ALMOST STRONGLY UNITAL RINGS

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Abstract. In the recent article [10], the authors determine all rings R for which every subring of R has an identity (which need not be the identity of R), calling such rings *strongly unital*. In this note, we extend this work to determine the rings S for which every proper subring of S has an identity, yet S does not, calling such rings *almost strongly unital*. We conclude the paper by classifying the rings T for which a subring R of T has an identity if and only if there is a subring S of T such that $R \subsetneq S \subsetneq T$.

1. Introduction

Of interest in the literature over the past several decades are mathematical structures which *almost* possess a certain property P. There are many ways in which one might make this precise; a natural way is the following: say that a mathematical structure $\mathfrak{U}$ *almost has property* P provided $\mathfrak{U}$ does not have property P, but every *proper* substructure (or quotient structure) has property P. We begin with the following interesting example.

Example 1. *Let p be a prime. The* ***quasi-cyclic group or Prüfer group*** *of type p^∞, denoted $C(p^\infty)$, is the direct limit of the cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$ as $n \to \infty$. The group $C(p^\infty)$ is infinite, yet every proper subgroup of $C(p^\infty)$ is finite (see ([3])). Moreover, if G is any infinite abelian group for which every proper subgroup of G has smaller cardinality than G, then $G \cong C(p^\infty)$ ([11]).*

Following the terminology introduced in the above example, we say that if G is a countably infinite abelian group, then G is *almost finite* if and only if $G \cong C(p^\infty)$ for some prime p. An infinite (possibly nonabelian) group G with the property that every proper subgroup of G has smaller cardinality than G is called a *Jónsson group*. By Example 1 above, every abelian Jónsson group is countable. On the other hand, nonabelian Jónsson groups exist. Indeed, Shelah constructs a Jónsson group of cardinality $\aleph_1$ in [12].

In the case of rings and modules, in [13], Weakley calls a module M over a commutative ring R with identity *almost finitely generated* provided M is not finitely generated, yet every proper submodule of M is finitely generated. Gilmer and Heinzer call a module M over a commutative ring R with identity a *Jónsson module* if M is infinite, yet every proper submodule of M has smaller cardinality than M ([5]). The dual notion, that of an infinite module M (again, over a commutative ring with identity) for which $|M/N| < |M|$ for every nonzero submodule N of M is studied in [10]. In [9], Laffey completely characterizes those rings R (not necessarily commutative or unital) which are "almost finite." For further reading on Jónsson algebras, we refer the reader to the excellent survey article [1].

2010 Mathematics Subject Classification. Primary: 13A99; Secondary: 13M99, 13E10.

Key Words and Phrases. Artinian ring, commutative ring, nilpotent element, reduced ring, strongly unital ring.

In this note, we study rings R (not assumed commutative) with the property that R does not have an identity, but every proper subring S of R has an identity. In other words, we classify the "almost unital" rings. To do this, we make heavy use of the main result of [10], where the rings R for which every subring of R has an identity are determined (again, the subrings of R need *not* contain the identity of R, only *an* identity). The authors call such rings *strongly unital*. For consistency with terminology introduced in [10], we call a ring R *almost strongly unital* if R does not have an identity, yet every proper subring of R has an identity. We conclude the article by pushing the almost strongly unital condition down an additional step, that is, we classify the rings T for which a subring R of T has an identity if and only if there is a subring S of T such that $R \subsetneq S \subsetneq T$. We conclude the introduction by mentioning that throughout this note, a ring with *zero multiplication* is a ring R for which $xy = 0$ for all $x, y \in R$.

2. Results

We begin by introducing an elementary example of the rings studied in [10].

Example 2. *Let $R := \mathbb{Z}/15\mathbb{Z}$. The subrings of R are $S_1 := \{\overline{0}\}$, $S_2 := \{\overline{0}, \overline{5}, \overline{10}\}$, $S_3 := \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}\}$ and $S_4 := R$. Then the identity of S_1 is $\overline{0}$, the identity of S_2 is $\overline{10}$, the identity of S_3 is $\overline{6}$, and the identity of S_4 is $\overline{1}$. Hence all subrings of R have an identity.*

Below, we state a result (Proposition 1) which will be heavily used throughout the paper. First, recall that a field F is *absolutely algebraic* if F is algebraic over its prime subfield (the subfield generated by 1).

