

THE THEORY OF INTEGRALLY CLOSED DOMAINS IS NOT FINITELY AXIOMATIZABLE

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ABSTRACT. It is well-known that the theory of algebraically closed fields is not finitely axiomatizable. In this note, we prove that the theory of integrally closed integral domains is also not finitely axiomatizable.

All rings in this paper are assumed commutative with identity.

Let L be the language of rings, that is, the language whose signature consists of equality, binary function symbols $+$ and \cdot , and constants $\mathbf{0}$ and $\mathbf{1}$. The theory \mathcal{T}_{AC} of algebraically closed fields is the set of consequences of the collection Σ of sentences comprised of

- (1) the conjunction β_F of the field axioms, and
- (2) for every positive integer n , a sentence β_n asserting that every polynomial of degree n has a root.

It is well-known that any two algebraically closed fields of the same uncountable cardinality and characteristic are isomorphic (this follows immediately from a famous theorem of Steinitz). Therefore, by the Los-Vaught Test (as is also well-known), the theory of algebraically closed fields of characteristic p is complete, where p is either 0 or a prime. It is also known that \mathcal{T}_{AC} is not finitely axiomatizable ([1], Theorem 3.22). We present a simplified proof of Theorem 3.22 below.

Proposition 1 ([1], Theorems 3.21 and 3.22). *The theory of algebraically closed fields is not finitely axiomatizable.*

Sketch of proof. By the Compactness Theorem, it suffices to prove that every finite subset of Σ has a model which is not an algebraically closed field. Toward this end, we need only show that for every positive integer k , there exists a field F which is not algebraically closed, but for which every polynomial of positive degree $d \leq k$ has a root in F . Clearly we may assume $k > 1$. Let S be the multiplicative semigroup generated by the collection of all primes $p \leq k$. Now fix an arbitrary prime q , and

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let $\overline{\mathbb{F}}_q$ be an algebraic closure of \mathbb{F}_q . Finally, set

$$(0.1) \quad F := \bigcup_{n \in S} \mathbb{F}_{q^n},$$

where each \mathbb{F}_{q^n} is the unique subfield of $\overline{\mathbb{F}}_q$ of q^n elements. One checks easily that F has the required property. \square

Remark 1. The previous argument shows that the theory of algebraically closed fields of characteristic q , q a prime, is not finitely axiomatizable. Further, by taking q to be sufficiently large and applying the Compactness Theorem, it follows that the theory of algebraically closed fields of characteristic 0 is also not finitely axiomatizable ([1], Corollaries 3.23 and 3.24).

The purpose of this note is to prove a related result, namely, that the theory of integrally closed domains is not finitely axiomatizable. We begin with some standard definitions. Let S be a ring extension of a ring R , and suppose that $s \in S$. Recall that s is *integral over R* provided $f(s) = 0$ for some monic polynomial $f(x) \in R[x]$. The set $\overline{R} := \{s \in S \mid s \text{ is integral over } R\}$ is a subring of S containing R known as the *integral closure of R in S* . In case R is a domain and S is the quotient field of R , then \overline{R} is called the *integral closure of R* . If $R = \overline{R}$, the domain R is said to be *integrally closed*. Integrally closed domains are ubiquitous in commutative ring theory. Indeed, most of the domains studied in multiplicative ideal theory are integrally closed. This massive class properly contains the classes of Prüfer domains and GCD domains, to name but two (we refer the reader to [2] for further details).

It is not hard to see that the theory of integrally closed domains is axiomatizable. Toward this end, we introduce a new definition.

Definition 1. Let D be a domain with quotient field K , and let n be a positive integer. Say that D is *n -integrally closed* provided for all $\alpha \in K$: if $f(\alpha) = 0$ for some monic polynomial $f(x) \in D[x]$ of degree n , then $\alpha \in D$.

