Calculating complex cosine clusters

Niles Johnson* Alan Loper† Greg Oman‡

June 23, 2022

Abstract

The purpose of this note is to investigate the symmetry of limit points arising from iterating complex powers of the cosine function (and to a lesser degree, the sine and exponential functions). We invite the interested reader to explore mathematical explanations for the data we present.

1 Introduction

The motivation for this paper stems from very humble beginnings. Indeed, the second author was bored while proctoring an exam as a graduate student at the University of Wisconsin, and decided to play around with his calculator (*Angry Birds* had yet to be invented, so his entertainment options were limited). He entered the cosine of 1 (radian) into his calculator. He then took the cosine of the output. Then the cosine of that output, and then the cosine of that output, and so on. It seemed that the outputs were beginning to get closer and closer to some fixed value which is approximately .739. Similarly, starting with the square of the cosine function evaluated at 1 and iterating as above, it also appears that the outputs start clustering around a real number. But when he moved to cubes, the outputs began to accumulate around two real numbers. We will give arguments to verify these conclusions below. Extrapolating from these results, we are led to the following more general question:

Question 1. *Let p be a complex number. Setting $a_0 := 1$, $a_1 := \cos^p(a_0)$, $a_2 := \cos^p(a_1)$, $a_3 := \cos^p(a_2), \cdots$, how many complex numbers z have the property that there are terms of this sequence that get arbitrarily close to, but not equal to, z (that is, how many* ***cluster points*** *does the sequence have)?*

Using the computational program *Sage* ([6]), we obtain data to assist us in conjecturing an answer to the above question. Initially, we weren't expecting the answer to be particularly illuminating or interesting, but after crunching the numbers, we see an inordinate amount of very striking symmetry. The purpose of this note is to present the data along with many sub-questions which we hope will inspire further research into pinning down exactly what is going on. Indeed, some of these questions may be good jumping-off points for undergraduate research projects; this is our hope, anyway. We explore analogs of Question 1 for the sine and exponential functions, though our main focus will be on the cosine function.

Toward this end, define a function from the complex plane to the set of natural numbers as follows:

$$N(p) := \text{number of cluster points of the sequence } (a_n) \text{ (defined as in Question 1).}$$

Now mark the pixel in the complex plane at point p with color $N(p)$ according to the following scheme, where the numbers below represent the outputs of $N(p)$.

*The Ohio State University Newark

†The Ohio State University Newark

‡The University of Colorado, Coloardo Springs

Figure 1.1. *Colors associated to outputs of N; the color black is used for outputs of N which Sage was unable to calculate.*

Using this color scheme, we present a graph of N.

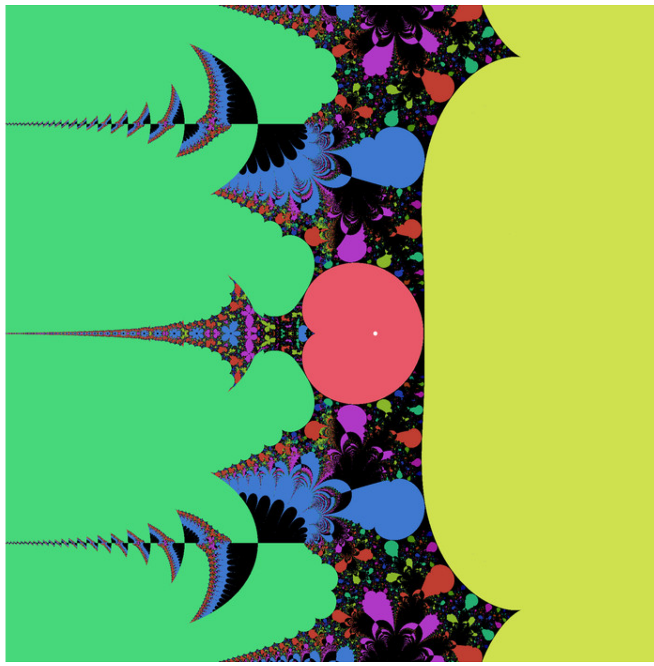

Figure 1.2. *Graph of N with domain $[-8, 8] \times [-8i, 8i]$, produced by Sage. The point $p = 1$ is marked with a white dot.*

There is (perhaps surprisingly) an abundance of rich symmetry and patterns, for which we invite the reader to explore mathematical explanations. As stated above, our purpose is to simply present some observations and questions for further research. We begin by giving a terse but self-contained introduction to complex exponentiation as well as complex-valued trig functions. We then proceed to present more graphs and observations; we close the paper with a final list of questions.

