Problem Proposal #87

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Problem Let n be a non-negative integer, and consider the ring $R := \mathbb{Q}[X_0, \ldots, X_n]$ of polynomials (via usual polynomial addition and multiplication) in the (commuting) variables $X_0, \ldots X_n$ with coefficients in $\mathbb{Q}$. It is well-known that R is a Noetherian ring, and so every ideal of R is finitely generated. Since R is countable, and there are but countably many finite subsets of a countable set, we deduce that R has but countably many ideals and thus, in particular, countably many maximal ideals. Next, let $X_0, X_1, X_2, \ldots$ be a countably infinite collection of indeterminates. Observe that (to within isomorphism) $\mathbb{Q}[X_0] \subseteq \mathbb{Q}[X_0, X_1] \subseteq \mathbb{Q}[X_0, X_1, X_2] \subseteq \cdots$. Let $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ be the union of the this increasing chain. How many maximal ideals does the ring $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ have? (More precisely, what is the cardinality of the set of maximal ideals of $\mathbb{Q}[X_0, X_1, X_2, \ldots]$?)[1]

[1]Recall that an ideal J of a ring R with identity is *maximal* if J is a proper ideal of R and J is not properly contained in any proper ideal of R.