa$. $\square$

Lemma 2. *Let R be a ring, and let I and J be left ideals of R. Further, let κ be a nonzero cardinal number (which may be infinite). If J is generated by κ elements as a left ideal, then $|R/J| \leq |R/IJ| \leq |R/I|^\kappa \cdot |R/J|$.*

Proof. Suppose that R is a ring and that I and J are left ideals of R. Further, let κ be a nonzero cardinal. Suppose that J is generated by κ elements as a left ideal. Note that by Fact 1, we have

$$|R/IJ| = |J/IJ| \cdot |(R/IJ)/(J/IJ)| = |J/IJ| \cdot |R/J|. \tag{3.4}$$

This proves that $|R/J| \leq |R/IJ|$. To finish the proof, it suffices to show that $|J/IJ| \leq |R/I|^\kappa$. Since J is a κ-generated left ideal of R, clearly J/IJ is a κ-generated left R-module. Let $K := ann_R(J/IJ)$. Then K is a two-sided ideal of R, and J/IJ is κ-generated over the ring R/K as a left R/K-module. Note trivially that $I \subseteq K$, whence

$$|R/K| \leq |R/I|. \tag{3.5}$$

Let $\bigoplus_\kappa R/K$ denote the direct sum of κ copies of R/K. Since J/IJ is κ-generated over R/K, there exists a surjection $\varphi : \bigoplus_\kappa R/K \to J/IJ$. Hence

$$|J/IJ| \leq |\bigoplus_\kappa R/K| \leq |R/K|^\kappa \leq \text{ (via (3.5)) } |R/I|^\kappa. \tag{3.6}$$

This concludes the proof. $\square$

Armed with the previous two lemmas, we prove our first theorem.

Theorem 1. *Let R be an infinite ring, and let I and J be left ideals of R.*

(*a*) *Suppose that I and J are two-sided ideals of R. Suppose further that $I \cap J$ can be generated by κ many elements as a two-sided ideal of R. Then either R/IJ is finite or $|R/IJ| \leq |R/(I \cap J)| \cdot \kappa$. Thus if $I \cap J$ is large and $\kappa < |R|$, then IJ is also large.*

(*b*) *Suppose that I and J are large and that J is finitely generated as a left ideal. Then IJ is large. Further, if R/I and R/J are finite, then so is R/IJ; if $|R/I| = |R/J| = \beta$ and β is infinite, then also $|R/IJ| = \beta$.*

Proof. Let R be an infinite ring, and let I and J be left ideals of R.

(a) Assume that I and J are two-sided and that $I \cap J$ can be generated by κ elements as a two-sided ideal of R. Then (analogous to (3.4))

(3.7) $\quad |R/IJ| = |(I \cap J)/IJ| \cdot |(R/IJ)/((I \cap J)/IJ)| = |(I \cap J)/IJ| \cdot |R/(I \cap J)|.$

Note that $(I \cap J)/IJ$ is a bimodule over R which is annihilated on the left and right by $I \cap J$. It follows that $(I \cap J)/IJ$ is naturally a bimodule over the ring $R/(I \cap J)$ which is κ-generated as an $R/(I \cap J)$-bimodule. If κ and $R/(I \cap J)$ are finite, then we deduce from Lemma 1 that $(I \cap J)/IJ$ is finite. It follows from (3.7) that R/IJ is finite. Now assume that either κ or $R/(I \cap J)$ is infinite. Then Lemma 1 implies $|(I \cap J)/IJ| \leq |R/(I \cap J)| \cdot \kappa$. Basic cardinal arithmetic along with (3.7) yields that $|R/IJ| \leq |R/(I \cap J)| \cdot \kappa$.

(b) The proof is immediate from Lemma 2. □

The following corollary is established by induction and (d) of Proposition 2. We omit the easy proof.

Corollary 3. *Let R be an infinite ring. If R satisfies the ascending chain condition (ACC) on ideals (respectively, is left Noetherian) and $I_1, I_2, \ldots, I_n$ are large ideals of R (respectively, large left ideals of R), then $I_1 I_2 \cdots I_n$ is a large ideal of R (respectively, a large left ideal of R).*

4. Prime Ideals and the Ascending Chain Condition

We begin this section by showing how to obtain some stronger results for rings which satisfy ACC on ideals, and begin with a trivial (but useful) observation. The easy proof is omitted.

Observation 1. *Let R be a ring, and let A, B, C be ideals of R. If $AB \subseteq C$, then $(C + A)(C + B) \subseteq C$.*

We use this observation along with some previous machinery to prove the following theorem.

Theorem 2. *Let R be a ring satisfying ACC on ideals.*

(*a*) *If there exists a proper ideal I of R of finite index in R, then there exists a prime ideal P of R containing I which is also of finite index in R.*

(*b*) *If I is an ideal of R of infinite index κ, then there exists a prime ideal P of R containing I which is also of index κ.*

Proof. Assume that R is a ring which satisfies ACC on ideals.