2 Complex exponentiation and trig functions

In this short section, we introduce the relevant definitions which will be needed to follow the results of this note.

In calculus, one learns that the function $f(x) = e^x$ is the unique differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) = 1$ and $f'(x) = f(x)$ for all real x. Moreover, this function is strictly increasing with range $(0, \infty)$, hence has an inverse function denoted by $\ln\colon (0, \infty) \to \mathbb{R}$. From this fact come the familiar identities $e^{\ln(x)} = x$ for every positive real number x and $\ln(e^x) = x$ for every real number x. These functions allow us to define a^b for every *positive* real number a and any real number b as follows: $a^b = (e^{\ln(a)})^b := e^{b\ln(a)}$ (note that if a is non-positive, then $\ln(a)$ is undefined).

Now, let's consider the problem of extending the exponential function to the complex plane.

Recall that the *imaginary number* i is defined by the equation $i^2 := -1$. A *complex number* is then a number of the form $z := a + ib$, where a and b are real numbers; moreover, the *real part* of z is a (denoted $Re(z) = a$) and the *imaginary part* of z is b (denoted $Im(z) = b$). Via the Pythagorean theorem, the *distance* between complex numbers $a + ib$ and $c + id$ is given by $\sqrt{(a-c)^2 + (b-d)^2}$ (see below, where we recall the usual geometric interpretation of complex numbers in the plane).

It turns out that, analogous to the real exponential function, there is a unique function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(0) = 1$ and $f'(z) = z$ for every complex number z; set $f(z) := e^z$ for complex z. For real x, let $g(x) := \frac{\cos(x)+i\sin(x)}{e^{ix}}$. A direct computation shows that $g'(x) = 0$ for all x and that $g(0) = 1$. Hence by differential calculus, $g(x) = 1$ for all real x, and so $e^{ix} = \cos(x) + i\sin(x)$ for all real x (this is the so-called *Euler's formula*).

Observe that the complex exponential function is *not* one-to-one: $e^0 = e^{i0} = \cos(0) + i\sin(0) = 1 = \cos(2\pi) + i\sin(2\pi) = e^{2\pi i}$. Thus we cannot simply take the inverse function to obtain a complex analog of the real natural logarithm function. Fortunately, there is a work-around. Consider a nonzero complex number $c := x + iy$. One may represent this number geometrically in the plane by the ordered pair (x, y) (we call the horizontal axis the *real axis* and the vertical axis the *imaginary axis*). Consider the associated (possibly degenerate) right triangle determined by $(0, 0)$, $(x, 0)$, and (x, y). By the Pythagorean theorem, the hypotenuse has length $\rho := \sqrt{x^2 + y^2}$. Moreover, $x = \rho\cos(\theta)$, and $y = \rho\sin(\theta)$ for some unique angle θ such that $-\pi < \theta \leq \pi$. Hence $c = x + iy = \rho\cos(\theta) + i\rho\sin(\theta) = \rho(\cos(\theta) + i\sin(\theta)) = \rho e^{i\theta}$. So now we set $\log(c) = \log(x + iy) = \log(\rho e^{i\theta}) := \ln(\rho) + i\theta$. This allows us to define complex exponentiation by $z^\omega := e^{\omega\log(z)}$ for complex numbers ω and z with $z \neq 0$.

Let's pause to do a few concrete computations.

Example 1. *Compute the following:*

1. i^3*,*

2. 3^i*, and*

3. i^i*.*

Solution: Let's tackle each in succession.

