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SOME RESULTS ON JÓNSSON MODULES OVER A COMMUTATIVE RING

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Abstract. Let M be an infinite unitary module over a commutative ring R with identity. M is called Jónsson over R provided every proper submodule of M has smaller cardinality than M; M is large if M has cardinality larger than R. Extending results of Gilmer and Heinzer, we prove that if M is Jónsson over R, then either M is isomorphic to R and R is a field, or M is a torsion module. We show that there are no large Jónsson modules of regular or singular strong limit cardinality. In particular, the Generalized Continuum Hypothesis (GCH) implies there are no large Jónsson modules. Necessary and sufficient conditions are given for an infinitely generated Jónsson module to be countable. As applications, we prove there are no large uniserial or Artinian modules. Under the GCH, we derive a new characterization of the quasi-cyclic groups.

In this paper, all rings are assumed to be commutative with identity, and all modules are assumed to be unitary.

1. Preliminaries

In this section, we acquaint the reader with some basic properties of Jónsson modules. Formally, an infinite module M over a commutative ring R is said to be *Jónsson over R* iff every proper submodule of M has smaller cardinality than M. We begin with a few examples:

Example 1. *Let F be an infinite field. Then F becomes a module over itself. Since F has only trivial ideals, it is easy to see that F is a Jónsson module. More*

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generally, if R is a ring, J a maximal ideal of R of infinite residue, then R/J is a Jónsson module over R.

Example 2 ([Fu2]). *Let p be a prime number. The subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all elements of the form $\frac{a}{p^n}$ where $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ (mod $\mathbb{Z}$) is the so-called quasi-cyclic group of type p^∞, denoted by $\mathbb{Z}(p^\infty)$. It is infinite, but all proper subgroups are cyclic of order p^n for some n.*

The next example is due to Gilmer and Heinzer ([GH]).

Example 3. *Let D be a one-dimensional Noetherian domain with a maximal ideal J of finite index. Let K be the fraction field of D, and let V be a valuation overring of D with center J on D. Then K/V is a faithful Jónsson module over D.*

It was first proved by Scott in [Sc] that the only Jónsson modules over $\mathbb{Z}$ are the quasi-cyclic groups. We will give a very short proof of this fact. We begin with the following proposition given in [GH], and include a different proof.

Proposition 1 (Proposition 2.5, [GH]). *Suppose that M is a Jónsson module over the ring R. Let $r \in R$ be arbitrary. Then:*
(1) Either $rM = M$ or $rM = 0$.
(2) Ann(M)= $\{s \in R : (\forall m \in M)(sm = 0)\}$ is a prime ideal of R.

Proof. We assume that M is a Jónsson module over R and let $r \in R$. Define $\varphi : M \to rM$ by $\varphi(m) = rm$. Clearly this is an R-module epimorphism. Let K be the kernel of this map. Then $rM \cong M/K$. In particular, this implies that $|K||rM| = |M|$. By elementary cardinal arithmetic, it is clear that either $|K| = |M|$ or $|rM| = |M|$. If $|K| = |M|$, then since M is a Jónsson module, $K = M$ and thus $rM = 0$. Otherwise $|rM| = |M|$ and it follows that $rM = M$. This establishes (1). As for (2), we suppose that r, s do not annihilate M. Then by (1), $rM = M$ and $sM = M$. Thus $rsM = M$ and so rs does not annihilate M. This proves (2). □

Before stating our next result, we note that a Jónsson module is indecomposable. To see this, suppose that the Jónsson module M has a direct sum decomposition $M = N \bigoplus P$. Then by elementary cardinal arithmetic, either $|N| = |M|$ or $|P| = |M|$. Since M is a Jónsson module, this forces either $N = M$ or $P = M$. We deduce the following corollary.

Corollary 1. *There are no Jónsson modules over a finite ring.*

Proof. Suppose that M is a Jónsson module over the finite ring R. Let P be the annihilator of M. Then M becomes a Jónsson module and a vector space over the finite field R/P, which is impossible since M is infinite and indecomposable. □

We now give a simple proof of the following old result of Scott in [Sc].