(a) (The proof does not require ACC) Suppose that I is a proper ideal of R which is of finite index in R. Let M be a maximal ideal of R containing I (the existence of M depends only on Zorn's Lemma). Since M is maximal, M is prime (this follows easily from the above observation). As $I \subseteq M$, we conclude that $|R/M| \leq |R/I|$. Hence R/M is finite.

(b) Suppose now that I is of infinite index κ. Let $\mathcal{S}$ be the collection of all ideals of R containing I which are of index κ. Since R satisfies ACC on ideals, there exists some maximal element $J \in \mathcal{S}$. It remains to show that J is prime (note that J is proper, lest J have index 1). Suppose by way of contradiction that there exist ideals A and B of R such that $AB \subseteq J$, but $A \nsubseteq J$ and $B \nsubseteq J$. As $J \subsetneq A + J := X$, we conclude that $|R/X| \leq |R/J|$. Since $J \subsetneq B + J := Y$, we have $|R/Y| \leq |R/J|$. By maximality of J, we obtain

$$|R/X| < \kappa \text{ and } |R/Y| < \kappa. \tag{4.1}$$

Observation 1 yields

$$XY \subseteq J. \tag{4.2}$$

Let $\varphi : R \to R/X \times R/Y$ be the natural map. Then $R/(X \cap Y) \cong \varphi[R]$, whence

$$|R/(X \cap Y)| \leq |R/X| \cdot |R/Y| < \kappa \text{ (by (4.1))}. \tag{4.3}$$

We conclude from (4.2) that

$$\kappa = |R/J| \leq |R/XY|. \tag{4.4}$$

Since κ is infinite, so is $|R/XY|$. We now invoke (a) of Theorem 1 to conclude that $|R/XY| \leq |R/(X \cap Y)|$. But by (4.3), $|R/(X \cap Y)| < \kappa$. Thus $|R/XY| < \kappa$, and this contradicts (4.4). The proof is now complete. ☐

Corollary 4. *If R is an infinite ring which satisfies ACC on ideals, then not every prime ideal of R is large.*

Proof. Suppose that R is an infinite ring and that R satisfies ACC on ideals. Note that the zero ideal has index $|R|$. By (b) of the previous theorem, there exists a prime ideal P of R such that $|R/P| = |R|$, and hence not every prime ideal of R is large. ☐

Before stating our next theorem, we pause to recall Cohen's famous theorem that a commutative ring R is Noetherian if and only if every prime ideal of R is finitely generated. A similar result (the Cohen-Kaplansky Theorem) states that every ideal of a commutative ring R is principal if and only if every prime ideal of R is principal. We refer the reader to Lam and Reyes [12] for a beautiful generalization of the theorems of Cohen and Kaplansky mentioned above. We now present a "Cohen-Kaplansky" type theorem for large prime ideals which generalizes Corollary 2.2 and Theorem 2.3 of [2] (where a special case of the following theorem was established for rings with ACC whose nonzero prime ideals have *finite* index).

Theorem 3. *Let R be a ring satisfying ACC on ideals. Then every nonzero ideal of R is large if and only if every nonzero prime ideal of R is large. If in addition R is infinite, then R is a prime ring.*

Proof. Assume that R satisfies ACC on ideals. Suppose further that every nonzero prime ideal of R is large, and let I be a nonzero ideal of R. We will show that I is large. If R is finite, this is patent. Thus assume that R is infinite. If I is not large, then $|R/I| = |R|$. But then by (b) of Theorem 2, there is a prime ideal P of R containing I such that $|R/P| = |R|$. Since I is nonzero and P contains I, also P is nonzero. But then P is nonzero and not large, and we have a contradiction. Now suppose that R is infinite. We will show that R is a prime ring. Since R satisfies ACC on ideals, we conclude from Corollary 4 that some prime ideal P of R is not large. Since every nonzero prime ideal of R is large, we conclude that $P = \{0\}$, and thus R is a prime ring. □

Remark 1. The ACC assumption in the previous theorem is not superfluous. Indeed, consider the ring $R := F[x_1, x_2, x_3, \ldots]/\langle x_1^2, x_2^2, x_3^2, \ldots\rangle$, where F is a finite field. It is easy to see that R is infinite and that $P := \langle x_1, x_2, x_3, \ldots\rangle/\langle x_1^2, x_2^2, x_3^2 \ldots\rangle$ is the unique prime ideal of R. Moreover, $R/P \cong F$. Thus every prime ideal of R is large. However, the nonzero ideal $\langle x_1, x_2^2, x_3^2, \ldots\rangle/\langle x_1^2, x_2^2, x_3^2 \ldots\rangle$ is not large.