1. Recall that by definition, $i^2 = -1$, so $i^3 = i^2 \cdot i = -i$.
2. $3^i = e^{i\log 3} = \cos(\ln(3)) + i\sin(\ln(3))$.
3. $i^i = e^{i\log i} = e^{i(\ln 1 + \frac{i\pi}{2})} = e^{-\frac{\pi}{2}}$ (which is real!). QED

We can use the results above to define the complex cosine and sine functions. Let y be a real number. Recall that $e^{iy} = \cos(y) + i\sin(y)$. Thus $e^{-iy} = \cos(-y) + i\sin(-y) = \cos(y) - i\sin(y)$. Adding, we see that $e^{iy} + e^{-iy} = 2\cos(y)$. Solving for $\cos(y)$, we get $\cos(y) = \frac{e^{iy}+e^{-iy}}{2}$. This extends to the definition on the entire complex plane, given by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}. \tag{1}$$

Similarly, we obtain

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2}$$

We close this section by referring the reader to [1] and [5] for a comprehensive introduction to complex analysis.

3 Before we begin...

We will shortly consider iterations (as in the Introduction) of functions $f_p(z) := f(z)^p$, where p is a complex number and $f_p(x) = \cos^p(x),\ \sin^p(x)$, or e^{xp}, as p ranges over the complex plane (our primary focus will be on the cosine function, however). In particular, we study the sequences (a_n) whose terms are given by

$$a_0 := 1, a_1 := f_p(a_0),\ a_2 := (f_p(a_1)),\ a_3 := (f_p(a_2)), \ldots,$$

The general question we explore is that of the number of *cluster points* of (a_n) (recall from the Introduction that this is the number of complex numbers z for which there exist terms of (a_n) which get arbitrarily close to (but not equal to) z. We pause to illustrate this definition with an example.

Example 2. *Let $a_n := \cos(n\pi) + \frac{i}{n+1}$. Then the cluster points of (a_n) are -1 and 1.*

Before presenting the results of this paper, we note that the functions f_p are generally not well-defined until one chooses a branch for the logarithm (which corresponds to restricting the angle θ as we did in the previous section to obtain a well-defined logarithmic function), resulting in discontinuities. This means that much of the general theory of complex dynamics does not apply. Nevertheless, it is clear from our observations below that, in spite of the discontinuities, there is interesting behavior similar to the dynamics of entire functions (that is, functions $f\colon \mathbb{C} \to \mathbb{C}$ which are differentiable on the entire complex plane). Recursively defined sequences as above are well-studied for families of rational functions, and some work has appeared on exponential functions of the form $z \mapsto ae^z + be^{-z}$ for complex numbers a and b (see [3], [7]). Our purpose is to present data on iterates of complex powers of the cosine function (and some on the sine and exponential functions) and invite the reader to explore mathematical explanations for the phenomena we observe.

4 Iterates of complex powers of cosine

We begin by presenting a few results on iterates of powers of cosine for small (real) p to motivate the observations made later in the article. We include a proof sketch.

Proposition 1. *Let p be a complex number, and consider the sequence (a_n) defined by $a_0 := 1$, and for $n \geq 0$, $a_{n+1} := \cos^p(a_n)$. Then the following hold:*

(1) If $p = 1$, then (a_n) has exactly one cluster point.

(2) If $p = 2$, then (a_n) also has exactly one cluster point.

(3) If $p = 3$, then (a_n) has two cluster points.

SKETCH OF PROOF: Let (a_n) be defined as above.

(1) Let $p = 1$. Observe first that $a_n \geq 0$ for all n. Moreover, for any $x, y \in [0,1]$, the Mean Value Theorem yields the existence of $c \in [0,1]$ such that $\cos(y) - \cos(x) = -\sin(c)(y-x)$. But then $|\cos(y) - \cos(x)| = |-\sin c||y-x| \leq \sin(1)|y-x|$, and $0 < \sin(1) < 1$. The conclusion now follows from the Banach Fixed Point Theorem (see [2] for standard results on fixed point theory), which implies that (a_n) converges to the unique real number r_0 such that $\cos(r_0) = r_0$ (that is, r_0 is a *fixed point* of cos), which implies that r_0 is the only cluster point of (a_n).