Theorem 1.1 (Scott)**.** *The only Jónsson modules over the ring $\mathbb{Z}$ of integers are the quasi-cyclic groups $\mathbb{Z}(p^\infty)$.*

Proof. Let M be a Jónsson module over $\mathbb{Z}$. The annihilator of M in $\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. If Ann(M)= (p) for some prime number p, then M becomes a Jónsson module over the finite ring $\mathbb{Z}/(p)$, contradicting Corollary 1. Thus Ann(M)= $\{0\}$. In particular, Proposition 1 implies that M is a divisible abelian group. It follows from the structure theorem for divisible abelian groups (see for example [Fu1], p. 64) that M is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ for various primes p. Since M is indecomposable, this forces $M = \mathbb{Q}$ or $M = \mathbb{Z}(p^\infty)$ for some prime p. Since $\mathbb{Q}$ is clearly not a Jónsson module, we get $M = \mathbb{Z}(p^\infty)$ and the proof is complete. □

2. Torsion-Free Jónsson Modules

In Theorem 3.1 of [GH], the authors prove (among other things) that every countably infinite, infinitely generated Jónsson module is a torsion module. We shall generalize this result, and begin with two preliminary lemmas. The first lemma is Proposition 2.2 of [GH]. The proof is a straightforward consequence of Proposition 1 and is omitted.

Lemma 1 (Proposition 2.2, [GH])**.** *Let R be an infinite ring. Then R is a Jónsson module over itself iff R is a field.*

Lemma 2. *Let M be a Jónsson module over the ring R, and suppose that N is a proper submodule of M. Then M/N is also a Jónsson module over R.*

Proof. We assume that M is a Jónsson module and that N is a proper submodule of M. Then $|N| < |M|$, and so $|M/N| = |M|$. Let P be a submodule of M/N of cardinality $|M/N|$, and let $\varphi : M \to M/N$ be the canonical map. Then it is easy to see that $\varphi^{-1}(P)$ has cardinality $|M|$. Since M is a Jónsson module, we get that $\varphi^{-1}(P) = M$. Hence $P = M/N$ and M/N is a Jónsson module. □

We now characterize the torsion-free Jónsson modules.

Proposition 2. *Let R be a ring, and suppose that M is an infinite torsion-free module over R. Then M is a Jónsson module over R iff R is a field and $M \cong R$.*

Proof. Suppose that R and M are as stated above. Since M is nontrivial and torsion-free, R is a domain. If R is a field and $M \cong R$, then trivially M is a Jónsson module. Thus we assume that M is a Jónsson module over R. We suppose first that $|M| \leq |R|$. Choose any nonzero $m \in M$. Since M is torsion-free, the mapping $r \mapsto rm$ is injective. In particular, this shows that $|R| \leq |M|$ and thus $|M| = |R|$. In particular, since M is a Jónsson module, we get $M = (m)$. If I is the annihilator of (m), then it is clear that $M \cong R/I$. Since M is torsion-free, we must have $I = \{0\}$. Thus R is a Jónsson module over itself, and so $M \cong R$ and R is a field by Lemma 1. We now suppose that $|M| > |R|$ and derive a contradiction. Let S denote the set of nonzero elements of R. Then $S^{-1}M$ becomes a vector space over the field $S^{-1}R$ of dimension $|M| = |S^{-1}M|$. Clearly, this implies that there exists a subset X of M of cardinality $|M|$ which is linearly independent over R. Since M is a Jónsson module, this implies that $M = \bigoplus_{x \in X} Rx$. This is a contradiction to the fact that M must be indecomposable, and the proof is complete. □

We use this result to prove the following theorem.

Theorem 2.1. *Suppose that M is a Jónsson module over the ring R. Then either $M \cong R$ and R is a field, or M is a torsion module.*

Proof. We suppose that M is a Jónsson module over the ring R. We first prove the theorem in the special case where R is a domain. We let T be the torsion submodule of M. If $|T| = |M|$, then $T = M$ and M is a torsion module. Thus we assume that $|T| < |M|$. By Lemma 2, M/T is a Jónsson module over R. Since M/T is torsion-free, it follows from Proposition 2 that R is a field. But then M is a Jónsson module over the field R. Since M is indecomposable, we are forced to conclude that $M \cong R$.

