STRONGLY JÓNSSON AND STRONGLY HS MODULES

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ABSTRACT. Let R be a commutative ring with identity and let M be an infinite unitary R-module. Then M is a *Jónsson module* provided every proper R-submodule of M has smaller cardinality than M. In this note, we strengthen this condition and call an R-module M (which may be finite) *strongly Jónsson* provided distinct R-submodules of M have distinct cardinalities. We present a classification of these modules, and then we study a sort of dual notion. Specifically, we consider modules M for which M/N and M/K have distinct cardinalities for distinct R-submodules N and K of M; we call such modules *strongly HS* (see the introduction for etymology). We conclude the paper with a classification of the strongly HS modules over an arbitrary commutative ring.

1. INTRODUCTION

Let R be a commutative ring with identity, and let M be an infinite unitary R-module. Then M is called a *Jónsson module* provided every proper R-submodule of M has smaller cardinality than M. Such modules have received attention in the literature; specifically, they have been studied by Robert Gilmer, Bill Heinzer, and the author (among others). We refer the reader to Gilmer and Heinzer ([5], [6], [7], [8]) and Oman ([12], [16], [17], [21]) for results on Jónsson modules and related structures. Dually, rings R for which R/I is finite for every nonzero two-sided ideal I of R were studied some time ago by Chew and Lawn ([2]); they call such rings *residually finite*. Many of their results were extended (in particular, to rings without identity) by Levitz and Mott in [13]. The notion of residual finiteness was generalized (in the commutative setting) by Salminen and the author. To wit, let R be a commutative ring with identity and let M be an infinite unitary R-module. Say that M is *homomorphically smaller* (HS for short; this terminology is due to Ralph Tucci) if $|M/N| < |M|$ for all nonzero R-submodules N of M. Various structural theorems on HS modules were obtained in Oman and Salminen [20]. Many of these results were subsequently generalized by Salminen and the author in [19].

In this article, we strengthen the cardinality assumptions in the definition of Jónsson and HS modules, but drop the requirement that the modules be infinite. Specifically, let R be a commutative ring with identity, and suppose that M is a unitary R-module. Say that M is *strongly Jónsson* provided $|N| \neq |K|$ for distinct R-submodules N and K of M. We present a structure theorem for such modules over an arbitrary commutative ring. We then study a sort of dual defined as follows: Call M *strongly HS* provided $|M/N| \neq |M/K|$ for distinct R-submodules N and K of M. In the final section of the paper, we classify the strongly HS modules.

2. Preliminaries

All rings in this paper are commutative with identity and all modules are unitary.

In this section, we collect some lemmas to which we will refer throughout the paper. Several of the following results appear in the literature. However, as all the proofs we present are short and instructive, we include them so as to keep the paper reasonably self-contained. Before presenting our first definition, we remark that there will be occasions throughout the paper when we will consider module structures on an abelian group over multiple rings. To help mitigate confusion, the notation "$M \cong_R N$" will always mean that R is a ring and M and N are isomorphic *as R-modules*. We will also be very careful to clearly differentiate between ring isomorphism and module isomorphism.

Definition 1 (Gilmer [4], p. 8)**.** *Let R be a ring, and let M be an abelian group which is a left module over both the rings R and S. Say that the structure of M as an R-module is essentially the same as the structure of M as an S-module if and only if $Rm = Sm$ for every $m \in M$.*

It is easy to see that if the structure of M as an R-module is essentially the same as the structure of M as an S-module, then the set of R-submodules of M is the same as the set of S-submodules of M.

Recall that if M is an R-module, then the *annihilator* of M in R is the set $ann_R(M) := \{r \in R : rM = \{0\}\}$. One checks easily that $ann_R(M)$ is an ideal of R. Further, M is naturally an $R/ann_R(M)$-module via the scalar product $\overline{r} \cdot m := rm$. The following lemma is trivial to verify:

Lemma 1. *Let R be a ring, and let M be an R-module. Then the structure of M as an R-module is essentially the same as the structure of M as an $R/ann_R(M)$-module.*

A ring R is *local* provided R has a unique maximal ideal. Unfortunately, this terminology is not universal. Some authors require a local ring to be Noetherian (calling general rings with a unique maximal ideal *quasi-local*), but we do not adopt that convention in this paper.

We now recall the following fundamental definition, and then prove a lemma which is perhaps not as well-known.

Definition 2. *Let R be a ring, I an ideal of R, and let M be an R-module. Say that M is I-primary provided that for every $m \in M$, there exists a positive integer k such that $I^k m = \{0\}$.*

Lemma 2. *Let R be a ring, and let J be a maximal ideal of R. Further, suppose that M is a J-primary R-module. For every $s \in R - J$ and every $m \in M$, there exists a unique $m' \in M$ such that $m = sm'$.*

Proof. Let R, J, and M be as in the statement of the lemma. Now let $s \in R - J$ and $m \in M$ be arbitrary. We first prove existence. By assumption, there exists a positive integer k such that $J^k m = \{0\}$. Since J is maximal and $s \notin J$, we conclude that $(J^k, s) = R$. But then $y + sx = 1$ for some $y \in J^k$ and $x \in R$. Multiplying through by m and using the fact that $ym = 0$, we get $sxm = m$. Thus $m = s(xm)$, proving existence. As for uniqueness, suppose that $m = sm' = sm''$ for some $m', m'' \in M$. Then $s(m' - m'') = 0$. Again, there is a positive integer k such that $J^k(m' - m'') = 0$. But then $(J^k, s) \subseteq ann_R(m' - m'')$. As above, $(J^k, s) = R$. Thus $1 \in ann_R(m' - m'')$, and we conclude that $m' = m''$. $\square$

Corollary 1. *Suppose that R is a ring and that M is a J-primary R-module for some maximal ideal J of R. Then M has a natural module structure over the local ring R_J (the localization of R at the maximal ideal J) given by $\frac{r}{s} \cdot m := \frac{rm}{s}$, where $\frac{rm}{s}$ denotes the unique $m' \in M$ such that $rm = sm'$.*

It is well-known that the cardinality of a finite local ring is a power of a prime (see Mcdonald [14], for example; we present the easy proof below). We will need the following more general result:

Lemma 3. *Let R be a local ring, and let M be a finite R-module (that is, the cardinality of M is finite). Then $|M|$ is a power of a prime.*

Proof. We begin by proving that every finite local ring has cardinality that is a power of a prime. Thus let (R, J) be a finite local ring (J is the unique maximal ideal of R). Then R/J is a finite field, whence $|R/J| = p^k$ for some prime p and positive integer k. Since R/J has characteristic p, it follows that $p \cdot 1 \in J$ (here, $p \cdot 1$ denotes $1 + 1 + \cdots + 1$ (p times)). Since R is finite, every prime ideal of R is maximal. We conclude that J is the unique prime ideal of R, whence $J = Nil(R)$, the nilradical of R. But since $p \cdot 1 \in J$, we conclude that $p^n \cdot 1 = 0$ for some positive integer n. Hence $p^n \cdot r = 0$ for every $r \in R$, and we see that the additive order of every element of R divides p^n. It follows that $|R| = p^j$ for some positive integer j.

More generally, assume that (S, J) is a local ring (which may be infinite) and that M is a finite S-module. We will prove that $|M|$ is a power of a prime. If $M = \{0\}$, the result is patent, so assume that M is nonzero. Fix $m_0 \in M - \{0\}$. Setting

$I := ann_S(Sm_0)$, we have $Sm_0 \cong_S S/I$. Since M is finite, so is Sm_0, whence S/I is a finite local ring. By what we just proved, $|S/I| = p^k$ for some prime p and positive integer k. Further, the module isomorphism theorems yield

$$(2.1) \qquad |S/J| \cdot |J/I| = |S/I| = p^k,$$

whence there exists a positive integer n such that

$$(2.2) \qquad |S/J| = p^n.$$

We will now prove that $|M|$ is a power of p. Let $m \in M - \{0\}$ be arbitrary, and set $I' := ann_S(Sm)$. As above, $Sm \cong_S S/I'$, and $|S/I'| = q^t$ for some prime q and positive integer t. But now (2.1) and (2.2) yield that $|S/J| = p^n$ is a factor of q^t. We deduce that $p = q$, and hence $|Sm| = p^t$. Thus the additive order of m is a power of p. As $m \in M - \{0\}$ was arbitrary, we conclude that $|M|$ is a power of p, and the proof is complete. $\qquad \square$

We conclude this section with a brief discussion of a class of rings which will play a pivotal role in the sequel. Let R be a ring. Then R is a *discrete valuation ring* (DVR) provided R is a principal ideal domain with a unique nonzero prime ideal (m). Since such an R is a unique factorization domain, every nonzero nonunit of R is of the form um^k for some unit u and positive integer k. Thus the set of proper nonzero ideals of R is precisely $\{(m^k) : k > 0\}$. It follows from this fact that the set of ideals of R is linearly ordered by inclusion. Moreover,

Lemma 4. *Let (V, m) be a DVR, and let K be the quotient field of V. For every positive integer k, let M_k be the V-submodule of K/V defined by $M_k := \{V + \frac{v}{m^k} : v \in V\}$. Then the modules M_k are precisely the proper nonzero V-submodules of K/V. Moreover, $M_k \cong_V V/(m^k)$ for every positive integer k.*

Proof. It is trivial to verify that each M_k is a proper nonzero V-submodule of K/V. Note that every nonzero element of K is of the form um^k for some unit $u \in V$ and some integer k. It follows that

(*) every nonzero element of K/V can be expressed in the form $V + \frac{u}{m^k}$ for some unit $u \in V$ and some positive integer k.