(2) It follows from calculus that $\cos^2$ has a unique fixed point r_0 and that $r_0 \in (0,1)$. Next, $a_0 = 1 > \cos^2(r_0) = r_0$. Since $\cos^2$ is strictly decreasing on $[0,1]$, we see that $0 < a_1 = \cos^2(a_0) = \cos^2(1) < \cos^2(r_0) = r_0$. Next, $a_2 = \cos^2(a_1) < \cos^2(0) = 1 = a_0$, and as $a_1 < r_0$, $a_2 = \cos^2(a_1) > \cos^2(r_0) = r_0$. We have shown that $a_0 > a_2 > r_0$. Continuing, since $a_2 > r_0$, $a_3 = \cos^2(a_2) < \cos^2(r_0) = r_0$. As $a_0 > a_2$ and $\cos^2$ is strictly decreasing on $[0,1]$, we see that

$a_1 < a_3$. Thus $a_1 < a_3 < r_0$. Continuing inductively, one shows that (a_{2n}) is strictly decreasing and (a_{2n+1}) is strictly increasing. Since both subsequences are bounded, they converge. Let L be the limit of (a_{2n}) and M be the limit of (a_{2n+1}). Then by continuity and the fact that $(a_{2n+2}) \to L$, we have $\cos^2(\cos^2(L)) = L$. Analogously, $\cos^2(\cos^2(M)) = M$. One can use differential calculus to show that $\cos^2 \circ \cos^2$ has a unique fixed point, which must then be equal to r_0. Hence (a_n) converges to r_0 when $p = 2$, and again, there is a unique cluster point.

(3) Using methods similar to those above, it can be shown that $\cos^3 \circ \cos^3$ has three fixed points. Moreover, (a_{2n}) converges to the fixed point of largest magnitude and (a_{2n+1}) converges to the fixed point of smallest magnitude. This yields two cluster points of (a_n). QED

Digging a bit deeper, for real positive $p \leq 2$, the sequence defined above converges to a single fixed point. For larger real p, this sequence oscillates between two distinct cluster points.[1] This behavior can be explained by elementary dynamics analogous to the proof of Proposition 1, which we omit. However, for complex p, the situation appears to be much more interesting.

Recall from the Introduction that we defined a function from the complex plane to the set of natural numbers as follows:

$$N(p) := \text{number of cluster points of the sequence } (a_n) \text{ (defined as in Proposition 1).}$$

We marked the pixel in the complex plane at point p with color $N(p)$ according to the following scheme, where the numbers below represent the outputs of $N(p)$.

Figure 4.1. *Colors associated to outputs of N; the color black is used for outputs of N which Sage was unable to calculate; see also Figure 1.1.*

Using this color scheme, we recall the graph of N from the Introduction.

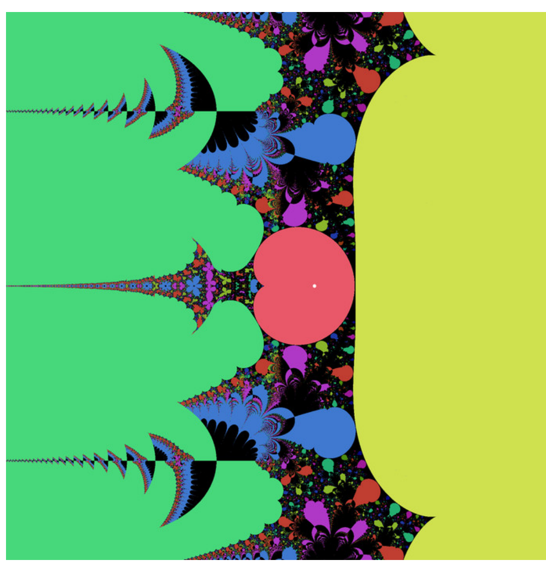

Figure 4.2. *Graph of N with domain $[-8, 8] \times [-8i, 8i]$, produced by Sage. The point $p = 1$ is marked with a white dot; see also Figure 1.2.*

[1]The transition from one to two cluster points occurs when $p \approx 2.188$.