Proposition 1 ([10], Theorem 1)**.** *Let R be a nontrivial ring. Then R is strongly unital if and only if $R \cong F_1 \times \cdots \times F_n$ for some fields $F_1, \ldots, F_n$, each of prime characteristic and each algebraic over its prime subfield.*

Our first objective is to study rings R which alre almost strongly unital, that is, R does not have an identity but every proper subring of R does. As we will show, this is a very narrow class of rings. Of interest is the proof, which in part generalizes one of the main results of [4]. We begin by showing that every almost strongly unital ring is commutative.

Proposition 2. *Suppose that R is a ring such that every proper subring of R is unital. Then R is commutative.*

Proof. Suppose that R is a ring for which every proper subring of R has an identity. We consider two cases.

Case 1. R is generated (as a ring) by some element $r \in R$. Then it is easy to see that we have $R = \{m_1 r + m_2 r^2 + \cdots + m_k r^k \colon m_i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$. Clearly R is commutative in this case.

Case 2. R is not generated by any member of R. Let $\alpha \in R \backslash \{0\}$. Then $S := \{m_1\alpha + m_2\alpha^2 + \cdots + m_k\alpha^k \colon m_i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$ is a proper, nontrivial subring of R, thus has an identity. Moreover, every subring of S is also a proper subring of R, thus has an identity. By Proposition 1, $S \cong F_1 \times \cdots \times F_n$ for some fields $F_1, \ldots, F_n$, each of prime characteristic and each algebraic over its prime subfield.

Thus we may suppose that $\alpha = (c_1, \ldots, c_n)$ where each $c_i \in F_i$. Let K_i be the prime subfield of F_i. Then $K_i(c_i)$ is a simple algebraic extension of the finite field K_i, and hence is finite, say of order m_i. Observe that if $c_i \neq 0$, then by Lagrange's Theorem, $c_i^{m_i-1} = 1$. Set $k := (m_1-1)\cdots(m_n-1)$. Then it is easy to see that $\alpha^{k+1} = \alpha$. Invoking Jacobson's Theorem ([7]), we see that R is commutative in this case as well. □

Next, we give several lemmas which will be used to prove Theorem 1, which is one of the main results of this note. First, we comment on notation. Let R be a commutative ring. By $R[X]$, we denote the polynomial ring in the variable X with coefficients in R. The subring of polynomials with zero constant term will be denoted by $XR[X]$. Finally, if S is any commutative ring and $s \in S$, then $\langle s \rangle$ denotes the ideal of S generated by s.

Our first lemma can be found in [6]. We give a short proof for the reader's convenience.

Lemma 1. *Suppose that R is a nontrivial ring such that $\{0\}$ and R are the only left ideals of R. Then either R is a division ring or $R \cong X\mathbb{F}_p[X]/\langle X^2 \rangle$ for some prime p, where $\mathbb{F}_p$ is the field with p elements.*

Proof. Let R be as stated above. We consider two cases.

Case 1. R possesses a nonzero element r which is a zero divisor. Without loss of generality, $xr = 0$ for some nonzero $x \in R$. It follows that the left ideal $\text{Ann}_R(r) := \{y \in R \colon yr = 0\}$ contains both 0 and x, and thus coincides with R. Now consider the abelian group $\mathbb{Z}r := \{mr \colon m \in \mathbb{Z}\}$, and note that since $\text{Ann}_R(r) = R$, $\mathbb{Z}r$ is a nonzero left ideal of R. By the condition on R, $\mathbb{Z}r = R$. Moreover, every additive subgroup of $\mathbb{Z}r$ is a left ideal of R, thus is either trivial or exhausts R. Therefore, $\mathbb{Z}r$ is a simple abelian group, hence (as is well-known) has p elements for some prime p. As $R = \mathbb{Z}[r] = \text{Ann}_R(r)$, the product of any two elements of R is zero. Now, by Lagrange's theorem, any group of order p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$; it follows that (as is well-known) any two groups of order p are isomorphic. Via this isomorphism, it is immediate that any two rings of order p with zero multiplication are isomorphic. Since $X\mathbb{F}_p[X]/\langle X^2 \rangle$ is a ring of order p with zero multiplication, we deduce that $R \cong X\mathbb{F}_p[X]/\langle X^2 \rangle$, as desired.