Thus $K/V = \bigcup_{k>0} M_k$. Now let L be a proper nonzero V-submodule of K/V. We will show that $L = M_k$ for some positive integer k. Since L is proper, there exists some $a > 0$ for which $M_a \not\subseteq L$. Let i be least with the property that $M_i \not\subseteq L$. It is clear from (*) that $i > 1$. We claim that $L = M_{i-1}$. By leastness of i, we must simply show that $L \subseteq M_{i-1}$. Consider an arbitrary nonzero element $V + \frac{u}{m^j}$ of L. Since $M_i \not\subseteq L$, we conclude that $j < i$. But then $V + \frac{u}{m^j} \in M_j \subseteq M_{i-1}$, whence $L \subseteq M_{i-1}$. Finally, for any $k > 0$, it is straightforward to verify that the map $\varphi : V \to M_k$ defined

by $\varphi(v) := V + \frac{v}{m^k}$ is a surjective V-module homomorphism with kernel (m^k). Thus $M_k \cong_V V/(m^k)$. $\qquad\square$

3. Strongly Jónsson Modules

Let R be a ring. Recall from the introduction that an R-module M is *strongly Jónsson* provided $|N| \neq |K|$ for distinct R-submodules N and K of M. In particular, every proper R-submodule of M has smaller cardinality than M. Thus an infinite, strongly Jónsson module is a Jónsson module. As a jumping-off point, we classify the strongly Jónsson abelian groups (the following proposition may appear somewhere in the literature as an exercise). We remind the reader that for any prime p, the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is the group $\mathbb{Q}/\mathbb{Z}_{(p)}$ (here $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p)).

Proposition 1. *Let G be an abelian group. Then G is strongly Jónsson (i.e. G is a strongly Jónsson $\mathbb{Z}$-module) if and only if $G \cong \mathbb{Z}(p^\infty)$ for some prime p or $G \cong \mathbb{Z}/(n)$ for some positive integer n.*

Proof. It is known that for any prime p, every proper subgroup of $\mathbb{Z}(p^\infty)$ is finite of order p^n for some non-negative integer n. Moreover, for every non-negative integer n, $\mathbb{Z}(p^\infty)$ possesses a *unique* subgroup of cardinality p^n (see Fuchs [3], pp. 23-25). It follows that distinct subgroups of $\mathbb{Z}(p^\infty)$ have distinct cardinalities. It is also well-known that $\mathbb{Z}/(n)$ enjoys this property for every positive integer n (Hungerford [10], p. 37). Thus $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}/(n)$ are strongly Jónsson.

Conversely, suppose that G is a strongly Jónsson abelian group. If G is infinite, then G is an abelian Jónsson group, whence $G \cong \mathbb{Z}(p^\infty)$ for some prime p by an old result of Scott (Scott [22]). Now assume that G is finite. Then by The Fundamental Theorem of Finitely Generated Abelian Groups, G is a finite direct sum of cyclic groups of prime power order. Clearly, no two distinct summands can have cardinality a power of the same prime p, lest G possess two distinct subgroups of order p. We conclude that $G \cong \mathbb{Z}/(n)$ for some positive integer n. $\qquad\square$

Employing a classical theorem of Baer, we can show that the conclusion of the previous proposition holds even without assuming that G is abelian.

Proposition 2. *Let G be a group with the property that distinct subgroups of G have distinct cardinalities. Then G is abelian.*

Proof. Assume that distinct subgroups of G have distinct cardinalities, and suppose by way of contradiction that G is nonabelian. We first claim that every subgroup of G is normal. Indeed, let $H < G$, and let $g \in G$ be arbitrary. Then clearly $|H| = |gHg^{-1}|$, whence by the condition on G, $H = gHg^{-1}$, and H is normal. Hence G is a Hamiltonian group (that is, G is nonabelian and all subgroups of G are normal).

An old result of Baer[1] (Baer [1]) yields that $G \cong Q_8 \times P$, for some torsion abelian group P that has no elements of order 4 (Q_8 denotes the quaternion group of order 8). But then we are forced to conclude that Q_8 inherits the property that distinct subgroups have distinct cardinalities. However, Q_8 has three subgroups of order 4, a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The goal of this section is to generalize Proposition 1 to modules over an arbitrary ring. We begin by reminding the reader that a ring R is a *principal ideal ring* provided every ideal of R is principal. A domain D is a *Dedekind domain* if D admits unique factorization of ideals, that is, if every proper nonzero ideal of D is uniquely a finite product of prime ideals. We will need several alternative characterizations of Dedekind domains. The following assertion is an amalgam of Theorem 37.1, Theorem 37.8, and Theorem 38.5 of Gilmer [4].

Fact 1. *Let D be a domain which is not a field. Then the following are equivalent:*

(a) *D is a Dedekind domain.*
(b) *D is one-dimensional, Noetherian, and integrally closed.*
(c) *If A is any ideal of D and if a is a nonzero element of A, then there is an element b of A such that $A = (a, b)$.*
(d) *Every proper homomorphic image of D is a principal ideal ring.*
(e) *D is Noetherian and for every maximal ideal J of D, the localization D_J is a discrete valuation ring.*

We now prove a technical lemma which will be of use to us throughout the remainder of the paper.

Lemma 5. *Let D be a Dedekind domain which is not a field, and let P be a prime ideal of D. Then for every positive integer n,*

$$|D/P^n| = |D/P|^n \ (\text{the cardinalities may be infinite}).$$

Proof. Assume that D is a Dedekind domain which is not a field and that P is a prime ideal of D. If $P = \{0\}$, then the assertion of the lemma reduces to $|D| = |D|^n$. Since D is not a field, D is infinite, whence $|D| = |D|^n$ holds via basic cardinal arithmetic. Suppose now that P is nonzero. Then (b) of Fact 1 implies that P is maximal. We now proceed by induction. Clearly the assertion is true for $n = 1$. Assume for some positive integer n that

$$(3.1) \qquad\qquad\qquad |D/P^n| = |D/P|^n.$$

To prove the assertion for $n + 1$, note first that $|D/P^{n+1}| = |D/P^n||P^n/P^{n+1}|$. Thus it suffices by the inductive hypothesis to prove that $|P^n/P^{n+1}| = |D/P|$. Toward this

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[1] We thank Professor Arturo Magidin for the reference to Baer's paper.

end, observe that $P^n \neq P^{n+1}$, lest unique factorization be violated. Thus P^n/P^{n+1} becomes a nonzero vector space over the field D/P. To finish the proof, it suffices to show that P^n/P^{n+1} is one-dimensional over D/P. Choose any nonzero element $a \in P^{n+1}$. By (c) of Fact 1, there exists $b \in P^n$ such that $P^n = (a, b)$. But then P^n/P^{n+1} can be generated by $b \pmod{P^{n+1}}$ over D/P. We conclude that P^n/P^{n+1} is one-dimensional over D/P. $\qquad\square$

We use the previous lemma to give an example of a class of finite rings which are strongly Jónsson as modules over themselves. For brevity, let us call a ring R which is strongly Jónsson as a module over itself a *strongly Jónsson ring*. We also agree to denote the set of cardinalities of the ideals of a ring R by $\mathcal{C}(R)$ (that is, $\mathcal{C}(R) := \{|I| : I \text{ is an ideal of } R\}$).

Example 1. *Let (V, m) be a DVR. Suppose further that $V/(m)$ is a finite field of order p^α, and let k be a positive integer. Set $R := V/(m^k)$. Then R is a strongly Jónsson ring. Furthermore, $\mathcal{C}(R) = \{1, p^\alpha, p^{2\alpha}, \ldots, p^{k\alpha}\}$.*

Proof. The set of ideals of R is precisely $\{(m^k)/(m^k), (m^{k-1})/(m^k), \ldots, (m^0)/(m^k)\}$. To finish the proof, it suffices to show that for each i, $0 \leq i \leq k$, one has $|(m^{k-i})/(m^k)| = p^{i\alpha}$. To verify this, note that

$$(3.2) \qquad |V/(m^k)| = |V/(m^{k-i})||(m^{k-i})/(m^k)|.$$

Invoking Lemma 5, (3.2) becomes

$$(3.3) \qquad p^{k\alpha} = p^{(k-i)\alpha}|(m^{k-i})/(m^k)|.$$

Solving, we get $|(m^{k-i})/(m^k)| = p^{i\alpha}$, as required. $\qquad\square$

To illustrate with a specific example, let p be a prime, and let α be a positive integer. Further, let $\mathbb{F}_{p^\alpha}$ be the field of order p^α. Then the power series ring $(\mathbb{F}_{p^\alpha}[[t]], (t))$ is a discrete valuation ring with residue field (isomorphic to) $\mathbb{F}_{p^\alpha}$.