Remark 1. *To obtain the above graph, Sage computes $a_1 - a_{500}$, and then looks at the next 30 terms. It returns the minimum n for which the subsequences $a_{501}, a_{502}, \ldots, a_{500+n}$ and $a_{500+n+1}, \ldots, a_{500+2n}$ $(1 \leq n \leq 15)$ have the property that a_{500+k} and $a_{500+n+k}$ are within $\epsilon := .001$ for $1 \leq k \leq n$. If no such n is found, it computes 500 more iterates and tries again. This is repeated 6 times. If still no such n is found, the pixel is colored black.*

Below, we note several observations.

Observations 1.

1. *The graph of N appears to be symmetric about the real axis.*
2. *The transition from $N(p) = 1$ to $N(p) = 2$ on the real axis occurs at $p \approx 2.188...$, and this can be confirmed via elementary dynamics of $\cos^p$ on the real line, as mentioned earlier.*
3. *The region $N(p) = 1$ appears to be a cardioid around 0 with cusp point at $p \approx -\frac{1}{2}$.*

4.1 Zooming in

Now we zoom in a bit on Figure 4.2 and make some similar observations.

Figure 4.3. *The upper boundary of the region $N(p) = 1$.*

Observations 2.

1. *There appear to be contact points along the boundary of $N(p) = 1$ with the regions $N(p) = 3, 5, 7, 9, 11, 13,$ and 15.*
2. *Regions where $N(p) = 4, 6, 8, 10, 12,$ or 14 appear along the boundary of the region $N(p) = 2$.*

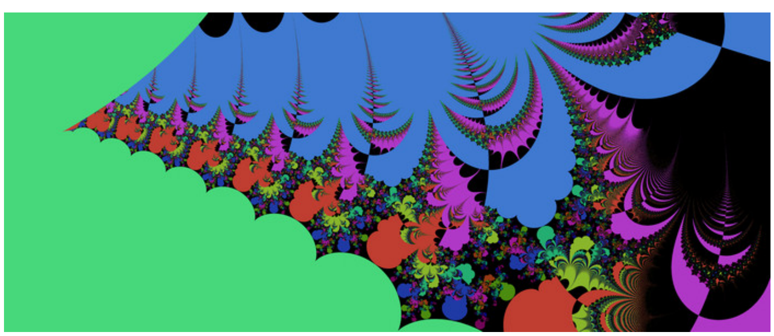

Figure 4.4. *A portion of the boundary of the region $N(p) = 3$, extending at an angle roughly $\frac{2\pi}{3}$ from the origin.*

Observation 1. *There appears to be self-similar behavior: bumps in the region $N(p) = 3$ tangent to regions $N(p) = 6, 9, 12$. This behavior ends abruptly on the left side of the picture.*

4.2 Zooming out

Next, we zoom out by increasing the domain of N as noted below.