Corollary 5. *Let R be an infinite ring which satisfies ACC on ideals, and let $J(R)$ be the Jacobson radical of R (the ideal of R obtained as the intersection of all maximal left (right) ideals of R), and let $Nil_*(R)$ denote the lower radical of R (the ideal of R obtained as the intersection of all prime ideals of R). Then,*

(*a*) *It is possible for $J(R)$ to be small, large, or neither. However,*
(*b*) *$Nil_*(R)$ is never large.*

Proof. (a) The ring $\mathbb{Z}$ of integers is Jacobson semisimple. Thus, trivially, $J(\mathbb{Z})$ is small. Now let F be a finite field. The power series ring $R := F[[t]]$ is an infinite local ring with maximal ideal $\langle t\rangle$. Thus $J(R) = \langle t\rangle$. Further, $F[[t]]/\langle t\rangle \cong F$, whence $J(R)$ is large. Finally, set $R := \mathbb{R}[[t]]$. As in the previous example, $J(R) = \langle t\rangle$. It follows from basic set theory that $|\langle t\rangle| = |R| = |R/J(R)| = 2^{\aleph_0}$. Thus $J(R)$ is neither small nor large.

(b) By Corollary 4, there exists some prime ideal P of R such that $|R/P| = |R|$. Since $Nil_*(R) \subseteq P$, we deduce that $|R| = |R/P| \leq |R/Nil_*(R)| \leq |R|$. Thus $|R/Nil_*(R)| = |R|$, and the proof is complete. □

5. Small maximal ideals and large minimal ideals

In this section, we consider the following natural question (and a sort of dual): Let R be an infinite ring. By definition, every maximal left ideal of R is maximal with respect to being properly contained in R. Must every maximal left ideal M of R also be maximal in cardinality? In other words, if M is a maximal left ideal of R, must it be the case that $|M| = |R|$? The answer is clearly no, as witnessed by division rings. Moreover, it is not hard to construct infinite rings with nonzero small maximal left ideals:

Example 2. *Let D be an infinite division ring, and suppose that S is a ring with $|D| > |S|$. Now set $R := D \times S$. Then $\{0\} \times S$ is a small maximal left ideal of R.*

We now investigate the structure of rings which possess a small maximal left ideal, and begin with the following proposition.

Proposition 5. *Let R be an infinite ring which is not a division ring. Suppose further that M_0 is a small maximal left ideal of R. Then:*

(a) *R is not local* (*that is, there exists a maximal left ideal M_1 of R different from M_0*).

(b) *If I is any left ideal of R not contained in M_0, then I is large. Thus every maximal left ideal $M \neq M_0$ is large.*

Proof. Assume that R is infinite and not a division ring and that M_0 is a small maximal left ideal of R.

(a) Suppose by way of contradiction that M_0 is the only maximal left ideal of R. Then every proper left ideal of R is contained in M_0, whence every proper left ideal of R is small. But then Proposition 4 implies that R is a division ring, contradicting our assumption that R is not a division ring.

(b) Suppose that I is a left ideal of R not contained in M_0. We will show that I is large. Toward this end, note that as $I \nsubseteq M_0$ and M_0 is maximal, it follows that I and M_0 are comaximal. Thus $I + M_0 = R$. It is trivial to verify that the map $x \mapsto I + x$ is a surjection from M_0 onto $(I + M_0)/I$. Thus $|(I + M_0)/I| \leq |M_0| < |R|$. But since $I + M_0 = R$, we deduce that $|R/I| < |R|$, and hence I is large. ☐

Corollary 6. *Let R be an infinite ring which possesses a small maximal left ideal. The following are equivalent:*

(a) *R is a division ring.*
(b) *R does not possess any* (*nonzero*) *right zero divisors.*
(c) *R is local.*

Proof. : We assume that R is an infinite ring which possesses a small maximal left ideal M. Clearly (a) implies (c). Proposition 5 shows that (c) implies (a). Clearly (a) implies (b). It remains to show that (b) implies (a). Thus assume (b). It suffices to show that $M = \{0\}$. If $M \neq \{0\}$, then choose any nonzero $m \in M$. The map $r \mapsto rm$ is an injection from R into M since m is not a right zero divisor. But this clearly implies that $|M| = |R|$, contradicting that M is small. ☐

Having established some ideal-theoretic properties of rings with a small maximal left ideal, we now narrow our focus to commutative rings. In this setting, we will show that stronger results can be obtained. We begin with a lemma.

Lemma 3. *Let R be an infinite commutative ring which is not a field. Then $R = F \times S$ for some field F and ring S such that $|F| > |S|$ (see Example 2) if and only if R possesses a small maximal ideal M such that $ann_R(M) \nsubseteq M$.*

Proof. Let R be an infinite commutative ring which is not a field. Suppose first that $R = F \times S$ for some field F and ring S such that $|F| > |S|$. Then clearly $M := \{0\} \times S$ is a small maximal ideal of R. Moreover, $(1, 0) \in ann_R(M) - M$.