It is easy to see that the direct sum of the groups $\mathbb{Z}/(n)$ and $\mathbb{Z}/(m)$ is strongly Jónsson (as a $\mathbb{Z}$-module) if and only if m and n are relatively prime. A natural question is the following: When is a direct product of strongly Jónsson rings strongly Jónsson? It is possible for the direct product of two strongly Jónsson rings to be strongly Jónsson even if the rings are powers of the same prime, as the ring $R := \mathbb{F}_p \times \mathbb{F}_{p^2}$ witnesses. On the other hand, $S := \mathbb{F}_p \times \mathbb{F}_{p^2} \times \mathbb{F}_{p^3}$ has two distinct ideals of cardinality p^3, whence is not strongly Jónsson. We now pause to collect some additional terminology which will aid us in answering this question.

Let S be a commutative semigroup, and let $S_1, S_2, \ldots, S_k$ be subsets of S. Then the *product set* $S_1 S_2 \cdots S_k$ is defined by $S_1 S_2 \cdots S_k := \{s_1 s_2 \cdots s_k : s_i \in S_i \text{ for } 1 \leq i \leq k\}$. Suppose further that each S_i is a finite set. Let us say that the collection $\{S_1, S_2, \ldots, S_k\}$ is *product-maximal* provided $S_1 S_2 \cdots S_k$ is as large as possible, that

is, if $|S_1 S_2 \cdots S_k| = |S_1 \times S_2 \times \cdots \times S_k|$. The following simple lemma gives a useful characterization of the product-maximal property.

Lemma 6. *Let S be a commutative semigroup, and let $S_1, S_2, \ldots, S_k$ be finite subsets of S. Then $\{S_1, S_2 \ldots, S_k\}$ is product-maximal if and only if the following property holds:*

(P) If $x_1 x_2 \cdots x_k = y_1 y_2 \cdots y_k$ with each $x_i, y_i \in S_i$, then $x_i = y_i$ for all $i, 1 \leq i \leq k$.

Proof. Assume that S is a commutative semigroup and that $S_1, S_2 \ldots, S_k$ are finite subsets of S. Suppose first that $\{S_1, S_2 \ldots, S_k\}$ is product-maximal. We will verify property (P). Define $\varphi : S_1 \times S_2 \cdots \times S_k \to S_1 S_2 \cdots S_k$ by $\varphi(s_1, s_2 \ldots, s_k) := s_1 s_2 \cdots s_k$. Clearly φ is onto. Since $\{S_1, S_2 \ldots, S_k\}$ is product-maximal, it follows that φ is a surjective map between two finite sets of the same cardinality. We conclude that φ is one-to-one. Property (P) now follows. We omit the easy proof of the converse. $\square$

Our interest in the previous lemma will be in the context of the semigroup $(\mathbb{Z}^+, \cdot)$ of positive integers under multiplication. In this setting, the reader may recall that our property (P) above is somewhat related to the following well-studied concept (due to Erdös) in additive number theory: A subset $S \subseteq \mathbb{Z}^+$ is a *multiplicative Sidon set* if and only if $ab = cd$ implies that $\{a, b\} = \{c, d\}$ for all $a, b, c, d \in S$. As far as we know, there is no literature on general product-maximal collections of subsets of $\mathbb{Z}^+$. Therefore, we pause to present two examples.

Example 2. *Let $S_1 = \{2, 3\}$ and $S_2 = \{4, 6\}$. Then $\{S_1, S_2\}$ is not product-maximal since $|S_1 S_2| = 3 \neq 4 = |S_1 \times S_2|$.*

Example 3. *If $S_1, S_2, \ldots S_k$ are pairwise relatively prime finite sets of positive integers (that is, if $i \neq j$ and $x \in S_i$, $y \in S_j$, then x and y are relatively prime), then $\{S_1, S_2, \ldots, S_k\}$ is product-maximal.*

We now give necessary and sufficient conditions for a finite product of finite strongly Jónsson rings to be strongly Jónsson. In what follows below, we remind the reader that $\mathcal{C}(R) := \{|I| : I$ is an ideal of $R\}$.

Lemma 7. *Let $R_1, R_2, \ldots, R_k$ be finite rings. Then $R_1 \times R_2 \times \cdots \times R_k$ is a strongly Jónsson ring if and only if*

(a) Each R_i is strongly Jónsson, and
(b) $\{\mathcal{C}(R_1), \mathcal{C}(R_2), \ldots, \mathcal{C}(R_k)\}$ is product-maximal (in the semigroup $(\mathbb{Z}^+, \cdot)$).

Proof. Assume that $R_1, R_2, \ldots, R_k$ are finite rings, and let $R := R_1 \times R_2 \times \cdots \times R_k$. Suppose first that R is strongly Jónsson. It is obvious that (a) holds via the natural injection of R_i into R. As for (b), we will prove that $\{\mathcal{C}(R_1), \mathcal{C}(R_2), \ldots, \mathcal{C}(R_k)\}$ has property (P). To see this, suppose that $a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_k$ where each $a_i, b_i \in \mathcal{C}(R_i)$. For each i, let I_i be an ideal of R_i of cardinality a_i and let J_i be an ideal of

R_i of cardinality b_i. Then since $a_1 a_2 \cdots a_k = b_1 b_2 \cdots b_k$, clearly $|I_1 \times I_2 \times \cdots \times I_k| = |J_1 \times J_2 \times \cdots \times J_k|$. As R is a strongly Jónsson ring, we conclude that $I_1 \times I_2 \times \cdots \times I_k = J_1 \times J_2 \times \cdots \times J_k$. But then $I_i = J_i$ for each i, whence $a_i = b_i$ for each i.

Conversely, suppose that (a) and (b) hold. We will show that R is strongly Jónsson. Thus we assume that $|I_1 \times I_2 \times \cdots \times I_k| = |J_1 \times J_2 \times \cdots \times J_k|$, where each I_i, J_i is an ideal of R_i. We will prove that $I_i = J_i$ for each i. We have that

$$(3.4) \qquad |I_1| \cdot |I_2| \cdots \cdot |I_k| = |J_1| \cdot |J_2| \cdots \cdot |J_k|.$$

Let $1 \le i \le k$ be arbitrary. We conclude from (b) that $|I_i| = |J_i|$. By (a), the ring R_i is a strongly Jónsson ring. Thus $I_i = J_i$, and the proof is complete. $\square$

We are almost ready to characterize the finite strongly Jónsson modules. First, we need one more technical lemma.

Lemma 8. *Let R be a ring, and let $I_1, I_2, \ldots, I_n$ be ideals of R. Suppose further that there exist positive integers $k_1, k_2, \ldots, k_n$ and distinct maximal ideals $J_1, J_2, \ldots, J_n$ of R such that*

(∗) $J_i^{k_i} \subseteq I_i$ *for all i, $1 \le i \le n$.*

Set $S := R/I_1 \times R/I_2 \times \cdots \times R/I_n$. Then the structure of the ring S as an R-module is essentially the same as the structure of S as an S-module (whence the R-submodules of S and the ideals of S coincide).

Proof. As in the statement of the lemma, set $S := R/I_1 \times R/I_2 \times \cdots \times R/I_n$, and let $x := (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n) \in S$ be arbitrary. We will show that $Rx = Sx$. It is clear that $Rx \subseteq Sx$. To prove the converse, it suffices to show that for each i, $(\bar{0}, \bar{0}, \ldots, \bar{r}_i, \bar{0}, \ldots, \bar{0}) \in Rx$. Without loss of generality, we may assume that $i = 1$ and that $n > 1$. Since the J_i are distinct maximal ideals, we conclude that $J_2^{k_2} \cap J_3^{k_3} \cap \ldots \cap J_n^{k_n} \nsubseteq J_1$. Let $\alpha \in (J_2^{k_2} \cap J_3^{k_3} \cap \ldots \cap J_n^{k_n}) - J_1$. It follows from (∗) that $\alpha \cdot x = (\overline{\alpha r}_1, \bar{0}, \ldots, \bar{0})$. Since $\alpha \notin J_1$, we see that $(\alpha, J_1^{k_1}) = R$. Thus there exists $\beta \in R$ and $y \in J_1^{k_1}$ such that $\beta \alpha + y = 1$. But via (∗), we conclude that (modulo I_1), $\overline{\beta \alpha} = \bar{1}$. Thus $(\bar{r}_1, \bar{0}, \ldots, \bar{0}) = (\beta \alpha) x \in Rx$. $\square$

Finally, we are able to characterize the finite strongly Jónsson modules. We will make use of the following well-known result on Artinian modules (see Weakley [24], Lemma 1.7).

Fact 2. *Let R be a ring, and suppose that M is an Artinian R-module. For every maximal ideal J of M, let $M[J] := \{m \in M : J^k m = \{0\}$ for some positive integer $k\}$ ($M[J]$ is called the J-torsion submodule of M). Then there exist finitely many maximal ideals $J_1, J_2, \ldots, J_n$ of R such that*

$$M = \bigoplus_{1 \leq i \leq n} M[J_i].$$

Proposition 3. *Let R be a ring, and let M be a finite nontrivial R-module. Then M is a strongly Jónsson R-module if and only if there exist discrete valuation rings $(V_1, m_1), (V_2, m_2), \ldots, (V_n, m_n)$, each with finite residue fields, and positive integers $k_1, k_2, \ldots, k_n$, such that if the ring $S := V_1/(m_1^{k_1}) \times V_2/(m_2^{k_2}) \times \cdots \times V_n/(m_n^{k_n})$, then*

(a) $M \cong_R S$. Moreover, the structure of S as an R-module is essentially the same as the structure of S as an S-module, and
(b) $\{\mathcal{C}(V_1/(m_1^{k_1})), \mathcal{C}(V_2/(m_2^{k_2})) \ldots, \mathcal{C}(V_n/(m_n^{k_n}))\}$ is product-maximal.