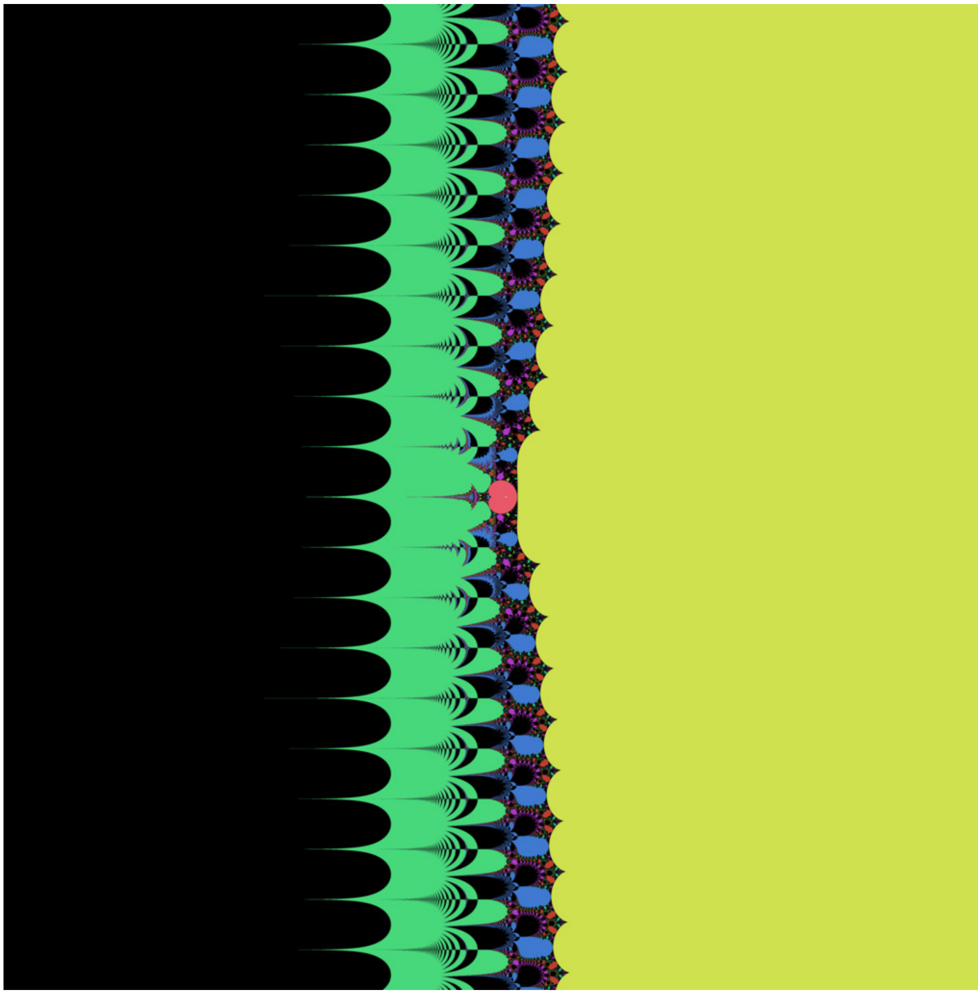

Figure 4.5. *Graph of N with domain $[-50, 50] \times [-50i, 50i]$.*

Observations 3.

1. *No period is computed for the black region. We do not know if this is a genuine feature, or a limitation of our software.*
2. *Horizontal bands appear to repeat; the gap between bands is between 5.08 and 5.11. We conjecture the exact value to be $\frac{\pi}{\ln(\frac{1}{\cos 1})}$.*
3. *The region $N(p) = 2$ appears to be asymptotic to a line somewhere between $Re(p) = 7.5$ and $Re(p) = 8.5$.*
4. *The graph of N appears in places to be somewhat erratic. For example, $N(-10) = 22$ (this cannot be gleaned from the graph, but was calculated by Sage) but $N(-10 + .001i) = 4$.*
5. *Increasing the domain of N to $[-150, 150] \times [-150i, 150i]$ does not show any new phenomena.*

5 Related functions

5.1 Taylor approximations

Here we consider the Taylor approximations $1 - \frac{z^2}{2}$ and $1 - \frac{z^2}{2} + \frac{z^4}{24}$ to $\cos(z)$. We apply the same method as above, replacing $\cos(z)$ with $1 - \frac{z^2}{2}$ and $1 - \frac{z^2}{2} + \frac{z^4}{24}$, respectively, and the resulting graphs are very similar to Figure 4.2.