Conversely, suppose that M is a small maximal ideal of R such that $ann_R(M) \nsubseteq M$. Let $I := ann_R(M)$. Since $I \nsubseteq M$ and M is maximal, we see that I and M are comaximal. The Chinese Remainder Theorem yields

$$R/(M \cap I) \cong R/M \times R/I. \tag{5.1}$$

Since I and M are comaximal, $M \cap I = MI = IM = \{0\}$ (since I annihilates M). Thus (5.1) reduces to

$$R \cong R/M \times R/I. \tag{5.2}$$

But now (b) of Proposition 5 implies that I is large. Setting $F := R/M$ and $S := R/I$, the proof is complete. □

We now present a (fairly mild) sufficient condition which guarantees that a small maximal ideal M of a commutative ring R satisfies $ann_R(M) \nsubseteq M$.

Proposition 6. *Let R be an infinite commutative ring of cardinality κ which is not a field. Further, suppose that M is a maximal ideal of R of cardinality $\alpha < \kappa$ which can be generated by β-many elements. If $\alpha^\beta < \kappa$ (which holds, in particular, if M is finitely generated), then $R = F \times S$ for some field F and ring S such that $|F| > |S|$.*

Proof. We suppose that R is an infinite commutative ring of cardinality κ which is not a field and that M is a maximal ideal of R of size $\alpha < \kappa$. Assume further that M can be generated by β-many elements and that $\alpha^\beta < \kappa$. It suffices by Lemma 3 to prove that $ann_R(M) \nsubseteq M$. Toward this end, let $X := \{x_i : i < \beta\}$ be a set of generators for M. Now consider the map $\varphi : R \to \prod_{i<\beta} Rx_i$ defined by

$$\varphi(r) := (rx_i : i < \beta).$$

Let K be the kernel of this map. Then it is clear that $K = ann_R(M)$. Furthermore, we have

$$R/K \cong_R \varphi[R]. \tag{5.3}$$

Moreover, $|\varphi[R]| \leq |\prod_{i<\beta} Rx_i| \leq \alpha^\beta < \kappa$. We deduce from (5.3) that $|R/K| < \kappa$, whence $K = ann_R(M)$ is large. Since M is small, it follows from Corollary 1 and (c) of Proposition 2 that $ann_R(M) \nsubseteq M$, and the proof is complete. □

We immediately obtain the following corollary.

Corollary 7. *Let R be an infinite commutative Noetherian ring which is not a field. Then R possesses a small maximal ideal if and only if $R = F \times S$ for some field F and (Noetherian) ring S such that $|F| > |S|$.*

We conclude the section by exploring the dual question: What can be said of rings which possess a minimal large left ideal? Again, division rings are the canonical examples. And again, there are others. For example, let $R := \mathbb{Q} \times \mathbb{R}$. Then R is Noetherian (Artinian, even) and $\{0\} \times \mathbb{R}$ is a minimal large left ideal of R. Note that $\mathbb{Q} \times \{0\}$ is a small maximal left ideal of R. In general, we have the following:

Proposition 7. *Let R be an infinite ring which is not a division ring, and suppose that I is a large minimal left ideal of R.*

(*a*) *If M is a maximal left ideal of R not containing I, then M is small.*

(*b*) *There is at most one maximal left ideal of R not containing I.*

(*c*) *If R is commutative, then $R = F \times R_0$ for some field F and ring R_0 with $|F| > |R_0|$.*

Proof. Assume that R is an infinite ring which is not a division ring, and suppose that I is a large minimal left ideal of R.

(a) Suppose that M is a maximal left ideal of R which does not contain I. We will show that M is small. Toward this end, since $I \nsubseteq M$, the maximality of M implies that $I + M = R$. But then $(I + M)/I = R/I$. Since I is large, we have $|(I+M)/I| = |R/I| < |R|$. But as I is minimal and $I \nsubseteq M$, we deduce that the sum $I + M$ is direct. We conclude that $(I + M)/I \cong_R M$, and hence $|M| < |R|$.

(b) This follows immediately from (a) and Proposition 5.

(c) Suppose further that R is commutative. We first show that there exists a maximal ideal M_0 not containing I. Assume by way of contradiction that $I \subseteq M$ for all maximal ideals M of R. Then $I \subseteq J(R)$ (the Jacobson radical of R). Since I is large, it follows from (c) of Proposition 2 that $J(R)$ is large. As I is minimal, we see that I is principal, whence certainly is finitely generated. We deduce from (b) of Theorem 1 that $J(R) \cdot I$ is large. But this clearly implies that $J(R) \cdot I$ is nonzero. As $J(R) \cdot I \subseteq I$ and since I is minimal, we conclude that $J(R) \cdot I = I$. Nakayama's Lemma implies that $I = \{0\}$, and we have reached a contradiction. Thus there is a maximal ideal M_0 not containing I. This implies that $M_0 + I = R$ and (by minimality of I) $M_0 \cap I = \{0\}$. The Chinese Remainder Theorem yields that

$R \cong R/M_0 \times R/I$. By (a), M_0 is small. Corollary 1 implies that $|R/M_0| = |R|$. Since I is large, $|R/I| < |R| = |R/M_0|$. This completes the proof. □

We close with the following corollary (compare to Corollary 6).