Proof. Let R be a ring, and let M be a finite nontrivial R-module.

Assume first that (a) and (b) hold. It follows immediately from Example 1 and Lemma 7 that the ring $S := V_1/(m_1^{k_1}) \times V_2/(m_2^{k_2}) \times \cdots \times V_n/(m_n^{k_n})$ is a strongly Jónsson ring, whence by (a), M is a strongly Jónsson R-module.

Conversely, suppose that M is a strongly Jónsson R-module. We will show that (a) and (b) hold. Since M is finite, M is certainly Artinian. Thus by Fact 2, there exist maximal ideals $J_1, J_2, \ldots, J_n$ of R such that

$$(3.5) \qquad\qquad M = \bigoplus_{1 \leq i \leq n} M[J_i].$$

Let i be arbitrary. Since $M[J_i]$ is J_i-primary, $M[J_i]$ is naturally a module over the local ring R_{J_i} by Corollary 1. Lemma 3 implies that $|M[J_i]|$ is a power of a prime; say that $|M[J_i]| = p^\lambda$. We will show that $M[J_i]$ is cyclic. Suppose not. Then $M[J_i]$ is the union of its proper R-submodules. Since each R-submodule of $M[J_i]$ has cardinality a power of p and since for every j, $0 \leq j < \lambda$, there is *at most one* R-submodule of $M[J_i]$ of cardinality p^j, we conclude that

$$(3.6) \qquad\qquad p^\lambda \leq 1 + p + p^2 + \cdots + p^{\lambda-1}.$$

However, $1 + p + p^2 + \cdots + p^{\lambda-1} = \frac{p^\lambda - 1}{p-1} < p^\lambda$, and we have a contradiction. Thus $M[J_i]$ is cyclic. The same argument can be applied to show that every R-submodule of $M[J_i]$ is cyclic. Let I_i be the annihilator of $M[J_i]$ in R. Then (3.5) and the module isomorphism theorems yield

$$(3.7) \qquad\qquad M \cong_R \bigoplus_{1 \leq i \leq n} R/I_i.$$

Again, let i be arbitrary. Since $I_i + 1$ is annihilated by a power of J_i, we see that I_i contains a power of J_i. Set $S := R/I_1 \times R/I_2 \times \cdots \times R/I_n$. Lemma 8 now

implies that the structure of the ring S as an R-module is essentially the same as the structure of S as a module over itself (and thus the R-submodules of S are the same as the ideals of S). To finish the proof, we must show that each R/I_i is a proper homomorphic image of a discrete valuation ring with a finite residue field. Toward this end, recall that $R/I_i \cong_R M[J_i]$ and that every R-submodule of $M[J_i]$ is cyclic. It follows that R/I_i is a finite principal ideal ring. Since I_i contains a power of J_i, R/I_i is local. Cohen's structure theorems for complete local rings yield that R/I_i is a proper homomorphic image of a discrete valuation ring (V_i, m_i) (this is stated explicitly as part of Theorem 3.3 of McLean [15]). Thus $R/I_i \cong V_i/(m_i^{k_i})$ (as *rings*) for some positive integer k_i. Lemma 5 implies that $V_i/(m_i)$ is finite, and hence (a) holds. That (b) holds follows immediately from (a) and Lemma 7 (note that (a) implies that S is a strongly Jónsson ring). $\square$

We easily establish the following corollary, which will be of great use to us in the following section.

Corollary 2. *Let R be a ring, and suppose that M is a finite strongly Jónsson R-module. Then every R-submodule of M is cyclic.*

Proof. Assume that M is a finite strongly Jónsson R-module. If M is trivial, the result is clear, so assume that M is nontrivial. The previous proposition yields discrete valuation rings $(V_1, m_1), (V_2, m_2), \ldots, (V_n, m_n)$, each with finite residue fields, and positive integers $k_1, k_2, \ldots, k_n$, such that if the ring $S := V_1/(m_1^{k_1}) \times V_2/(m_2^{k_2}) \times \cdots \times V_n/(m_n^{k_n})$, then $M \cong_R S$. Moreover, the structure of S as an R-module is essentially the same as the structure of S as an S-module. Each $V_i/(m_i^{k_i})$ is a principal ideal ring. It is easy to check that this implies that S too is a principal ideal ring. Since the structure of S as an R-module is essentially the same as the structure of S as an S-module, we deduce that every R-submodule of S is cyclic. As $M \cong_R S$, it follows that every R-submodule of M is cyclic. This concludes the proof. $\square$

Having classified the finite strongly Jónsson modules, we move on to those which are countably infinite. Recall that if M is an R-module, then the *lattice of R-submodules of M*, denoted $\mathcal{L}_R(M)$, is the collection of all R-submodules of M ordered by set-theoretic inclusion. We now describe the order on $\mathcal{L}_R(M)$, where M is a countably infinite strongly Jónsson R-module which is not cyclic.

Proposition 4. *Let R be a ring, and suppose that M is a countably infinite strongly Jónsson R-module which is not cyclic. Then $\mathcal{L}_R(M)$ is order isomorphic to $\omega + 1$ with the usual ordinal ordering (ω denotes the first infinite ordinal).*

Proof. Let R and M be as stated. Since every proper R-submodule of M is finite, it is clear that M is Artinian. Hence by Fact 2, there exist maximal ideals $J_1, J_2, \ldots, J_n$ of R such that

$$M = \bigoplus_{1 \leq i \leq n} M[J_i].$$

Since M is countably infinite, it follows that some $M[J_i]$ is also infinite. But as M is strongly Jónsson, we deduce that $M = M[J_i]$. Setting $J := J_i$, we see that M is J-primary. Now suppose that N is a proper nonzero R-submodule of M. Then N is J-primary and finite. The proof of Proposition 3 shows that $N \cong_R R/ann_R(N)$ and that the ring $R/ann_R(N)$ is a proper homomorphic of a discrete valuation ring. It follows that $R/ann_R(N)$ is a uniserial R-module, whence N is also a uniserial R-module. We now claim that M itself is uniserial. Indeed, let A and B be arbitrary R-submodules of M. We will show that either $A \subseteq B$ or $B \subseteq A$. Clearly we may assume that A and B are proper R-submodules of M, hence finite. We may also assume that A and B are nonzero. But then the R-module $A + B$ is finite as well, whence by the above argument is uniserial. Since $A \cup B \subseteq A + B$, we conclude that $A \subseteq B$ or $B \subseteq A$.

Now set $M_0 := \{0\}$. Choose any $m \neq 0$ in M. Then Rm is finite (since M is not cyclic and strongly Jónsson) and nonzero. Since M is Artinian and uniserial, there exists a unique minimal finite R-submodule M_1 which properly contains M_0. Continuing recursively, we obtain an infinite sequence of finite R-submodules

$$(3.8) \qquad\qquad\qquad M_0 \subsetneq M_1 \subsetneq M_2 \cdots$$

such that M_i/M_{i-1} is simply for $i > 0$. To finish the proof, it suffices to show that the M_i are precisely the proper R-submodules of M. Toward this end, let N be a proper R-submodule of M. Then as M is strongly Jónsson, we see that N is finite, whence N cannot contain every M_i. Let i be least such that $M_i \nsubseteq N$. As $M_0 = \{0\}$, clearly $i > 0$. We claim that $N = M_{i-1}$. By leastness of i, we have $M_{i-1} \subseteq N$. If $M_{i-1} \subsetneq N$, then since M_i is minimal with respect to properly containing M_{i-1} (and since M is uniserial), we conclude that $M_i \subseteq N$, which is a contradiction. Thus $N = M_{i-1}$, and the proof is complete. $\square$

We now recall the following result of Hirano and Mogami:

Lemma 9 ([9], Theorem 8 and Theorem 10). *Let R be a ring and let M be an R-module. Suppose further that $\mathcal{L}_R(M)$ is order isomorphic to $\omega+1$. Let $S := End_R(M)$ be the endomorphism ring of M over R. Then the structure of M as an S-module is essentially the same as the structure of M as an R-module. Moreover, S is a (commutative) complete discrete valuation ring, and if K is the quotient field of S, then $M \cong_S K/S$.*

We need one more result from the literature and then we can classify the countably infinite strongly Jónsson modules. Recall from the introduction that an infinite

module M over a ring R is a Jónsson module if and only if every proper R-submodule of M has smaller cardinality than M.