Now for the general case. Let P be the annihilator of M in R. By Proposition 1, P is a prime ideal and hence R/P is a domain. M is naturally a Jónsson module over R/P. Hence from the above work, M is either a torsion or a torsion-free module over R/P. If M is a torsion module over R/P, then it is clear that M is a torsion module over R. Thus we assume that M is a torsion-free module over R/P. If M is torsion-free over R, we're done. Thus we suppose that M is not torsion-free over R. Then there exists a nonzero $m \in M$ and a nonzero $r \in R$ with $rm = 0$. But viewing M as an R/P-module, we get that $\overline{r}m = 0$. As M is a torsion-free R/P module, this forces $r \in P$. But recall that P is the annihilator of M in R, and thus r annihilates all of M, so that M is a torsion module. This completes the proof. □

3. Large Jónsson Modules

Let us agree to call an infinite R-module *large* if M has cardinality greater than R. In this section, with Theorem 2 in hand, we investigate the existence of large Jónsson modules. We begin with the following immediate corollary of Theorem 2.

Corollary 2. *Every large Jónsson module is torsion.*

Before proving our next proposition, we recall some definitions from set theory.

Definition 1. Let κ be an infinite cardinal. The *cofinality* cf(κ) of κ is the least cardinal λ such that κ is the sum of λ many cardinals, each smaller than κ. The cardinal κ is called *regular* if cf(κ)=κ and *singular* if cf(κ)< κ.

The regular cardinals include $\aleph_0$ as well as every successor cardinal; that is, every cardinal of the form $\aleph_{\alpha+1}$ for some ordinal α. It is also well-known that cf(κ) is a regular cardinal for every infinite κ.

We now prove the following result which generalizes Corollary 1.

Proposition 3. *Suppose* $|R| <$*cf*$(|M|)$. *Then* M *is not Jónsson over* R.

Proof. By contraposition. Suppose M is Jónsson over R. By Theorem 2.1, either M is isomorphic to R (and R is a field), or M is torsion. In the former case, $|R| = |M| \geq$cf$(|M|)$. In the latter case, since M is Jónsson, modding out by the annihilator, we may assume M is faithful and R is a domain. So the submodule $M[r] = \{m \in M : rm = 0\}$ has cardinality less than M. Since M is torsion, $M = \bigcup_{r\in R-\{0\}} M[r]$, and it follows that $|R| \geq$cf$(|M|)$. The proof is complete. □

Corollary 3. *There does not exist a large Jónsson module of regular cardinality.*

The previous proof shows that, in fact, if there exists a cardinal $\lambda < |M|$ such that for every $r \in R$, $M[r]$ has cardinality at most λ, then M is not a large Jónsson module over R.

We consider next the question of the existence of large Jónsson modules. We show that their existence cannot be proved in ZFC, but the question of whether their nonexistence can be proved in ZFC remains open. We begin with the following result of Ecker in [Ec]:

Proposition 4 ([Ec]). *Let R be an infinite ring and I a maximal independent set in an R-module M. Then we have the following facts:*

(1) If $|I| = 0$, then $M = \{0\}$.
(2) If $|I| = 1$, then $|M| \leq 2^{|R|}$.
(3) If $|I| > 1$, then $|M| \leq |I|^{|R|}$.

Before proving our next result, we recall the following definition.

Definition 2. Let κ be an infinite cardinal. κ is called a *strong limit cardinal* provided that for every $\lambda < \kappa$, one has $2^{\lambda} < \kappa$.

Note that if κ is a strong limit, and $\alpha, \beta < \kappa$, then $\alpha^{\beta} \leq (2^{\alpha})^{\beta} = 2^{\alpha \cdot \beta} < \kappa$. We now prove the following theorem:

Theorem 3.1. *Assume that every singular cardinal is a strong limit. Then there are no large Jónsson modules.*

Proof. Suppose by way of contradiction that M is a large Jónsson module over R and that every singular cardinal is a strong limit. By Corollary 3, the cardinality of M is singular. Let X be a maximal independent set in M. By Ecker's result, since $|M|$ is a strong limit cardinal, it follows that $|X| = |M|$. Hence $M = \bigoplus_{x \in X} Rx$, contradicting the fact that M is indecomposable. □

We immediately obtain the following corollary.