Corollary 8. *Let R be an infinite commutative ring with a large minimal ideal. The following are equivalent:*

(*a*) *R is a field.*
(*b*) *R is an integral domain.*
(*c*) *R is local.*

Proof. Assume that R is an infinite commutative ring with a large minimal left ideal I. Clearly (a) implies (c). As for the converse, suppose that R is local. Then (c) of Proposition 7 fails. We conclude that R is a field. It is also clear that (a) implies (b). Conversely, if R is an integral domain, then (as is well-known) the minimality of I implies that $I = R$. Thus the only ideals of R are $\{0\}$ and R. We deduce that R is a field, and the proof is complete. □

6. Sizes of Ideals in Commutative Noetherian Rings

In this final section, we explore the existence (and non-existence) of small and large ideals in commutative Noetherian rings. We begin with an example showing that it is easy to construct commutative Noetherian rings with residue rings and ideals of arbitrary prescribed cardinality.

Example 3. *Let κ and λ be cardinals with $0 < \kappa \leq \lambda$ and λ infinite. There exists a commutative Artinian* (*hence Noetherian*) *ring R of cardinality λ with both an ideal of index κ and an ideal of cardinality κ.*

Proof. Let S be a commutative Artinian ring of cardinality κ (if κ is finite, we may take S to be $\mathbb{Z}/\langle\kappa\rangle$; if κ is infinite, we may take S to be any field of size κ), and let F be a field of cardinality λ. Now set $R := S \times F$. Then R is clearly a commutative Artinian ring of cardinality λ. Further, $\{0\} \times F$ is an ideal of index κ and $S \times \{0\}$ is an ideal of cardinality κ . □

Note that the ring R constructed in the previous example is not local (that is, R has more than one maximal ideal). Much more difficult questions are the following:

Questions 1. *Let λ be an infinite cardinal. Does there exist a commutative Noetherian* ***local*** *ring R of cardinality λ with a proper large ideal? Does such an R exist with a nonzero small ideal? What are the possible indexes of such a large ideal* (*respectively, possible cardinalities of such a small ideal*)*?*

We kick things off by answering these questions for commutative Artinian local rings. Specifically, we show that a (infinite) commutative Artinian local ring does not possess nontrivial small or large ideals.

Proposition 8. *Let R be an infinite commutative Artinian local ring. Then $\{0\}$ is the unique small ideal of R and R is the unique large ideal of R.*

Proof. Let R be an infinite commutative Artinian local ring. It suffices to show that for every nonzero, proper ideal I of R, we have $|R/I| = |R| = |I|$. Let I be such an ideal. Since R is Artinian, R is also Noetherian and zero-dimensional (see Lang [13], pp. 443-444); as R is local, R has a unique prime ideal P (which is, of course, also maximal). We conclude from Corollary 4 that P is not large, whence $|R/P| = |R|$. Since $I \subseteq P$, we deduce that $|R| = |R/P| \leq |R/I| \leq |R|$. Thus $|R/I| = |R|$. It remains to show that $|I| = |R|$. Toward this end, if I were small, then by Proposition 3, P would be large, contradicting what we proved above. □

Remark 1 There is an alternative proof of Proposition 8. As above, we let P be the unique prime ideal of R. Suppose by way of contradiction that P is large. Then Corollary 3 yields that P^n is large for every positive integer n. But R is Noetherian and P is the nilradical of R, whence P is nilpotent. Thus $P^k = \{0\}$ for some positive integer k. We deduce that $\{0\}$ is large, and this is absurd. The remainder of the proof proceeds as above.

Answering the previous questions for commutative Noetherian local rings in general is much more difficult. Some work has already been done in this direction. Specifically, Kearnes and the author determined the possible sizes of residue fields of Noetherian integral domains in [9]. Further, Gilmer and Heinzer have shown ([7]) that if R is a commutative Noetherian (not necessarily local) ring and n is a positive integer, then there are but finitely many ideals I of R of index n.

We now recall some basic results on idealization, which will play an important role in what is to follow. Let R be a commutative ring and let M be an (unitary) R-module. The *idealization* of R and M (also known as the *trivial extension of R by M* or the *Dorroh extension of R by M*), denoted by $R(+)M$, is the ring on the Cartesian product $R \times M$ defined by the following operations:

(i) $(r_1, m_1) + (r_2, m_2) := (r_1 + r_2, m_1 + m_2)$, and
(ii) $(r_1, m_1) \cdot (r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$.