Case 2. R has no (nonzero) zero divisors. Let $r \in R$ be nonzero. Then Rr is a nonzero left ideal of R and hence $Rr = R$. Thus $r = er$ for some $e \in R$. Multiplying through by e, we see that $er = e^2r$, and hence $(e - e^2)r = 0$. Since r is not a zero divisor, $e - e^2 = 0$, so $e = e^2$. Note that since $r = er$ and $r \neq 0$, also $e \neq 0$, and so e is not a zero divisor. We claim that e is the multiplicative identity of R. To see this, let $x \in R$ be arbitrary. Then $ex = e^2x$, so $e(x - ex) = 0$. Because e is not a zero divisor, $x - ex = 0$, so $ex = x$. Similarly, $xe = xe^2$. By an analogous argument, $xe = x$. This proves that R has an identity $e := 1$. Now let $y \in R$ be nonzero. Then by the condition on R, $Ry = R$. Hence $1 = xy$ for some (nonzero) $x \in R$. Hence every nonzero $y \in R$ has a left inverse. Now since $1 = xy$, $x = xyx$. Multiply on the left by the left inverse z of x to obtain $1 = yx$. Hence R is a division ring in this case, and the proof is complete. □

A generalization of our next lemma is in the literature, and can be found in [8, p. 22], where the author establishes that a left Artinian ring which has no nonzero nilpotent left ideals is necessarily semisimple with identity. We give a modified proof in the commutative setting. We remind the reader that an element r of a ring R is *nilpotent* if $r^n = 0$ for some positive integer n. Further, R is *reduced* if R has no nonzero nilpotent elements.

Lemma 2. *A commutative reduced Artinian ring has an identity.*

Proof. Let R be a commutative reduced Artinian ring. We may assume of course that R is nontrivial. It is a well-known consequence of Zorn's Lemma that every nontrivial ring with identity has a maximal, hence prime, ideal. However, we have not assumed that R has an identity. Yet we shall still establish the existence of maximal ideals of R. We begin by showing the following:

$$\text{if } r \in R \backslash \{0\}, \text{ then there is a prime ideal } P \text{ of } R \text{ for which } r \notin P. \tag{2.1}$$

To see this, let $S := \{r^n : n \in \mathbb{Z}^+\}$. Next, let $\mathcal{I}$ denote the collection of all ideals of R which are disjoint from S. Observe that $\mathcal{I}$ is a nonempty set since $\{0\} \in \mathcal{I}$ (because r is not nilpotent). By Zorn's Lemma, there is a member I of $\mathcal{I}$ which is maximal with respect to set inclusion. Note that I is proper since $r \notin I$. We claim that I is a prime ideal. Suppose not. Then there are $x, y \in R$ such that $xy \in I$, yet $x \notin I$ and $y \notin I$. By maximality of I, it follows that $\langle I, x\rangle$ and $\langle I, y\rangle$ intersect S nontrivially (here, $\langle I, x\rangle$ and $\langle I, y\rangle$ are the ideals of R generated by I and x and I and y, respectively). We conclude that $i_1 + rx + mx = r^a$ and $i_2 + sy + ny = r^b$ for some $i_1, i_2 \in I$, $r, s \in R$, $m, n \in \mathbb{Z}$, and $a, b \in \mathbb{Z}^+$. Multiplying the left sides of these equations together and using the fact that $xy \in I$, it follows that the product lies in I. But then $r^a \cdot r^b = r^{a+b} \in I$, contradicting that I and S are disjoint. This proves (2.1).

Next we claim that

$$\text{if } R \text{ has no zero divisors, then } R \text{ is a field.} \tag{2.2}$$

To see this, suppose R has no zero divisors, and let $r \in R$ be nonzero. Then observe that $\cdots \subseteq Rr^n \subseteq Rr^{n-1} \subseteq \cdots \subseteq Rr$. Because R is Artinian, there is a positive integer n such that $Rr^{n+1} = Rr^n$. Thus $r^{n+1} \in Rr^n = Rr^{n+1}$, so $r^{n+1} = er^{n+1}$ for some $e \in R$. Because R has no zero divisors and $r \neq 0$, we see that $r = er$, and so $er = e^2r$. Again, because R has no zero divisors, $e = e^2$ (and $e \neq 0$). By the proof included in Case 1 of Lemma 1, we see that $e := 1$ is the identity of R. But now since $Rr^n = Rr^{n+1}$ and R has an identity, $r^n = yr^{n+1}$ for some $y \in R$, and so $1 = yr$. This proves that R is a field. The following is immediate from (2.2), using the easy fact that the Artinian property passes to factor rings:

$$\text{every prime ideal of } R \text{ is maximal.} \tag{2.3}$$

Next, we show that

$$R \text{ has but finitely many prime ideals.} \tag{2.4}$$

Our proof of this fact is standard. Let $\mathcal{S}$ be the collection of all finite intersections of prime ideals of R. By (2.1), this collection is nonempty. Because R is Artinian, there exist minimal elements of $\mathcal{S}$; let $P_1 \cap \cdots \cap P_n$ be such a minimal element. We claim that $\{P_1, \ldots, P_n\}$ is the set of prime ideals of R. For suppose that P_{n+1} is another prime ideal. Then $P_1 \cap \cdots \cap P_n \cap P_{n+1} \subseteq P_1 \cap \cdots \cap P_n$. By minimality, we have $P_1 \cap \cdots \cap P_n \cap P_{n+1} = P_1 \cap \cdots \cap P_n$. Thus $P_1 \cap \cdots \cap P_n \subseteq P_{n+1}$. Because P_{n+1} is *prime*, a straightforward proof by contradiction shows that $P_i \subsetneq P_{n+1}$ for some i, $1 \leq i \leq n$. But by (2.3), P_i is maximal, and we have a contradiction. This proves (2.4).

Let $P_1, \ldots, P_n$ be the (distinct) prime ideals of R. By (2.2), it suffices to prove that

$$R \cong R/P_1 \times \cdots \times R/P_n, \tag{2.5}$$

since then R is isomorphic to a finite product of rings with identity (the summands being fields), hence R is unital as well. This follows immediately from the Chinese Remainder Theorem, but we include an argument since in many sources, the Chinese Remainder Theorem assumes the existence of a multiplicative identity. Define $\varphi\colon R \to R/P_1 \times \cdots \times R/P_n$ by $\varphi(r) := (\overline{r}, \ldots, \overline{r})$, where the ith $\overline{r}$ denotes the coset $P_i + r$. Clearly φ is both a ring and an R-module homomorphism with kernel $P_1 \cap \cdots \cap P_n$. Since R is *reduced*, it follows from (2.1) that $\ker(\varphi) = \{0\}$. It remains to show that φ is onto. Since $\varphi(R)$ is closed under addition, it clearly suffices to prove that for every i, $1 \leq i \leq n$ and every $x \in R$, $(\overline{0}, \ldots, \overline{x}, \overline{0}, \ldots, \overline{0})$ is in the range of φ, where the ith entry in the above sequence is $\overline{x}$. Without loss of generality, let $i = 1$ and let $x \in R$ be arbitrary. We must find $r \in R$ such that $r - x \in P_1$ and $r \in P_2 \cap \cdots \cap P_n$ (clearly we may assume that $n > 1$). If $P_2 \cap \cdots \cap P_n \subseteq P_1$, then as above, since P_1 is prime, $P_j \subsetneq P_1$ for some j, $2 \leq j \leq n$. But this is a contradiction to the maximality of P_j. Now, since P_1 is maximal, we see that $P_1 + (P_2 \cap \cdots \cap P_n) = R$. Hence $s + r = x$ for some $s \in P_1$ and $r \in P_2 \cap \cdots \cap P_n$. Thus $r - x = -s \in P_1$, and the proof is complete. □

We require a final lemma before proving the main result of this note. Our lemma is a generalization of one of the main results of [4], where the authors use the theory of Jónsson modules to show that if T is a ring with identity 1_T which admits a proper subring S with $1_T \in S$ and every proper subring[1] of T is Artinian, then T is Artinian. We present a generalization of this result using only basic principles.

Lemma 3 ([4], Corollary 2)**.** *Let T be a commutative ring with a proper subring S which contains a nonzero idempotent e. Suppose further that every proper subring of T is Artinian. Then T is Artinian.*

Proof. Let e, S, and T be as stated. Suppose by way of contradiction that T is not Artinian. Then there exists an infinite strictly decreasing sequence

$$\cdots \subsetneq I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 \tag{2.6}$$

of ideals of T. Let $R := \{me\colon m \in \mathbb{Z}\}$. Because e is idempotent, R is a subring of T. Moreover, $R \cong \mathbb{Z}/\langle n\rangle$ for some non-negative integer n. If $n = 0$, then $R \cong \mathbb{Z}$, and hence R is not Artinian. But $R \subseteq S \subsetneq T$, and this is a contradiction. It follows that

$$R \text{ is a finite ring, say of cardinality } n. \tag{2.7}$$

[1]by "subring", the authors mean "unital subring", that is, a subring which contains 1_T.