Corollary 4. *The Generalized Continuum Hypothesis implies that there are no large Jónsson modules.*

Thus it is impossible to prove in ZFC that large Jónsson modules exist. R.G. Burns, F. Okoh, H. Smith, and J. Wiegold have shown in [Bu] that if R is Noetherian and M is a large module over R, then M possesses an independent subset of size $|M|$. Hence the nonexistence of large Jónsson modules can be proved in ZFC over Noetherian rings.

4. Necessary and Sufficient Conditions for a Jónsson Module to be Countable

In this section, we give two necessary and sufficient conditions in order for an infinitely generated Jónsson module to be countable. The next lemma will allow us to prove a very useful result.

Lemma 3. *Let M be an infinite R-module, and let $r \in R, n \in \mathbb{N}$. Suppose that r^n annihilates M. Let $M[r]$ denote the submodule of M consisting of the elements of M annihilated by r. Then $|M[r]| = |M|$.*

PROOF. We prove this by induction on $n \in \mathbb{N}$. The case when $n = 1$ is trivially true. Thus we assume the lemma is true for some $n \in \mathbb{N}$. Suppose that M is an infinite R-module, $r \in R$, $n \in \mathbb{N}$, and r^{n+1} annihilates M. It is clear that $M/M[r] \cong rM$. Hence we get that $|M| = |rM||M[r]|$. As M is infinite, it follows that either $|M[r]| = |M|$ or $|rM| = |M|$. If $|M[r]| = |M|$, then we have what we want and we are done. Otherwise $|rM| = |M|$. Recall that r^{n+1} annihilates M, and therefore r^n annihilates rM. By the inductive hypothesis, we have $|(rM)[r]| = |rM| = |M|$. Clearly $(rM)[r] \subseteq M[r]$, and thus $|M[r]| = |M|$. This completes the proof. □

Proposition 5. *Suppose that M is a faithful Jónsson module over the ring R. Further, suppose that $r \in R$ is nonzero and that every element of M is annihilated by some power of r. Then M is countable.*

PROOF. For each positive integer n, we let M_n be the collection of elements of M annihilated by r^n. Clearly $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ and M is the union of the M_n's as n ranges over the positive integers. We claim that $M[r] = M_1$ is finite. Suppose by way of contradiction that M_1 is infinite. Then it follows from Lemma 3 that $|M_n| = |M_1|$ for every positive integer n. But since M is the union of the M_n's, it is clear that $|M| = |M_1|$. As M is Jónsson over R, $M = M_1 = M[r]$, contradicting that M is faithful. Thus M_1 is finite. It follows from Lemma 3 that M_n is finite for every positive integer n. This completes the proof. □

Recall that a module M is Artinian provided that the descending chain condition on submodules holds. M is almost Noetherian if M is not finitely generated, but every proper submodule of M is finitely generated. By modding out the annihilator, there is no loss of generality in restricting our attention to faithful Jónsson modules over a domain. Gilmer and Heinzer have shown that a finitely generated Jónsson module is cyclic, and since the faithful cyclic Jónsson modules are torsion-free, we already have complete information about them. Thus we focus on infinitely generated faithful Jónsson modules over a domain. Using the previous results, we prove the following equivalence.

Theorem 4.1. *Suppose that M is an infinitely generated faithful Jónsson module over the domain D. The following are equivalent:*

(a) M is countable.
(b) M is Artinian.
(c) M is almost Noetherian.

PROOF. We prove the equivalence of (a) and (b) and of (a) and (c):

(a)⇒ (b): this is trivial.

(b)⇒ (a): suppose that M is Artinian. It is well-known (see [WK], Lemma 1.7, for example) that $M = \bigoplus_{i=1}^{n} M[J_i]$ where the J_i are maximal in R, $n \in \mathbb{N}$, and $M[J_i]$ is the set of elements of M annihilated by a power of J_i. Since M is a Jónsson module, M is indecomposable, and we see that $M = M[J_i]$ for some i. In particular, this means that M is J_i-primary. Since M is infinitely generated, D cannot be a field (since then M would be isomorphic to D and would thus be cyclic). Hence J_i is nonzero. Pick any nonzero $r \in J_i$. It then follows that M is r-primary. We now conclude that M is countable by Proposition 5.

(a)⇒ (c): this is trivial.

(c)⇒ (a): suppose that M is almost Noetherian. Since M is infinitely generated, M cannot be torsion-free (lest M be cyclic). Let $m \in M$ be a nonzero torsion element annihilated by some nonzero $r \in R$. We claim that $\bigcup_{n=1}^{\infty} Ann_M(r^n) = M$. Let n be a positive integer. Since D is a domain and M is faithful, it follows easily from Proposition 1 that $r^n M = M$. In particular, we have $r^n x = m$ for some $x \in M$. But this means that $x \in Ann_M(r^{n+1})$, but $x \notin Ann_M(r^n)$, and hence the union $\bigcup_{n=1}^{\infty} Ann_M(r^n)$ is strictly ascending. In particular, the union isn't finitely generated. Since M is almost Noetherian, this implies that $M = \bigcup_{n=1}^{\infty} Ann_M(r^n)$. In particular, it follows that M is r-primary. It now follows from Proposition 5 that M is countable. This completes the proof. □

5. Applications

In this section we give some applications of the results of this paper. We furnish elementary proofs that there can be no large uniserial or Artinian modules. We then provide a new characterization of the quasi-cyclic groups $\mathbb{Z}(p^{\infty})$.

Proposition 6. *Large uniserial modules do not exist.*

Proof. Suppose by way of contradiction that there exists a ring R and an infinite uniserial R-module M of cardinality greater than $|R|$. Suppose that N is an arbitrary proper submodule of M. Pick any $m \in M$ not in N. Then since M is uniserial, we see that $N \subseteq (m)$, and thus $|N| \leq |R|$. We suppose first that R is finite. Then M is infinite but (by the previous remark) has all proper submodules finite. It follows that M is a Jónsson module over the finite ring R. This contradicts Corollary 1. Thus we are forced to conclude that R is infinite. Since M is a large module, we see that $|M| \geq |R|^+$. This implies that M possesses a submodule P of cardinality $|R|^+$. Since every proper submodule of M has size $\leq |R|$, this implies that $M = P$, and so $|M| = |R|^+$. Again, recall that every proper submodule of M has cardinality $\leq |R|$, and hence M is a Jónsson module.

But $|M| = |R|^+$, a regular cardinal, so we have a contradiction to Proposition 3. This completes the proof. □

We now prove that the same result holds for Artinian modules.

Proposition 7. *Large Artinian modules do not exist.*

Proof. Suppose by way of contradiction that there exists a ring R and an infinite Artinian module M of cardinality greater than $|R|$. We claim that M possesses a Jónsson module N of the same cardinality as M. If M is a Jónsson module, we are clearly done. Otherwise there exists a proper submodule M' of the same cardinality as M. If M' is a Jónsson module, we're done. Otherwise we can find a proper M''. Since M is Artinian, this process must terminate after finitely many steps, producing the desired submodule. This implies immediately that R cannot be finite (lest there exist an infinite Jónsson module over R). Now, since M has larger cardinality than R and R is infinite, clearly there exists a submodule N of M of cardinality $|R|^+$. But now N is also Artinian and by the above argument, there exists a Jónsson module over R of cardinality $|R|^+$, a regular cardinal. This contradicts Corollary 3 and completes the proof. □

The mathematician R. McKenzie proved the remarkable result, assuming the Generalized Continuum Hypothesis, that every Jónsson semigroup (the definition is the obvious one) is actually a group. Using his result, we can give an equally remarkable characterization of the quasi-cyclic groups $\mathbb{Z}(p^\infty)$. We first state a simple lemma needed in our characterization. The proof is analogous to the proof of Lemma 2 and is omitted.

Lemma 4. *Suppose that G is a Jónsson group, and H is a proper normal subgroup of G. Then G/H is also a Jónsson group.*

Our last lemma is a result of Strunkov. We refer the reader to [Str] for a proof.

Lemma 5 (Strunkov)**.** *Suppose that G is an infinite non-abelian group which is generated by more than two elements. Suppose further that the set of prime divisors of the orders of the elements of G is finite. Then G possesses a proper infinite subgroup.*

We are now in position to prove our characterization theorem. First, for a semigroup S, we let S^* denote the subsemigroup of S generated by all commutators $aba^{-1}b^{-1}$. Of course, an arbitrary semigroup may have no commutators at all, in which case we put $S^* := \emptyset$. We present the following theorem.

Theorem 5.1. *Assume the Generalized Continuum Hypothesis. Let S be an infinite semigroup. Then $S \cong \mathbb{Z}(p^\infty)$ for some prime number p iff S satisfies the following two conditions:*

(1) S is a Jónsson semigroup

(2) $S^* \neq S$

Proof. We first quickly verify that $\mathbb{Z}(p^\infty)$ satisfies (1) and (2). Note that any proper subsemigroup S of $\mathbb{Z}(p^\infty)$ is actually a subgroup of $\mathbb{Z}(p^\infty)$ since $\mathbb{Z}(p^\infty)$ is a torsion group. Thus S has smaller cardinality, since $\mathbb{Z}(p^\infty)$ is a Jónsson group. Thus (1) holds. (2) is trivial since $\mathbb{Z}(p^\infty)$ is abelian.

Conversely, suppose that S is an arbitrary semigroup satisfying (1) and (2) above. Then McKenzie's result shows that S is a group. We assume S is uncountable and derive a contradiction. Since S is a Jónsson semigroup, it follows that S^* has smaller cardinality than S (note that since S is a group, S^* is actually a subsemigroup). But then it is easy to see that the commutator subgroup S' must also have smaller cardinality than S. By Lemma 4, S/S' is a Jónsson group. Since S/S' is abelian, it follows from Theorem 1.1 that $S/S' \cong \mathbb{Z}(p^\infty)$. But $|S/S'| = |S|$, and S is uncountable, so this is impossible. Thus we are forced to conclude that S is countable. Note that the condition on S clearly implies that every element of S has finite order (lest $S \cong \mathbb{Z}$). If $S^* = \{e\}$, then S is abelian and $S \cong \mathbb{Z}(p^\infty)$ for some prime number p, which is what we wanted to show. Thus we suppose that $S^* \neq \{e\}$ and derive a contradiction. By (2), it is clear that S^* must be finite. Since S is a torsion group, we see that S^* must be a group, and hence $S^* = S'$. Next we invoke Strunkov's result. Since S' is nontrivial, S is nonabelian. Further, since S/S' is abelian, $S/S' \cong \mathbb{Z}(p^\infty)$ for some prime number p. Note that $\mathbb{Z}(p^\infty)$ is not finitely generated, and thus neither is S. We also have that every element of $\mathbb{Z}(p^\infty)$ has order p^n for some natural number n. It follows that for any $x \in S$, $x^{p^n} \in S'$ for some natural number n. Since S' is finite, it follows that the set of prime divisors of the orders of the elements of S must be finite. By Strunkov's result, we see that S possesses a proper infinite subgroup, contradicting the fact that S is a countable Jónsson semigroup (and since S is a group, S is a countable Jónsson group). This contradiction completes the proof. □

The following result follows easily from the above proof (and does not depend on GCH). We leave its proof to the reader.

Corollary 5. *Let G be a Jónsson group with derived subgroup G'. Then $G' = \{e\}$ or $G' = G$.*

Thus a nonabelian Jónsson group is, in some sense, highly nonabelian.

6. Open Problems

We close the paper by stating two open problems we feel are interesting. We first give a short historical account to motivate interest. In model theory, a *Jónsson model* is a model $\mathbf{X} = (X, R, F)$ where R and F are countable collections of finitary relations and operations on the set X such that every elementary submodel of $\mathbf{X}$ has smaller cardinality. A Jónsson group is thus a group in which all proper subgroups have smaller cardinality. It was conjectured by Kurosh in the sixties that uncountable Jónsson groups exist. In 1980, Shelah proved the existence of a Jónsson group of size $\aleph_1$ ([Sh]). We ask an analogous question about modules.

Question 1. *Does there exist an infinitely generated uncountable Jónsson module?*

Erdös and Hajnal showed in [Er] that the axiom of constructibility ($V = L$) implies that there are Jónsson models of every infinite cardinality. Since $V = L$ implies GCH, it follows from our results that $V = L$ implies the nonexistence of large Jónsson modules. We would like to know the answer to the following question.

Question 2. *Is ZFC sufficient to prove there does not exist a large Jónsson module M over R such that cf($|M|$)$\leq |R|$?*

For an introduction to Jónsson models and algebras, we refer the reader to the excellent survey article [Co] and to the most recent edition of the book [Ch] and the survey [Eis]. For more advanced results, see [Sh2] and his subsequent papers on Jónsson models.

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