It is readily checked that the identity of $R(+)M$ is $(1, 0)$ and that for any submodule N of M, $0(+)N := \{(0, n) : n \in N\}$ is an ideal of $R(+)M$. Further,

Fact 2. *Let R be a commutative ring, and let M be an R-module. Then:*

(a) *If R is Noetherian and M is finitely generated over R, then $R(+)M$ is Noetherian.*
(b) *$R(+)M$ is local if and only if R is local.*

Proof. These results can be found, respectively, in Proposition 2.2 and Theorem 3.2 of Anderson [1]. □

If a ring R has an ideal of index λ, then certainly R need not possess an ideal of cardinality λ. To see how badly this fails, one need only note that every ideal of a commutative domain D has cardinality either 1 or $|D|$. To illustrate, there are numerous infinite domains with proper ideals of finite index (for example, $\mathbb{Z}$, $F[t]$, and $F[[t]]$, where F is a finite field). Despite this, we do have:

Lemma 4. *Let R be an infinite commutative Noetherian local ring of cardinality κ and suppose that R has an ideal of index λ. Then there exists a commutative Noetherian local ring of cardinality κ with an ideal of cardinality λ.*

Proof. Let R be an infinite commutative Noetherian local ring of cardinality κ, and let I be an ideal of index λ. Now set $S := R(+)R/I$. As $|R| = \kappa$ is infinite, clearly $|S| = \kappa$. It follows immediately from Fact 2 that S is a Noetherian local ring. Further, the ideal $0(+)R/I$ of S has cardinality λ. $\square$

We need one more technical lemma before proving our next theorem. It is well-known that the cardinality of any finite field is a power of a prime. Less well-known is that, more generally, the cardinality of a finite commutative local ring is a power of a prime (this result can be found in McDonald [15], for example). We give the short proof.

Fact 3. *The cardinality of any finite commutative local ring is a power of a prime.*

Proof. Let R be a finite local ring with maximal ideal M. As R/M is a field, $|R/M| = p^n$ for some prime p and positive integer n. It follows that R/M has characteristic p. In particular, $p \cdot 1$ (the identity 1 of R added to itself p times) is in M. But since every prime ideal of a finite ring is maximal, we conclude that M is the unique prime ideal of R. It follows that $Nil(R) = M$. As $p \cdot 1 \in M$, we conclude that $p^k \cdot 1 = 0$ for some positive integer k. Hence $p^k x = 0$ for all $x \in R$. It follows that $(R, +)$ is a finite p-group, whence $|R|$ is a power of p. $\square$

Fact 3 can be strengthened as follows (this generalization will be needed in the proof of our next theorem):

Lemma 5. *Let R be a commutative local ring. If I is a finite nonzero ideal of R, then $|I|$ is the power of a prime.*

Proof. Assume that R is local with maximal ideal M, and suppose that I is a finite, nonzero ideal of R. Proposition 3 implies that the field R/M is finite, whence

$$|R/M| = p^n \text{ for some prime } p \text{ and positive integer } n. \tag{6.1}$$

Now let x be an arbitrary nonzero element of I. We will show that $|Rx|$ is a power of p. Toward this end, note first that

$$|R/M| \cdot |M/ann_R(x)| = |R/ann_R(x)| = |Rx|. \tag{6.2}$$

Observe also that $R/ann_R(x)$ is a finite local ring, whence by Fact 3, $|R/ann_R(x)| = q^m$ for some prime q and positive integer m. However, (6.1) and (6.2) above yield that $p^n | q^m$. We deduce that $p = q$, whence $|Rx| = p^m$. As the abelian group $(Rx, +)$ has order p^m, we conclude that the additive order of x is a power of p. Since $x \in I - \{0\}$ was arbitrary, we conclude that $(I, +)$ is a finite p-group, whence $|I|$ is a power of p. □

Finally, we are in position to prove our next theorem. We mention that the following theorem is an adaptation of the proof of Lemma 2.1 of [9], where the possible cardinalities of residue fields of an infinite Noetherian domain were obtained (Lemma 2.1 says nothing useful about cardinalities of ideals in this context, since all nonzero ideals of a domain have the same cardinality).

Theorem 4. *Let $1 < \lambda < \kappa$ be cardinals with κ infinite. Suppose further that R is a commutative Noetherian local ring of cardinality κ with an ideal of index λ. Then the following hold:*

(1) *If λ is finite, then λ is a prime power.*
(2) $\kappa \leq \lambda^{\aleph_0}$.

Further, both (1) *and* (2) *above remain true if "index" is replaced with "cardinality".*

Proof. Assume that $1 < \lambda < \kappa$ are cardinals and that κ is infinite. Further, let R be a commutative Noetherian local ring of cardinality κ and let M be the maximal ideal of R.