Clearly a domain D is integrally closed if and only if D is n -integrally closed for every positive integer n . Moreover, the property of being n -integrally closed can be expressed in first order logic (in the language of rings) via the following sentence φ_n :

$$\forall d_0 \cdots \forall d_{n-1} \forall a \forall b ((b \neq 0 \wedge d_0 b^n + d_1 a b^{n-1} + \cdots + d_{n-1} a^{n-1} b + a^n = 0) \Rightarrow \exists c (a = bc)).$$

It follows that the theory \mathcal{T}_{IC} of integrally closed domains is axiomatized by $\{\varphi_D\} \cup \{\varphi_n : n > 0\}$, where φ_D is the conjunction of the (commutative) integral domain axioms. We are now ready to prove the main result of this note, namely:

Theorem 1. *The theory of integrally closed domains is not finitely axiomatizable.*

Proof. Let n be a positive integer. It suffices to prove the existence of an integral domain D which is m -integrally closed for $1 \leq m \leq n$, but not integrally closed. Toward this end, let q be an arbitrary prime, and let p be a prime number larger than n . Let $F := \mathbb{F}_{q^p}$ be the field with q^p elements, and E be the prime subfield of F . Further, let t be a generator of the multiplicative group F^\times . Now let $\varphi: E[x] \rightarrow F$ be the evaluation map defined by $\varphi(f(x)) := f(t)$. Finally, set $D := \varphi^{-1}(E)$. Note that since p is prime,

(0.2) there are no intermediate fields properly between E and F .

Now pick any nonzero $f \in \ker(\varphi)$. Then $\{f, xf\} \subseteq \ker(\varphi) \subseteq D$. We deduce that the quotient field of D contains x . Letting $E(x)$ denote the field of rational functions over E , it is evident that

(0.3) $E(x)$ is the quotient field of D .

Next, let m be an integer with $1 \leq m \leq n$. We prove that

(0.4) D is m -integrally closed.

Toward this end, suppose that $\alpha \in E(x)$ satisfies

(0.5) $d_0 + d_1\alpha + d_2\alpha^2 + \cdots + \alpha^m = 0$,

where $d_0, d_1, \dots, d_{m-1} \in D$. Since $D \subseteq E[x]$, it follows that α is integral over $E[x]$. But $E[x]$ is a UFD, hence is integrally closed (cf. [3], p. 397). We deduce from (0.3) that $\alpha \in E[x]$. Applying φ to both sides of (0.5), we see that $\varphi(\alpha) \in F$ is algebraic over E of degree at most m . It follows from (0.2) above that $E(\varphi(\alpha)) = E$ or $E(\varphi(\alpha)) = F$. The latter is impossible since then $\varphi(\alpha)$ would have degree p over E and $m < p$. We deduce that $\varphi(\alpha) \in E$. But then by definition of D , we see that $\alpha \in D$. This proves (0.4). Finally, note that $t^{q^p-1} = 1$. Therefore $f(x) := x^{q^p-1} \in D$. We conclude that x is a root of $g(y) := y^{q^p-1} - f(x) \in D[y]$, yet $x \notin D$ (since t is a generator of F^\times , it follows that $t \notin E$). Hence D is not integrally closed, and the proof is complete. \square

Remark 2. The previous proof shows that the theory of integrally closed domains of characteristic q is not finitely axiomatizable, where q is an arbitrary prime (that the theory of integrally closed domains of characteristic 0 is not finitely axiomatizable follows at once from the fact that the theory of fields of characteristic 0 is not finitely axiomatizable).

Corollary 1. *Let β be a sentence in the language of rings such that every model of β is an integrally closed domain. Then there exists some integrally closed domain D which is not a model of β .*

Recall that a cancellative, commutative monoid M is *root closed* provided for all $g \in \mathcal{Q}(M)$ (the group of fractions of M) and all positive integer n : if $g^n \in M$, then $g \in M$. We close this note with a short proof of Theorem 1 of [4].

Corollary 2. *The theory of root closed monoids is not finitely axiomatizable.*

Proof. Let D be the domain constructed in the proof of Theorem 1, and consider the multiplicative monoid $M := D - \{0\}$. Then M is m -root closed for all m , $1 \leq m \leq n$, but M is not root closed, since $x^{q^p-1} \in D$, yet $x \notin D$. \square

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