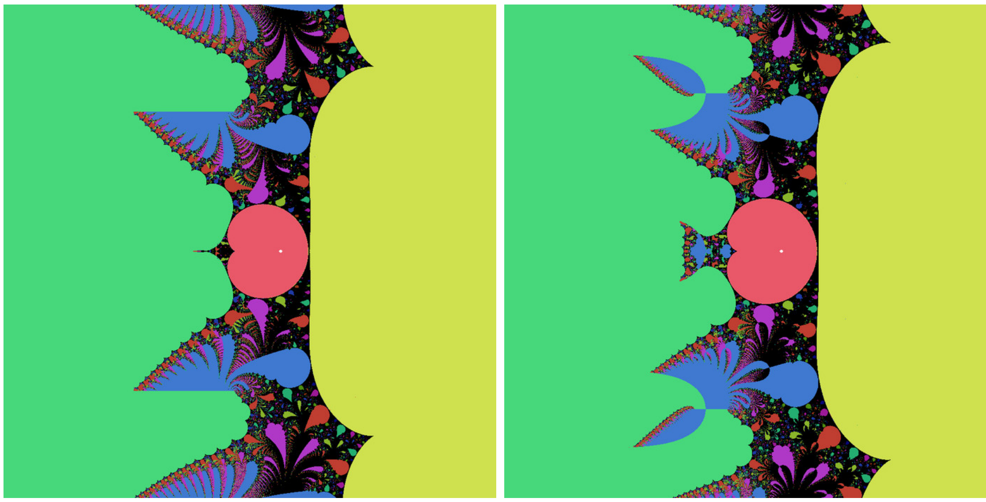

Figure 5.1. *Graphs of $N(p)$ obtained via iteration of the function $f_p(z) = \left(1 - \frac{z^2}{2}\right)^p$ (left) and $f_p(z) = \left(1 - \frac{z^2}{2} + \frac{z^4}{24}\right)^p$ (right), with $p = 1$ marked with a white dot. As before, the graphs give the number of cluster points of the sequence (a_n) determined by the f_p functions above for complex p.*

5.2 The sine and exponential functions

Similar behavior appears to arise for the sine and exponential functions, although we have not explored these other cases as broadly as we did for the cosine function.

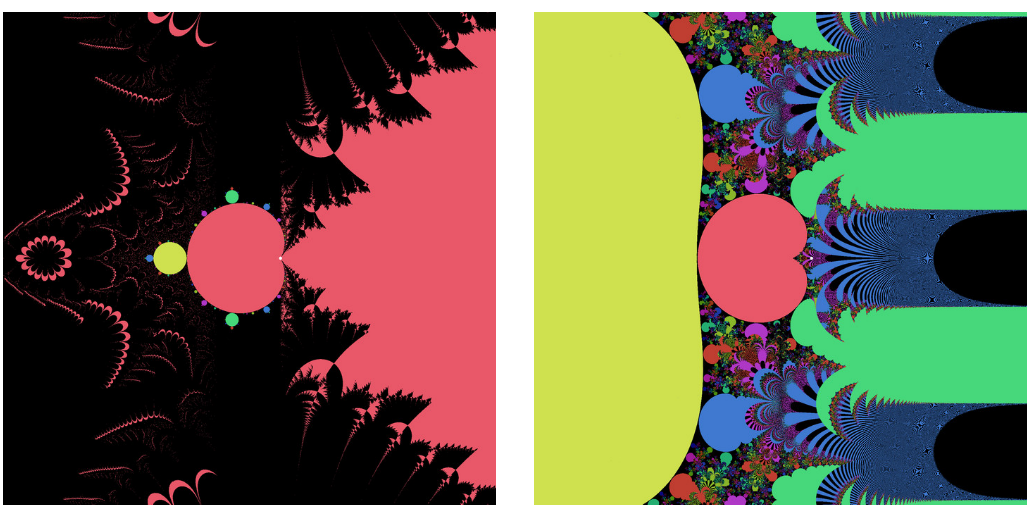

Figure 5.2. *Graphs of $N(p)$, using the same color scheme, for iterates of $f_p(z) = \sin^p(z)$ (left) and $f_p(z) = e^{pz}$ (right). In each case, a white dot marks the point $p = 1$.*

We pause to give the reader an idea of why the cosine function is our main area of focus in this paper. When p is real and positive (the setting considered at the genesis of the paper, many decades ago), the sine function yields $N(p) = 1$ (see Figure 5.2 above). Moreover, the sole cluster point of the corresponding sequence (a_n) is zero. For real $p > 0$, the exponential function has $(a_n) \to \infty$.

All of the questions we've asked above have analogs for the sine and exponential functions; we have one additional question specific to the sine function:

Question 2. *Why does the left graph in Figure 5.2 appear to match the Mandelbrot set so closely (more closely than for the cosine and exponential functions)? Are there connections to the universality of the Mandelbrot set (see [4])?*

6 Closing questions concerning iterates of cosine

We conclude the paper with a list of questions for further exploration.