Suppose first that I is an ideal of R and $|R/I| = \lambda$. Clearly R/I is also local, whence if λ is finite, then Fact 3 implies that λ is a prime power. Thus (1) is verified. It remains to show that $\kappa \leq \lambda^{\aleph_0}$. Toward this end, it follows from Krull's Intersection Theorem that $\bigcap_{n=1}^{\infty} M^n = \{0\}$. Since $I \subseteq M$, we conclude that

$$\bigcap_{n=1}^{\infty} I^n = \{0\}. \tag{6.3}$$

Thus the natural map $\varphi : R \to \prod_{n=1}^{\infty} R/I^n$ is injective. We deduce that

$$\kappa = |R| \leq |\prod_{n=1}^{\infty} R/I^n|. \tag{6.4}$$

We now distinguish two cases.

Case 1: λ is finite. Then (b) of Theorem 1 yields that R/I^n is finite for every positive integer n. Hence

$$\kappa = |R| \leq |\prod_{n=1}^{\infty} R/I^n| \leq |\prod_{n=1}^{\infty} \mathbb{N}| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \lambda^{\aleph_0}.$$

Case 2: λ is infinite. Then again, (b) of Theorem 1 applies, and we conclude that $|R/I^n| = \lambda$ for every positive integer n. Thus

$$\kappa = |R| \leq |\prod_{n=1}^{\infty} R/I^n| = |\prod_{n=1}^{\infty} R/I| = \lambda^{\aleph_0}.$$

We have now verified (1) and (2) in the case that I has index λ. Suppose now that $|I| = \lambda$. If λ is finite, then Lemma 5 implies that λ is the power of a prime. As for (2), Proposition 3 yields that $1 < |R/M| \leq |I| = \lambda < \kappa$. By what we just proved above, it follows that $\kappa \leq |R/M|^{\aleph_0} \leq |I|^{\aleph_0} = \lambda^{\aleph_0}$. This concludes the proof. □

A natural question arises: If λ and κ are cardinals satisfying the conditions of the previous theorem, does there exist a commutative Noetherian local ring of cardinality κ with an ideal of index (respectively, cardinality) λ? The answer is 'yes'. The majority of the ideas used in the proof of the following proposition already appear in the literature (see Lemma 2.2 of [9]). However, we reproduce them in the interest of keeping the paper self-contained.

Proposition 9. *Suppose that $1 < \lambda < \kappa \leq \lambda^{\aleph_0}$ are cardinals such that κ is infinite and if λ is finite, then λ is a prime power. Then there exists a commutative Noetherian local ring R and ideal I of R such that R has cardinality κ and I has index* (*respectively, cardinality*) *λ.*

Proof. Let F be a field of cardinality λ, and let $F[[t]]$ be the ring of formal power series over F in the variable t. The underlying set of $F[[t]]$ is the set of all functions from $\mathbb{N}$ into F, whence $|F[[t]]| = \lambda^{\aleph_0}$. The quotient field of $F[[t]]$ is the field $F((t))$ of formal Laurent series in the variable t. It is easy to see that there is a field K such that

$$F(t) \subseteq K \subseteq F((t)) \text{ and } K \text{ has cardinality } \kappa. \tag{6.5}$$

Thus $F[[t]]$ is a DVR on $F((t))$ (that is, $F[[t]]$ is a DVR with quotient field $F((t))$), $K \subseteq F((t))$, and $F[[t]] \cap K$ is not a field (since t is not invertible in $F[[t]]$). It follows that $F[[t]] \cap K$ is a DVR on K (whence also has cardinality κ and is Noetherian and local) with maximal ideal $M := \langle t \rangle \cap K$. It is easy to check that F maps injectively into $(F[[t]] \cap K)/M$ and $(F[[t]] \cap K)/M$ maps injectively into $F[[t]]/\langle t \rangle \cong F$. It follows that $|(F[[t]] \cap K)/M| = |F| = \lambda$. This completes the proof (note that "index" may be replaced with "cardinality" by Lemma 4). □

We have established necessary and sufficient conditions on an infinite cardinal κ for there to exist a commutative Noetherian local ring of cardinality κ with either a proper large ideal or a nonzero small ideal. We now show (in ZFC) that there are arbitrarily large cardinals κ for which there exists a commutative Noetherian local ring of size κ with a proper large ideal (respectively, nonzero small ideal). We also show that there are arbitrarily large cardinals for which this is not the case.

Proposition 10. *There are arbitrarily large cardinals κ for which there exists a commutative Noetherian local ring of size κ with a proper large ideal (respectively, nonzero small ideal). There are also arbitrarily large cardinals κ with the property that if R is a commutative Noetherian local ring of size κ and I is a proper, nonzero ideal of R, then $|R/I| = |R| = |I|$.*

Proof. Let a be an arbitrary ordinal, and consider the cardinal $\aleph_{a+\omega}$. Then of course

$$\aleph_{a+\omega} = \bigcup_{n\in\omega} \aleph_{a+n},$$

whence $\aleph_{a+\omega}$ has cofinality $\aleph_0$ (that is, $\mathrm{cf}(\aleph_{a+\omega}) = \aleph_0$). König's Theorem[2] now implies that $(\aleph_{a+\omega})^{\aleph_0} > \aleph_{a+\omega}$. Thus we have

$$\aleph_{a+\omega} < (\aleph_{a+\omega})^{\aleph_0} \leq (\aleph_{a+\omega})^{\aleph_0}. \tag{6.6}$$

Proposition 9 yields the existence of a commutative Noetherian local ring R of size $(\aleph_{a+\omega})^{\aleph_0}$ with an ideal of index (respectively, cardinality) $\aleph_{a+\omega}$.