Lemma 10 ([8], Proposition 2.2). *Let R be an infinite ring. Then R is Jónsson as a module over itself if and only if R is a field.*

Proposition 5. *Let R be a ring and let M be a countably infinite R-module. Then M is strongly Jónsson if and only if one of the following holds:*

(a) *There exists a maximal ideal J of R such that $M \cong_R R/J$.*

(b) *$End_R(M) := (V, m)$ is a complete discrete valuation ring with a finite residue field, and the structure of M as an R-module is essentially the same as the structure of M as a V-module. Moreover, if K is the quotient field of V, then $M \cong_V K/V$.*

Proof. We assume that R is a ring and that M is a countably infinite R-module. Suppose first that there exists a maximal ideal J of R such that $M \cong_R R/J$. Then M is simple, whence clearly is strongly Jónsson. Now suppose that (b) holds. It suffices to show that K/V is a strongly Jónsson V-module. Let L and N be distinct V-submodules of K/V. We will show that $|L| \neq |N|$. Lemma 4 and Lemma 5 imply that every proper V-submodule of K/V is finite. Thus we may assume that L and N are both proper and nonzero. Then (in the notation of Lemma 4), we see that $L = M_i$ and $N = M_j$ for some positive integers $i \neq j$. But then $L \cong_V V/(m^i)$ and $N \cong_V V/(m^j)$. We now invoke Lemma 5 again to conclude that $|L| \neq |N|$.

Conversely, suppose that M is a countably infinite strongly Jónsson R-module. We distinguish two cases.

Case 1: M is cyclic. Then $M \cong_R R/I$ for some ideal I of R. Hence R/I is a Jónsson module over R. Since $ann_R(R/I) = I$, it follows that R/I is a Jónsson module over R/I. Thus by Lemma 10, R/I is a field, whence I is a maximal ideal of R. We conclude that (a) holds.

Case 2: M is not cyclic. Then Proposition 4 and Lemma 9 show that $End_R(M) := (V, m)$ is a complete discrete valuation ring, and the structure of M as an R-module is essentially the same as the structure of M as a V-module. Moreover, if K is the quotient field of V, then $M \cong_V K/V$. It remains to show that $V/(m)$ is finite. Let x be any nonzero element of K/V. Then $ann_V(x)$ is clearly a proper, nonzero ideal of V. Thus $Vx \cong_V V/(m^i)$ for some positive integer i. Since M is not cyclic, $M \cong_V K/V$, and K/V is a strongly Jónsson V-module, we conclude that Vx is finite. Lastly, we invoke Lemma 5 to conclude that $V/(m)$ is finite. $\qquad\square$

We complete the classification of strongly Jónsson modules by determining those which are uncountable. Again, we will require some preliminary results.

Lemma 11. *Let F be a field, and let M be an infinite F-vector space. Then M is a Jónsson module over F if and only if $M \cong_F F$.*

Proof. We suppose that F is a field and that M is an infinite vector space over F. If $M \cong_F F$, then since M is simple, it is clear that M is a Jónsson module over F. Conversely, suppose that M is a Jónsson module over F. There is a nonempty index set I such that

$$M \cong_F \bigoplus_{i \in I} F.$$

We claim that $|I| = 1$. If not, then by deleting one summand, one obtains a proper F-submodule of M of the same cardinality as M, a contradiction.   □

Lemma 12 ([21], Proposition 5). *Suppose that M is a faithful Jónsson module over the ring R (that is, $ann_R(M) = \{0\}$). Assume further that there exists a nonzero ideal I of R such that M is I-primary. Then M is countable.*

We establish one more lemma and then we characterize the uncountable strongly Jónsson modules.

Lemma 13. *Let R be a ring. Every strongly Jónsson R-module is Artinian.*

Proof. Suppose by way of contradiction that M is a strongly Jónsson R-module which is not Artinian, and let $\{M_i : i > 0\}$ be a strictly descending chain of R-submodules of M. Now let i be arbitrary. Since $M_{i+1} \subsetneq M_i$ and since M is strongly Jónsson, we conclude that $|M_{i+1}| < |M_i|$. But then $\{|M_i| : i > 0\}$ is an infinite, strictly decreasing sequence of cardinal numbers, contradicting the fact that the cardinal numbers (more generally, the ordinal numbers) are well-ordered.   □

Proposition 6. *Let R be a ring, and let M be an uncountable R-module. Then M is strongly Jónsson if and only if there exists a maximal ideal J of R such that $M \cong_R R/J$.*

Proof. We let R be a ring and M be an uncountable R-module. If $M \cong_R R/J$ for some maximal ideal J of R, then M is simple, whence strongly Jónsson. Conversely, suppose that M is strongly Jónsson. We will show that there is a maximal ideal J of R such that $M \cong_R R/J$. Let I be the annihilator of M in R. Then M is strongly Jónsson and faithful over the ring $S := R/I$. Lemma 13 implies that M is an Artinian S-module. As in the proof of Proposition 4, we see that there exists some maximal ideal J of S such that M is a J-primary S-module. But then Lemma 12 yields that $J = \{0\} \pmod I$. We conclude that $J = I$, and thus M is a strongly Jónsson module over the field R/J. We now invoke Lemma 11 to conclude that $M \cong_{R/J} R/J$. This clearly implies that $M \cong_R R/J$, and the proof is complete.   □

We conclude the section with a summary of our results on strongly Jónsson modules.

Theorem 1. *Let R be a ring, and let M be a nonzero R-module. Then M is strongly Jónsson if and only if one of the following holds:*

(***I***) *There exist discrete valuation rings $(V_1, m_1), (V_2, m_2), \ldots, (V_n, m_n)$, each with finite residue fields, and positive integers $k_1, k_2, \ldots, k_n$, such that if the ring $S := V_1/(m_1^{k_1}) \times V_2/(m_2^{k_2}) \times \cdots \times V_n/(m_n^{k_n})$, then*

(a) *$M \cong_R S$. Moreover, the structure of S as an R-module is essentially the same as the structure of S as an S-module, and*
(b) *$\{\mathcal{C}(V_1/(m_1^{k_1})), \mathcal{C}(V_2/(m_2^{k_2})) \ldots, \mathcal{C}(V_n/(m_n^{k_n}))\}$ is product-maximal.*

(***II***) *$End_R(M) := (V, m)$ is a complete discrete valuation ring with a finite residue field, and the structure of M as an R-module is essentially the same as the structure of M as a V-module. Moreover, if K is the quotient field of V, then $M \cong_V K/V$.*

(***III***) *There exists a maximal ideal J of R such that $M \cong_R R/J$.*

4. Strongly HS Modules

Having classified the strongly Jónsson modules, we now consider a sort of dual notion defined as follows: Let R be a ring, and let M be an R-module. Say that M is *strongly HS* provided $|M/N| \neq |M/K|$ whenever N and K are distinct R-submodules of M. Recall from the introduction that an infinite module M over a ring R is *homomorphically smaller* (HS for short) provided $|M/N| < |M|$ for every nonzero R-submodule N of M. Note that if M is an infinite strongly HS module and N is a nonzero R-submodule of M, then $|M/\{0\}| \neq |M/N|$. We conclude that $|M| \neq |M/N|$. Since $|M/N| \leq |M|$, we deduce that $|M/N| < |M|$. Thus every infinite strongly HS module is HS.

As in the previous section, we proceed by first classifying the strongly HS abelian groups ($\mathbb{Z}$-modules). We remark that this problem is closely related to a famous old textbook problem of Kaplansky (Kaplansky [11]): Show that $\mathbb{Z}$ is the unique infinite abelian group G with the property that G/H is finite for every nonzero subgroup H of G.

Proposition 7. *Let G be an abelian group. Then G is strongly HS if and only if $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}/(n)$ for some positive integer n.*

Proof. It is clear that $\mathbb{Z}$ is a strongly HS abelian group. Recall from Proposition 1 that $\mathbb{Z}/(n)$ is strongly Jónsson for any positive integer n, whence (as $\mathbb{Z}/(n)$ is *finite*) also strongly HS (see Lemma 14). Conversely, suppose that G is an arbitrary strongly HS abelian group. Again, if G is finite, then G is strongly Jónsson, whence isomorphic to $\mathbb{Z}/(n)$ for some positive integer n by Proposition 1. Suppose now that G is infinite. We will show that $G \cong \mathbb{Z}$. We first claim that G is countable. To see this, suppose by way of contradiction that G is uncountable, and let $g \in G - \{0\}$ be arbitrary. Note trivially that

$$|(g)| \cdot |G/(g)| = |G|.$$

Since the cyclic group (g) is certainly countable, it follows from basic cardinal arithmetic that $|G/(g)| = |G|$, and this is impossible since G is strongly HS. We conclude that G is countable, whence G/H is finite for every nonzero subgroup H of G. Now let $g_0 \in G - \{0\}$ be arbitrary. It is easy to see that $G = (g_0, X)$, where X is a complete set of coset representatives for G modulo (g_0). Since X is finite, it follows that G is finitely generated. Thus by The Fundamental Theorem of Finitely Generated Abelian Groups, G is a finite direct sum of cyclic groups. Since G is infinite, at least one summand must be isomorphic to $\mathbb{Z}$. There can be no other summands, lest $\mathbb{Z}$ be an infinite proper homomorphic image of G. Thus $G \cong \mathbb{Z}$, and the proof is concluded. $\square$

As in the previous section, our goal is to extend the previous proposition to modules over an arbitrary ring. We begin with a simple lemma.