For every positive integer k, let $R_k := R + I_k$. It is easy to see that each R_k is a subring of T. Moreover, for every $k \in \mathbb{Z}^+$, $\cdots \subsetneq I_{k+2} \subsetneq I_{k+1} \subsetneq I_k$ is an infinite, strictly decreasing sequence of ideals of R_k. Because every proper subring of T is Artinian, we deduce that

$$R_k = T \text{ for every positive integer } k. \tag{2.8}$$

Next, we claim that for every positive integer k,

$$\text{there is a surjective ring homomorphism } f\colon T/I_{k+1} \to T/I_k \text{ which is not injective.} \tag{2.9}$$

Indeed, define $f\colon T/I_{k+1} \to T/I_k$ by $f(I_{k+1} + t) := I_k + t$. It is immediate from (2.6) that f is well-defined and obvious that f is a surjective ring homomorphism. To see that f is not injective, pick $t \in I_k \backslash I_{k+1}$. Then $I_{k+1} + t \in T/I_{k+1}$ is nonzero, yet $I_k + t \in T/I_k$ is zero. It is immediate from (2.9) that

$$\text{there is no finite upper bound on the sizes of } T/I_k \text{ where } k \text{ ranges over } \mathbb{Z}^+. \tag{2.10}$$

Next, let $k \in \mathbb{Z}^+$. Recall from (2.7) that $|R| = n$. Thus trivially,

$$|\{I_k + r\colon r \in R\}| \leq n. \tag{2.11}$$

Finally, recall from (2.8) that $R_k = T$. Hence $T/I_k = R_k/I_k = (R + I_k)/I_k = R/I_k$, and thus $|T/I_k| = |R/I_k| \leq n$ (from (2.11) above). This contradicts (2.10) and completes the proof. □

Remark. Note that we can easily recover Gilmer and Heinzer's result ([4], Corollary 2) as follows. Suppose that T is a ring with identity with a proper unital subring and with the property that every proper (unital) subring of T has an identity. Then the prime subring $P(T)$ generated by 1 cannot be all of T (lest T not have a proper unital subring), and so is Artinian. Hence $P(T) \ncong \mathbb{Z}$, so $P(T) \cong \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 1$. It follows that 1 is a nonzero idempotent of $R := P(T)$. The above argument shows that T is Artinian.

We are now equipped to classify the rings R for which every proper subring of R has an identity.

Theorem 1. *Let R be a nonzero ring. Then every proper subring of R has an identity if and only if one of the following holds:*

(1) *$R \cong F_1 \times \cdots \times F_n$ for some fields $F_1, \ldots F_n$, each of prime characteristic and each algebraic over its prime subfield, or*
(2) *$R \cong X\mathbb{F}_p[X]/\langle X^2 \rangle$ for some prime number p.*

Proof. Let R be a nonzero ring. If R is a finite direct product of absolutely algebraic fields of prime characteristic, then Proposition 1 implies that every subring of R has an identity, so every proper subring of R has an identity. Now suppose that $R \cong X\mathbb{F}_p[X]/\langle X^2 \rangle$. Then R is a ring with p elements and zero multiplication. Thus R does not have an identity. Moreover, the only proper subring of R is trivial, and thus obviously has an identity.

Conversely, suppose that every proper subring of R has an identity. If R has an identity, then we invoke Proposition 1 to deduce that R is a finite direct product of absolutely algebraic fields of prime characteristic, and we are done. Finally, suppose that R does not have an identity. Invoking Proposition 2, R is commutative. We claim that R is *not* reduced. Suppose by way of contradiction that R is reduced. We consider two cases and obtain a contradiction in each case.

Case 1. R has no proper, nontrivial subring. Then (as ideals are subrings) the only ideals of R are $\{0\}$ and R. Since R is reduced, Lemma 1 implies that R is a field. It follows that R has an identity. This is a contradiction.

Case 2. R has a proper, nontrivial subring. Let S be any such subring. Then S has an identity and every subring of S is a proper subring of R, thus has an identity. We conclude from Proposition 1 that S is a finite direct product of fields. Let e be the identity of S. Then e is a nonzero idempotent of S. Since every proper subring of R is a finite direct product of fields, every proper subring of R is Artinian. Invoking Lemma 3, R is Artinian. So now R is a reduced Artinain ring. Applying Lemma 2, R has an identity, a contradiction.

Led to contradictions in both cases above, we deduce that R is not reduced; let α be a nonzero nilpotent element of R. Without loss of generality, we may assume that $\alpha^2 = 0$. Since every proper, nontrivial subring of R is a finite direct product of fields, it follows that every proper, nontrivial subring of R is reduced. In other words, every nonzero subring of R which is not reduced must coincide with R. Let $S := \{m\alpha\colon m \in \mathbb{Z}\}$. