GETTING TO THE ROOT OF THE ROOT LAW FOR LIMITS

Abstract. In this note, we present a short proof of the root law for real sequences suitable for both calculus students and math majors alike.

0.1. **Introduction.** One of the most fundamental ideas in calculus and analytic geometry is the concept of "getting arbitrarily close" to a certain real value; more formally, this is the notion of *convergence*. Indeed, this concept is ubiquitous throughout real analysis. We recall the usual definition of convergence of a sequence of real numbers.

Definition 1. Suppose that (x_n) is a sequence of real numbers (indexed by the set of positive integers) and let $L \in \mathbb{R}$. Then the sequence (x_n) **converges** to L provided that for every $\epsilon > 0$, there is $m \in \mathbb{Z}^+$ such that for $n \geq m$, $|x_n - L| < \epsilon$. If (x_n) converges to L, we often denote this fact by $(x_n) \to L$, and we call L the **limit** of the sequence (x_n).

The limit behaves particularly well with respect to the usual arithmetic operations on the reals. We recall the standard limit laws below, which are often presented in a first course in calculus.

Theorem 1 (Limit Laws)**.** *Suppose that (x_n) and (y_n) are sequences of real numbers converging to X and Y, respectively, and let $r \in \mathbb{R}$. Then the following hold:*

(1) $(rx_n) \to rX$,
(2) $(x_n \pm y_n) \to X \pm Y$,[1]
(3) $(x_n y_n) \to XY$, *and*
(4) $(\frac{x_n}{y_n}) \to \frac{X}{Y}$ *if* $y_n \neq 0$ *for all* n *and* $Y \neq 0$.

Of course, many authors augment this list by including other laws as well (which are sometimes stated for functions rather than sequences; as is well-known, there is a natural way to translate between these two settings). For instance, in [10], Stewart's Limit Law 10 on p. 36, after being converted from the realm of functions of a real variable to the setting of real sequences, asserts the following.

Theorem 2 (Root Law for Limits)**.** *Suppose that (x_n) is a real sequence converging to X and that k is a positive integer. Then $(\sqrt[k]{x_n}) \to \sqrt[k]{X}$, where if k is even, it is assumed that $X > 0$ (so as to avoid imaginary numbers).*

Stewart proves laws (1)–(4) above in his appendix, though no proof is given for the root law for limits. Moreover, after conducting a literature review including calculus and both recent and not-so-recent analysis texts (some well-cited and some not), we have not found any that include a proof of the root law. Indeed, [3] and [4] give the root law as an exercise and only for $k = 2$. The $k = 2$

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[1]Here, we intend for the same arithmetic operation to appear on both sides of the arrow.

case is proved in 12 lines on p. 66 of [1]; however, the texts [2],[5-9],[11-12] don't mention even a special case of the root law in the text or the exercises. Of course, from the continuity of the exponential function $f(x) := e^x$, the limit law for roots follows immediately. On the other hand, the exponential function is often introduced rigorously (in a real analysis course) after developing the theory of convergence for power series, a treatment of which is necessarily given *after* an introduction to limits.

The purpose of this note is to give a compact and simple proof of the root law. Before proceeding, we give a "standard proof" of the root law by extrapolating the argument for the case $k = 2$ in [1]. The complication of the algebra which ensues yields a possible reason why the proof is hard to find in a calculus textbook. Indeed, for calculus students struggling to understand the epsilon-delta definition of "limit", the following argument (including the question of how one would think to apply equation (0.1) below) may be difficult to intuit.

Proof of the root law. Suppose that $(x_n) \to X$ and let $k \in \mathbb{Z}^+$ be such that if k is even, then $X > 0$. We shall prove that $(\sqrt[k]{x_n}) \to \sqrt[k]{X}$.

Case 1. $X = 0$. Then k is odd. Now let $\epsilon > 0$ be arbitrary, and choose $m \in \mathbb{Z}^+$ such that if $n \geq m$, then $|x_n| < \epsilon^k$. Taking the kth root of both sides for $n \geq m$, we get $\sqrt[k]{|x_n|} = |\sqrt[k]{x_n}| < \epsilon$, and so $(\sqrt[k]{x_n}) \to 0 = \sqrt[k]{0}$.