Lemma 14. *Let R be a ring, and let M be an R-module.*

(a) *If M is finite, then M is strongly Jónsson if and only if M is strongly HS.*
(b) *If M is infinite and R is a field, then M is strongly HS if and only if $M \cong_R R$.*

Proof. Assume that R is a ring and that M is an R-module.

(a) Suppose that M is finite, and let N and K be R-submodules of M. Then simply observe that $|N| = |K|$ if and only if $|M|/|N| = |M|/|K|$ if and only if $|M/N| = |M/K|$. The result follows.

(b) The proof is analogous to the proof of Lemma 11 (instead of deleting a summand, mod out by it). $\square$

Analogous to strongly Jónsson rings, we define a ring R to be a *strongly HS ring* provided R is strongly HS as a module over itself. If R is in addition a domain, then we will say that R is a *strongly HS domain*. It is well-known that the class of Noetherian rings properly includes the class of Artinian rings, but that the class of Noetherian modules and the class of Artinian modules do not compare under $\subseteq$. Moreover,

Proposition 8. *The class of strongly HS rings properly includes the class of strongly Jónsson rings. However, the class of strongly HS modules and the class of strongly Jónsson modules do not compare under $\subseteq$.*

Proof. Let R be a strongly Jónsson ring. We will show that R is a strongly HS ring. If R is finite, then R is strongly HS by Lemma 14. Suppose now that R is infinite. Then Lemma 10 implies that R is a field, whence R is a strongly HS ring (again by Lemma 14). To see that the containment is proper, simply note that $\mathbb{Z}$ is a strongly

HS ring which is not strongly Jónsson. As for the second claim, $\mathbb{Z}(p^\infty)$ is a strongly Jónsson $\mathbb{Z}$-module which is not strongly HS, and $\mathbb{Z}$ is a strongly HS $\mathbb{Z}$-module which is not strongly Jónsson. $\square$

In light of Lemma 14 and our work in the previous section, we may restrict our study to infinite strongly HS modules over a ring R which is not a field. In fact, we can restrict our study even further by recalling the following result from [20]:

Lemma 15 ([20], Proposition 3.2). *Let M be an HS module over the ring R. Then $ann_R(M)$ is a prime ideal of R.*

Hence by modding out the annihilator, we may restrict our study to infinite faithful strongly HS modules over a domain D which is not a field. Recall from Lemma 13 that every strongly Jónsson module is Artinian. Here is the strongly HS-theoretic analog:

Lemma 16. *Every strongly HS module is Noetherian.*

Proof. Let R be a ring, and suppose that M is a strongly HS R-module. Assume by way of contradiction that M is not Noetherian, and let $\{M_i : i > 0\}$ be a strictly ascending chain of submodules of M. Let $i > 0$ be arbitrary. Since $M_i \subsetneq M_{i+1}$, there is a natural surjection $f : M/M_i \to M/M_{i+1}$. Hence $|M/M_{i+1}| \leq |M/M_i|$. Since M is strongly HS, we conclude that $|M/M_{i+1}| < |M/M_i|$. But then $\{|M/M_i| : i > 0\}$ is an infinite, strictly decreasing sequence of cardinals, and we obtain a contradiction to the fact that the cardinals are well-ordered. $\square$

We now present two more lemmas which will play an important role in the classification of the strongly HS modules.

Lemma 17. *Let D be a domain with quotient field K, and let M be an infinite, faithful D-module. If M is HS over D, then (up to isomorphism) $D \subseteq M \subseteq K$.*

Proof. This is a portion of Theorem 3.3 of [20]. $\square$

Lemma 18. *Let M be a strongly HS module over the ring R. Then every R-submodule of M is also strongly HS.*

Proof. Suppose that M is a strongly HS module over R, and let N be an arbitrary submodule of M. We will show that N is also strongly HS. Toward this end, assume that K and L are distinct R-submodules of N. We will prove that $|N/K| \neq |N/L|$. To see this, note that

$$(4.1) \qquad (M/K)/(N/K) \cong_R M/N, \text{ and}$$

$$(4.2) \qquad (M/L)/(N/L) \cong_R M/N.$$

The previous isomorphisms imply that

$$(4.3) \qquad |M/K| = |M/N| \cdot |N/K|, \text{ and}$$

$$(4.4) \qquad |M/L| = |M/N| \cdot |N/L|.$$

Since M is strongly HS, we conclude that $|M/K| \neq |M/L|$. this fact along with equations (4.3) and (4.4) above implies that $|N/K| \neq |N/L|$. $\qquad \square$

We now show that the classification of the strongly HS modules can be reduced to the classification of the strongly HS rings.

Proposition 9. *Let D be a domain which is not a field, and let M be an infinite faithful D-module. Then M is a strongly HS D-module if and only if D is a strongly HS domain and $M \cong_D I$ for some nonzero ideal I of D.*

Proof. Assume that D is a domain which is not a field and that M is an infinite faithful D-module. Let K be the quotient field of D.

Suppose first that M is a strongly HS D-module. Then M is HS. Lemma 17 implies that (up to isomorphism) $D \subseteq M \subseteq K$. It now follows from Lemma 18 that D is a strongly HS domain. Lemma 16 implies that M is finitely generated. Since $M \subseteq K$, there exists a nonzero element $d \in D$ such that $dM \subseteq D$. In particular, dM is an ideal of D. As $M \cong_D dM$, we see that M is isomorphic to a nonzero ideal of D.

Conversely, suppose that D is a strongly HS domain and that $M \cong_D I$ for some nonzero ideal I of D. Then Lemma 18 implies that M is strongly HS, and the proof is complete. $\qquad \square$

Given the previous proposition, we now focus our efforts on classifying the strongly HS domains. We will need the following lemma.

Lemma 19. *Let D be a domain, and let $d \in D - \{0\}$ be arbitrary. Then*

$$|D/(d^2)| = |D/(d)|^2.$$

Proof. Assume that D is a domain and that $d \in D - \{0\}$. Observe first that $D/(d) \cong_D (D/(d^2))/((d)/(d^2))$. We conclude that

$$(4.5) \qquad |D/(d^2)| = |D/(d)||(d)/(d^2)|.$$

Thus to finish the proof, it suffices to show that $|(d)/(d^2)| = |D/(d)|$. Toward this end, let $\varphi : D \to (d)/(d^2)$ be defined by $\varphi(x) := (d^2) + dx$. Note that $x \in \ker(\varphi)$ if and only if $dx \in (d^2)$ if and only if (since D is a domain and $d \neq 0$) $x \in (d)$. We conclude that $D/(d) \cong_D (d)/(d^2)$, and the conclusion follows. $\qquad \square$

Recall from the introduction that a ring R is residually finite provided R/I is finite for every nonzero ideal I of R.

Lemma 20. *Let D be a strongly HS domain. Then D is residually finite.*

Proof. Assume that D is a strongly HS domain, and let I be a nonzero ideal of D. We will show that D/I is finite. If $I = D$, the result is patent, so assume that I is proper. Let $d \in I - \{0\}$ be arbitrary. We first show that $D/(d)$ is finite. Suppose by way of contradiction that $D/(d)$ is infinite. Lemma 19 gives $|D/(d^2)| = |D/(d)|^2$. Since $D/(d)$ is infinite, it follows from basic cardinal arithmetic that $|D/(d)|^2 = |D/(d)|$. But then $|D/(d^2)| = |D/(d)|$. Since D is strongly HS, we conclude that $(d^2) = (d)$. Since D is a domain, we deduce that d is a unit, contradicting that I is proper. Thus $D/(d)$ is finite. The map $\varphi : D/(d) \to D/I$ defined by $\varphi((d) + x) := I + x$ is a well-defined surjection. We conlude that $|D/I| \leq |D/(d)|$, and hence D/I is finite, as required. $\square$

We are now able to establish a connection between strongly Jónsson rings and strongly HS rings. Namely:

Proposition 10. *Let D be a strongly HS domain, and suppose that I is a proper nonzero ideal of D. Then D/I is a finite strongly Jónsson ring.*

Proof. Let D and I be as stated. Lemma 20 implies that D/I is finite. It remains to show that D/I is strongly Jónsson. Toward this end, let J_1 and J_2 be ideals of D which contain I and suppose that $|J_1/I| = |J_2/I|$. We will prove that $J_1/I = J_2/I$. Toward this end, it follows as in the proof of Lemma 18 that

$$(4.6) \qquad |D/I| = |D/J_1| \cdot |J_1/I|, \text{ and}$$

$$(4.7) \qquad |D/I| = |D/J_2| \cdot |J_2/I|.$$

Thus we obtain

$$(4.8) \qquad |D/J_1| \cdot |J_1/I| = |D/J_2| \cdot |J_2/I|.$$

Recall that D/I is finite. Since J_1/I and J_2/I are ideals of D/I, it follows that J_1/I and J_2/I are finite. Since we have assumed that $|J_1/I| = |J_2/I|$, we may cancel them in (4.8) above to get

$$(4.9) \qquad |D/J_1| = |D/J_2|.$$

Since D is a strongly HS domain, we deduce that $J_1 = J_2$, whence $J_1/I = J_2/I$, and the proof is complete. $\square$

We easily obtain the following corollary.