Questions 1 (Cardioid)**.**

1. *Is the heart-shaped object in the middle of the graphs a cardioid?*
2. *If it is a cardioid parametrized by*

$$-r(1 - \cos(t))e^{it} - c,$$

then we must have $c \approx .5$ and $r \approx \frac{2.188+.5}{2} = 1.344$. Is there a nice closed form expression for these constants? (Note: a cardioid with these numerical constants has y-intercepts at roughly ± 1.658, and the computed region $N(p) = 1$ is consistent with these values.)

3. *Do points on the boundary of the cardioid yield a single cluster point (as is apparently the case for all points on the interior)?*

Questions 2 (Left half-plane)**.**

1. *Beyond a short distance from the center, the left half-plane appears as black (recall Figure 4.5). Calculating one point at a time in this region fails for most points, but not all. For example, it computes easily for $p = -157$. Similarly, other scattered points are computable. Is it the case that for these points, the numbers involved become too large for the software to handle or is there some other explanation?*
2. *At the start of the left half-plane it looks as if the region is divided into horizontal strips for which we have uniformly $N(p) = 3$. These strips are separated by very narrow lines in which it looks as if most anything can happen. Does this pattern continue for the entire half-plane?*
3. *It seems that the three cluster points for any point in the $N(p) = 3$ region consist of*
    - *a value very close to 0,*
    - *a value very close to 1, and*
    - *another point w.*

    As p moves vertically along a line $Re(p) = -t$, it seems that the absolute value of this third cluster point, w, is constant at $|w| = k^t$, with $k \approx 1.8508$. As p changes along this line, w appears to rotate around the circle of radius k^t. It takes two *bands to make a full rotation, thus the period appears to be roughly 10.2. Can any of these numerical values be confirmed?*

Questions 3 (Global questions)**.**

1. *Consider a complex number of the form $p := -a + ib$, where a is a positive real number and $b \neq 0$ is real. It seems that the moduli $\sqrt{Re(z)^2 + Im(z)^2}$ of the non-negligible terms of (a_n) approach $(\frac{1}{\cos 1})^a$. Can this be verified?*

2. *Does every natural number occur as a value of $N(p)$ for some point p?*

3. *Is $N(p)$ indeed symmetric with respect to the real axis, as appears to be the case from the graphs?*

4. *How do the graphs change if one changes the initial point a_0?*

5. *Is there any regular behavior in the set of cluster points for various p? For example, in numerical experiments it appears that, for each p with $N(p) > 1$, one of the cluster points is very close to 1. If $N(p) > 2$, another of the cluster points is very close to 0. Does this hold for all p with $N(p) > 2$?*

References

[1] Ahlfors, L., "Complex analysis. An introduction to the theory of analytic functions of one complex variable." Third edition. International Series in Pure and Applied Mathematics, McGraw-Hill Book Company, 1978. 3

[2] Granas, A. and Dugundji, J., "Fixed point theory." Springer Monographs in Mathematics, Springer-Verlag, 2003. 4

[3] Hubbard, J., Schleicher, D., and Shishikura, M., *Exponential Thurston maps and limits of quadratic differentials,* Journal of the American Mathematical Society, Vol. 1, No. 22, pp 77–117, 2009. 4

[4] McMullen, C., *The Mandelbrot set is universal,* In The Mandelbrot Set, Theme and Variations, ed. T. Lei, 1–18. Cambridge U.K.: Cambridge University Press. Revised 2007. 9

[5] Remmert, E., "Theory of complex functions." Translated from the second German edition by Robert B. Burckel. Graduate Texts in Mathematics, Vol. 122. Readings in Mathematics. Springer-Verlag, 1991. 3

[6] The Sage Developers, *SageMath, the Sage Mathematics Software System. Version 9.3 (2021).* https://sagemath.org. 1

[7] Schleicher, D., *Dynamics of entire functions. Holomorphic dynamical systems,* Lecture Notes in Mathematics, 1998, pp 295–339, Springer, 2010. 4

About the authors

Niles Johnson

Department of Mathematics, The Ohio State University, Newark, OH 43055 njohnson@ohio-state.edu

Alan Loper

Department of Mathematics, The Ohio State University, Newark, OH 43055 lopera@ohio-state.edu

Greg Oman

Department of Mathematics, The University of Colorado, Colorado Springs, CO 80918 goman@uccs.edu