Case 2. $X \neq 0$. Then we may assume without loss of generality that $x_n > 0$ for every $n \in \mathbb{Z}^+$ or $x_n < 0$ for every $n \in \mathbb{Z}^+$ as this is true for sufficiently large n. Before proceeding, recall that for any $a, b \in \mathbb{R}$ and any $m \in \mathbb{Z}^+$, we have the telescoping sum

$$a^m - b^m = (a - b) \sum_{i=0}^{m-1} a^i b^{m-1-i}. \tag{0.1}$$

Now let $\epsilon > 0$. Because $(x_n) \to X$, there is $m \in \mathbb{Z}^+$ such that $|x_n - X| < \sqrt[k]{|X|^{k-1}}\epsilon$ for $n \geq m$. Fix $n \geq m$, and set $a := \sqrt[k]{x_n}$ and $b := \sqrt[k]{X}$. By (0.1), we get

$$x_n - X = (\sqrt[k]{x_n} - \sqrt[k]{X}) \sum_{i=0}^{k-1} \sqrt[k]{x_n^i} \sqrt[k]{X^{k-1-i}}. \tag{0.2}$$

Our next claim is that

$$\text{for } 0 \leq i \leq k-1, \ \sqrt[k]{x_n^i} \sqrt[k]{X^{k-1-i}} > 0. \tag{0.3}$$

If x_n and X are positive, then (0.3) is clear. Suppose that x_n and X are negative. Then k is odd. Hence for $0 \leq i \leq n$, i and $k - 1 - i$ are both even or both odd, implying (0.3) above. Using (0.3) and solving (0.2) above for $\sqrt[k]{x_n} - \sqrt[k]{X}$, we obtain

$$
\begin{aligned}
|\sqrt[k]{x_n} - \sqrt[k]{X}| &= \left| \frac{x_n - X}{\sum_{i=0}^{k-1} \sqrt[k]{x_n^i}\sqrt[k]{X^{k-1-i}}} \right| \\
&= \frac{|x_n - X|}{\sum_{i=0}^{k-1} \sqrt[k]{|x_n|^i}\sqrt[k]{|X|^{k-1-i}}} \\
&\leq \frac{|x_n - X|}{\sqrt[k]{|X|^{k-1}}} \text{ (plugging in 0 for } i \text{ above)} \\
&< \epsilon.
\end{aligned} \tag{0.4}
$$

Remark 1. Now that the smoke has cleared, we would like to elaborate a bit on how one might be led to choose the quantity $\sqrt[k]{|X|^{k-1}}\epsilon$ in the inequality on the line following (0.1) above. Our goal is to show that we can make $|\sqrt[k]{x}_n - \sqrt[k]{X}|$ arbitrarily small with large enough n. Upon substituting $\sqrt[k]{x_n}$ for a and $\sqrt[k]{X}$ for b in (0.1), we may solve for $|\sqrt[k]{x}_n - \sqrt[k]{X}|$ in (0.4) above. This allows us to obtain $|x_n - X|$ in the numerator, which we can make as small as we like since $(x_n) \to X$. Moreover, since adding a positive number to a positive denominator makes a fraction with positive numerator smaller, we make a judicious choice to eliminate all but the first term of the denominator sum in (0.4); this yields the value $\sqrt[k]{|X|^{k-1}}\epsilon$.

0.2. **Easier Proof.** We now present a proof which is significantly easier to parse than the one presented above; the proof hinges upon the following simple observation.

Lemma 1. *Let (x_n) be a sequence of real numbers such that $(x_n) \to 1$. Then $(\sqrt[k]{x_n}) \to 1$ for any $k \in \mathbb{Z}^+$.*

Proof. Suppose that $(x_n) \to 1$ and $k \in \mathbb{Z}^+$. We may assume without loss of generality that each $x_n > 0$. Now let $0 < \epsilon < 1$ be arbitrary and choose $m \in \mathbb{Z}^+$ such that if $n \geq m$, then $1 - \epsilon < x_n < 1 + \epsilon$. Noting that $0 < 1 - \epsilon < 1$ and $1 + \epsilon > 1$, we see that $1 - \epsilon \leq \sqrt[k]{1 - \epsilon} < \sqrt[k]{x_n} < \sqrt[k]{1 + \epsilon} \leq 1 + \epsilon$. Thus $(\sqrt[k]{x_n}) \to 1$, as claimed. □

Short proof of the root law. Suppose that $(x_n) \to X$ and let $k \in \mathbb{Z}^+$ with the condition that if k is even, then $X > 0$. We show that $(\sqrt[k]{x}_n) \to \sqrt[k]{X}$ as follows: if $X = 0$, then we apply the argument given in Case 1 of the proof supplied in the previous section. Now assume that $X \neq 0$.[2] By limit law (1), we see that $(\frac{x_n}{X}) \to 1$. Applying Lemma 1, $\left(\frac{\sqrt[k]{x_n}}{\sqrt[k]{X}}\right) \to 1$. Invoking limit law (1) again, $(\sqrt[k]{x_n}) \to \sqrt[k]{X}$. □

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[2]Again, we may assume that each x_n and X have the same sign.

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