Corollary 3. *Let D be a strongly HS domain which is not a field. Then D is Dedekind.*

Proof. Suppose that D is a strongly HS domain. To show that D is Dedekind, it suffices (by (d) of Fact 1) to show that D/I is a principal ideal ring for every proper nonzero ideal I of D. If I is such an ideal, then the previous proposition yields that D/I is a finite strongly Jónsson ring. We now invoke Corollary 2 to conclude that D/I is a principal ideal ring. □

We are now in position to classify the strongly HS modules.

Theorem 2. *Let D be a domain which is not a field, and let M be an infinite faithful D-module (recall that we can restrict to this setting without loss of generality). Then M is a strongly HS D-module if and only if the following hold:*

(a) *D is a Dedekind domain with all residue fields finite,*
(b) *If P and Q are distinct maximal ideals of D, then D/P and D/Q have distinct (nonzero) characteristics, and*
(c) *$M \cong_D I$ for some nonzero ideal I of D.*

Proof. Assume that D is a domain which is not a field and that M is an infinite faithful D-module.

Suppose first that M is a strongly HS D-module. We will verify (a)–(c) above. Proposition 9 tells us that D is a strongly HS domain and $M \cong_D I$ for some nonzero ideal I of D, verifying (c). Corollary 3 yields that D is a Dedekind domain, and Lemma 20 implies that all residue fields of D are finite. Hence (a) is established. It remains to check (b). To wit, suppose that P and Q are distinct maximal ideals of D. Since D/P and D/Q are finite fields, both D/P and D/Q have nonzero characteristics. Suppose by way of contradiction that D/P and D/Q both have characteristic p. Then $|D/P| = p^m$ and $|D/Q| = p^n$ for some positive integers m and n. Lemma 5 implies that

$$p^{mn} = |D/P|^n = |D/P^n| = |D/Q|^m = |D/Q^m|.$$

Since D is strongly HS, we deduce that $P^n = Q^m$, and this is impossible as P and Q are distinct maximal ideals of D (alternatively, unique factorization is violated).

Conversely, suppose that (a)–(c) hold. We will show that M is a strongly HS D-module. It suffices by Proposition 9 to show that D is a strongly HS domain. We begin by showing that D is residually finite. Let I be any nonzero ideal of D. We will show that D/I is finite. This is clear if $I = D$, so assume that I is proper. Since D is Dedekind, $I = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}$ for some (distinct) prime ideals $P_1, P_2, \ldots, P_k$ of D and positive integers $n_1, n_2, \ldots, n_k$. Of course, each P_i is nonzero, whence maximal. It follows that

(4.10) $$D/I = D/(P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}) \cong D/P_1^{n_1} \times D/P_2^{n_2} \times \cdots \times D/P_k^{n_k}.$$

We now apply Lemma 5 to conclude that

(4.11) $\quad |D/I| = |D/P_1^{n_1} \times D/P_2^{n_2} \times \cdots \times D/P_k^{n_k}| = |D/P_1|^{n_1} \cdot |D/P_2|^{n_2} \cdots \cdot |D/P_k|^{n_k}.$

Since all residue fields of D are finite, we conclude that D/I is finite.

Now let $I \neq J$ be ideals of D. We will demonstrate that $|D/I| \neq |D/J|$, thus verifying that D is a strongly HS domain. This is clear if either $I = D$ or $J = D$. Suppose now that $I = \{0\}$. Then since $J \neq I$ and D is residually finite (and not a field), we see that D/I is infinite whereas D/J is finite. Finally, suppose that I and J are nonzero, proper ideals of D. Let $I = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}$ and $J = Q_1^{m_1} Q_2^{m_2} \cdots Q_l^{m_l}$ be prime factorizations of I and J (as products of prime ideals of D). We deduce (as in (4.11) above) that:

(4.12) $$|D/I| = |D/P_1|^{n_1} \cdot |D/P_2|^{n_2} \cdots \cdot |D/P_k|^{n_k}.$$

Analogously,

(4.13) $$|D/J| = |D/Q_1|^{m_1} \cdot |D/Q_2|^{m_2} \cdots \cdot |D/Q_l|^{m_l}.$$

Since $I \neq J$ and since distinct residue fields of D have cardinalities which are powers of distinct primes, the previous two equations clearly imply that $|D/I| \neq |D/J|$, and the proof is complete. $\qquad \square$

Before stating a corollary, we recall that for a ring R, $\mathrm{Max}(R)$ denotes the set of maximal ideals of R.

Corollary 4. *Let D be a domain which is not a field.*

(a) Suppose that D has prime characteristic p. Then D is a strongly HS domain if and only if D is a DVR with a finite residue field.

(b) Suppose that D has characteristic 0 (and hence $\mathbb{Z} \subseteq D$). Then D is a strongly HS domain if and only if D is a Dedekind domain with all residue fields finite and with the additional property that the map $P \mapsto P \cap \mathbb{Z}$ is an injection from $\mathrm{Max}(D)$ into $\mathrm{Max}(\mathbb{Z})$.

(c) If D is a strongly HS domain, then $|D| \leq 2^{\aleph_0}$.

Proof. Let D be a domain which is not a field.

(a) Suppose first that D has characteristic p. If D is a DVR with a finite residue field, then it follows immediately from the previous theorem that D is a strongly HS domain. Conversely, suppose that D is a strongly HS domain. Let J be an arbitrary maximal ideal of D. Since D has characteristic p, clearly so does D/J. Theorem 2

implies that J is the unique maximal ideal of D. Since D is a Dedekind domain with a finite residue field, we now invoke (e) of Fact 1 to conclude that $D = D_J$ is a DVR with a finite residue field.

(b) This follows easily from (b) of Theorem 2.

(c) Assume now that D is a strongly HS domain. Then D is Dedekind, whence Noetherian. Let P be a maximal ideal of D. Krull's Intersection Theorem yields that $\bigcap_{i=1}^{\infty} P^i = \{0\}$. It follows that D maps injectively into $\prod_{i=1}^{\infty} D/P^i$. But since D is a residually finite Dedekind domain, Lemma 5 yields that D/P^i is finite for every positive integer i. Thus $|D| \leq |\prod_{i=1}^{\infty} D/P^i| \leq |\prod_{i=1}^{\infty} \mathbb{N}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. $\square$

We conclude the paper by giving some examples of strongly HS domains and discussing their behavior with respect to integral extensions and overrings.

Example 4. *Let p be a prime and let κ be a cardinal number satisfying $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. There exists a strongly HS domain (which is not a field) of prime characteristic p and of cardinality κ.*

Proof. Let F be a finite field of characteristic p, and let $F[[t]]$ be the ring of formal power series over F in the variable t. The underlying set of $F[[t]]$ is the set of all functions from $\mathbb{N}$ into F, whence $|F[[t]]| = 2^{\aleph_0}$. The quotient field of $F[[t]]$ is the field $F((t))$ of formal Laurent series in the variable t. There is a field K of cardinality κ such that $F(t) \subseteq K \subseteq F((t))$. Note that $F[[t]]$ is a DVR on $F((t))$ (that is, $F[[t]]$ is a DVR with quotient field $F((t))$), $K \subseteq F((t))$, and $F[[t]] \cap K$ is not a field (since t is not invertible in $F[[t]]$). It follows that $F[[t]] \cap K$ is a DVR on K (whence also has cardinality κ) with maximal ideal $M := (t) \cap K$. It is obvious that $F[[t]] \cap K$ has characteristic p. It is also easy to check that F maps injectively into $(F[[t]] \cap K)/M$ and $(F[[t]] \cap K)/M$ maps injectively into $F[[t]]/(t) \cong F$. It follows that $|(F[[t]] \cap K)/M| = |F|$. We have shown that $F[[t]] \cap K$ is a DVR of characteristic p and of cardinality κ and that $F[[t]] \cap K$ has residue field isomorphic to F, which is finite. Part (a) of Corollary 4 yields that $F[[t]] \cap K$ is strongly HS, and the proof is complete. $\square$

Given the previous example, it is natural to enquire about the characteristic 0 case. Before giving a more general result, we remark that the ring J_p of p-adic integers is a DVR of characteristic 0 (of cardinality $2^{\aleph_0}$) with residue field isomorphic to $\mathbb{Z}/(p)$, whence J_p is a strongly HS ring.

Example 5. *Let κ be a cardinal number satisfying $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Further, let $p_1, p_2, \ldots, p_n$ be distinct primes and let $k_1, k_2, \ldots, k_n$ be positive integers. There exists a principal ideal domain D of cardinality κ with exactly n maximal ideals $M_1, M_2, \ldots, M_n$ with the property that $D/M_i \cong \mathbb{F}_{p_i^{k_i}}$ for each i, $1 \leq i \leq n$. Hence D is a strongly HS domain.*

Proof. The existence of such a D with the above properties is established in Theorem 2.6 of [12]. The construction is quite technical, and we suppress the details here (we remark that the ideas of the construction are due to C. Shah; see Theorem 2.3 of Shah [23]). We conclude immediately from Theorem 2 that D is a strongly HS domain. □

We now present a discussion of how the strongly HS property behaves with respect to integral extensions and overrings. We will show that the property is not well-preserved with respect to integral extensions, but that it is preserved in overrings.