To prove the second assertion, let α be an arbitrary infinite cardinal. Define the following cardinals κ_n by recursion on ω:

$$\kappa_0 := \alpha, \text{ and for } n \in \omega, \kappa_{n+1} := 2^{\kappa_n}. \tag{6.7}$$

Finally, set

$$\kappa := \bigcup_{n\in\omega} \kappa_n. \tag{6.8}$$

Now let R be a commutative Noetherian local ring of cardinality κ. We claim that if I is any proper, nonzero ideal of R, then $|R/I| = |R| = |I|$. By Theorem 4, it suffices to show that if λ is a cardinal with $\lambda < \kappa$, then $\lambda^{\aleph_0} < \kappa$. Indeed, suppose $\lambda < \kappa$. Then by definition of κ, we have $\lambda < \kappa_n$ for some $n \in \omega$. Now simply note that $\lambda^{\aleph_0} \leq (\kappa_n)^{\aleph_0} \leq (\kappa_n)^{\kappa_n} = 2^{\kappa_n} = \kappa_{n+1} < \kappa$. This completes the proof. $\square$

[2] König's Theorem states that if κ is an infinite cardinal, then $\kappa^{cf(\kappa)} > \kappa$; see Theorem 3.11 of Jech [8].

To conclude the paper, we remark that for many infinite cardinals κ, it is undecidable in ZFC whether there exists a commutative Noetherian local ring of size κ with a proper large ideal (respectively, nonzero small ideal). The smallest such infinite cardinal is $\aleph_2$, as we will shortly prove. First, we recall that the *Continuum Hypothesis* (CH) is the statement that there are no cardinals κ satisfying $\aleph_0 < \kappa < 2^{\aleph_0}$ (or more succinctly, that $\aleph_1 = 2^{\aleph_0}$). It is well-known (from Gödel and Cohen's work) that CH is independent of the usual axioms of ZFC (that is, if ZFC is consistent, then CH cannot be proved nor refuted from the ZFC axioms). The *Generalized Continuum Hypothesis* (GCH) is the statement that for any infinite cardinal λ, there is no cardinal κ satisfying $\lambda < \kappa < 2^{\lambda}$. Again, it is well-known that GCH is independent of ZFC. We will make use of the following fact:

Fact 4 ([8], Theorem 5.15)**.** *Assume that GCH holds, and suppose that κ and λ are infinite cardinals. If $\kappa < cf(\lambda)$, then $\lambda^{\kappa} = \lambda$.*

Proposition 11. *It is provable in ZFC that there exist commutative Noetherian local rings of cardinality $\aleph_0$ and $\aleph_1$, respectively, with proper large ideals* (*respectively, nonzero small ideals*)*. However, it is undecidable in ZFC whether there exists a commutative Noetherian local ring of size $\aleph_2$ with a proper large ideal* (*respectively, nonzero small ideal*)*.*

Proof. The ring $\mathbb{Z}_{\langle p \rangle}$ ($\mathbb{Z}$ localized at the prime ideal $\langle p \rangle$) is a commutative Noetherian local ring of size $\aleph_0$ with a finite residue field. As for $\aleph_1$, simply note that $2 < \aleph_1 \leq 2^{\aleph_0}$ (this is provable in ZFC) and apply Proposition 9.

We now show that it is undecidable in ZFC whether there exists a commutative Noetherian local ring R of size $\aleph_2$ with a proper large ideal (respectively, nonzero small ideal). Toward this end, assume first that CH fails. Then $2 < \aleph_2 \leq 2^{\aleph_0}$. The existence of such a ring R now follows from Proposition 9. Now assume that GCH holds. Suppose that $\lambda < \aleph_2$. To prove the nonexistence of such a ring R, it suffices by Theorem 4 to show that $\lambda^{\aleph_0} < \aleph_2$. Since $\lambda < \aleph_2$, we have $\lambda \leq \aleph_1$. Suppose first that $\lambda < \aleph_1$. Then $\lambda \leq \aleph_0$. Thus $\lambda^{\aleph_0} \leq \aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1 < \aleph_2$. Now suppose that $\lambda = \aleph_1$. Since $\aleph_1$ is a regular cardinal (this is well-known and provable in ZFC), it follows that $\aleph_0 < \aleph_1 = cf(\aleph_1)$. Thus by the above fact, we have $\lambda^{\aleph_0} = \aleph_1^{\aleph_0} = \aleph_1 < \aleph_2$. The proof is now complete. $\square$

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