Proposition 11. *Let D be a strongly HS domain, and suppose that R is a finite integral extension of D (that is, R is integral over D has has a finite basis as a D-module). Then R need not be strongly HS.*

Proof. Let $\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Then clearly $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$ and $\{1, i\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}[i]$. Recall that the function $N : \mathbb{Z}[i] \to \mathbb{N}$ given by $N(a + bi) := a^2 + b^2$ is a Euclidean norm (from which one proves that $\mathbb{Z}[i]$ is a Euclidean domain). Consider the ideals $(1 + 2i)$ and $(1 - 2i)$. Since $N(1 + 2i) = N(1 - 2i) = 5$, which is prime, we conclude that $1 + 2i$ and $1 - 2i$ are Gaussian primes. Thus $(1 + 2i)$ and $(1 - 2i)$ are maximal ideals of $\mathbb{Z}[i]$. The units of $\mathbb{Z}[i]$ are precisely the elements of $\mathbb{Z}[i]$ which have norm 1. It follows (and is well-known, of course) that the units of $\mathbb{Z}[i]$ are exactly $1, -1, i$, and $-i$. From this fact, it is clear that $1 + 2i$ and $1 - 2i$ are not associates, whence $(1 + 2i)$ and $(1 - 2i)$ are distinct maximal ideals of $\mathbb{Z}[i]$. Note that $5 \in (1 + 2i) \cap (1 - 2i)$, whence $(1 + 2i) \cap \mathbb{Z} = (1 - 2i) \cap \mathbb{Z} = 5\mathbb{Z}$. Corollary 4 part (b) implies that $\mathbb{Z}[i]$ is not a strongly HS domain. □

We conclude the paper with an investigation of how the strongly HS property behaves with respect to overrings. Recall that if D is a domain with quotient field K, then an *overring* of D is simply a ring R satisfying $D \subseteq R \subseteq K$. Of particular interest is a special kind overring of D called a *quotient ring*. Recall that a subset $S \subseteq D - \{0\}$ is *multiplicative* provided $1 \in S$ and whenever $x, y \in S$, also $xy \in S$. The ring $D_S := \{\frac{d}{s} : d \in D, s \in S\}$ is then a subring of K containing D called *the quotient ring of D with respect to the multiplicative system S*. We pause to recall the following well-known fact concerning maximal ideals in quotient rings:

Lemma 21 ([4], Corollary 4.6)**.** *Let D be a domain and let S be a multiplicative subset of D. Then every maximal ideal of D_S is of the form $P_S := \{\frac{p}{s} : p \in P, s \in S\}$, where P is an ideal of D maximal with respect to avoiding S (P is necessarily a prime ideal).*

We will shortly be able to prove that the strongly HS property is preserved in quotient ring extensions. We will need three more lemmas.

Lemma 22 ([4], Theorem 17.6)**.** *Suppose that V is a valuation ring on a field K. If $\{P_\lambda\}$ is the set of prime ideals of V, then $\{V_{P_\lambda}\}$ is the set of overrings of V.*

Lemma 23 ([4], Theorem 40.1). *Every overring of a Dedekind domain is a Dedekind domain.*

Lemma 24. *Suppose that D is a one-dimensional Noetherian domain, and let R be an overring of D. If I is a nonzero ideal of R, then R/I is finitely generated as a D-module.*

Proof. This claim is well-known and follows immediately from Theorem 40.8 and Theorem 40.9 of [4], for example. $\square$

Proposition 12. *Let D be a domain with quotient field K, and let S be a multiplicative subset of D. If D is a strongly HS domain, then so is D_S.*

Proof. Let D be a domain and suppose that S is a multiplicative subset of D. Assume further that D is a strongly HS domain. We will show that $R := D_S$ is also strongly HS. If R is a field, we are clearly done. Thus assume that R is not a field (hence also D is not a field). We consider two cases:

Case 1: D has prime characteristic p. Then (a) of Corollary 4 implies that D is a DVR. Lemma 22 yields that the only overrings of D are D and K. Since R is not a field, we deduce that $R = D$, whence R is strongly HS.

Case 2: D has characteristic 0. It suffices by (b) of Corollary 4 to show that R is a Dedekind domain of characteristic 0 with all residue fields finite and with the additional property that $P \mapsto P \cap \mathbb{Z}$ is an injective map from $\mathrm{Max}(R)$ into $\mathrm{Max}(\mathbb{Z})$. Since D has characteristic 0 and since $D \subseteq R$, clearly R has characteristic 0. Since D is strongly HS (and not a field), D is Dedekind, all residue fields of D are finite, and $P \mapsto P \cap \mathbb{Z}$ is an injective map from $\mathrm{Max}(D)$ to $\mathrm{Max}(\mathbb{Z})$. We deduce from Lemma 23 that R is Dedekind. We now show that all residue fields of R are finite. Toward this end, let J be a maximal ideal of R. Recall that R is not a field, whence $J \neq \{0\}$. Lemma 24 yields that R/J is a finitely generated D-module, and hence also a finitely generated $D/(J \cap D)$-module (since $J \cap D \subseteq ann_D(R/J)$). As $J \neq \{0\}$, clearly also $J \cap D \neq \{0\}$. But then $J \cap D$ is a maximal ideal of D. We conclude that R/J is a finite extension of the finite field $D/(J \cap D)$, whence R/J is finite. Lastly, suppose that J_1 and J_2 are distinct maximal ideals of R. We must prove that $J_1 \cap \mathbb{Z} \neq J_2 \cap \mathbb{Z}$. Toward this end, we conclude from Lemma 21 that $J_1 = P_S$ and $J_2 = Q_S$ for some distinct prime ideals P and Q of D which avoid S. Since J_1 and J_2 are nonzero, we see that P and Q are also nonzero. It follows that P and Q are distinct maximal ideals of D. Hence $P \cap \mathbb{Z} = \mathbb{Z}p$ and $Q \cap \mathbb{Z} = \mathbb{Z}q$ for some prime numbers p and q with $p \neq q$. One checks immediately that $P_S \cap D = P$ and $Q_S \cap D = Q$. Hence

$$(4.14) \qquad \mathbb{Z}p = P \cap \mathbb{Z} = (P_S \cap D) \cap \mathbb{Z} \subseteq P_S \cap \mathbb{Z} = J_1 \cap \mathbb{Z}, \text{ and}$$

(4.15) $$\mathbb{Z}q = Q \cap \mathbb{Z} = (Q_S \cap D) \cap \mathbb{Z} \subseteq Q_S \cap \mathbb{Z} = J_2 \cap \mathbb{Z}.$$

Thus $\mathbb{Z}p = J_1 \cap \mathbb{Z}$ and $\mathbb{Z}q = J_2 \cap \mathbb{Z}$. Since $p \neq q$, this concludes the proof. $\square$

We will shortly be able to prove something stronger, namely that every overring of a strongly HS domain is strongly HS. We first establish the following lemma.

Lemma 25. *Let D be a strongly HS domain. Then every prime ideal of D is the radical of a principal ideal of D.*

Proof. Assume that D is a strongly HS domain. If D is a field, the result is obvious, so assume that D is not a field. If D has characteristic $p > 0$, then D is a DVR by (a) of Corollary 4, and the assertion clearly holds. Suppose now that D has characteristic 0. Since D is Dedekind and not a field, D is one-dimensional. It suffices to show that every maximal ideal of D is the radical of a principal ideal. Let Q be an arbitrary maximal ideal of D. We deduce from Corollary 4 that $Q \cap \mathbb{Z} = q\mathbb{Z}$ for some prime number q. We claim that there exists a positive integer n such that $Q^n = Dq$. To see this, begin by noting that Dq is a proper nonzero ideal of D. Since D is Dedekind, we conclude that $Dq = J_1 J_2 \cdots J_n$ for some maximal ideals $J_1, J_2, \ldots, J_n$ of D. We claim that each $J_i = Q$. Fix an arbitrary i with $1 \leq i \leq n$, and simply note that $q \in Dq = J_1 J_2 \cdots J_n \subseteq J_i$, whence $\mathbb{Z}q = J_i \cap \mathbb{Z} = Q \cap \mathbb{Z}$. Corollary 4 yields that $J_i = Q$. Since i was arbitrary, we obtain $Dq = Q^n$. Thus Q^n is a principal ideal. As $Q = \sqrt{Q^n}$, the proof is complete. $\square$

We are almost ready to prove that every overring of a strongly HS domain is strongly HS. We recall that a domain D has *the QR property* if every overring of D is a quotient ring of D. Before stating the following result, we recall that a domain D is a *Prüfer domain* provided every (nonzero) finitely generated ideal I of D is invertible. It is well-known that every Dedekind domain is Prüfer.

Lemma 26 ([4], Proposition 27.4)**.** *Suppose that D is a Prüfer domain and that every prime ideal of D is the radical of a principal ideal of D. Then D has the QR property.*

We conclude the paper with the following theorem.

Theorem 3. *Let D be a strongly HS domain. Then every overring of D is also strongly HS.*

Proof. Of course, we assume that D is a strongly HS domain which is not a field. Let R be an overring of D. Since D is Dedekind, D is also a Prüfer domain. Lemma 25 says that every prime ideal of D is the radical of a principal ideal of D. Hence D has the QR property by the previous lemma, whence R is a quotient ring of D. Proposition 12 implies that R is strongly HS, and the proof is complete. $\square$

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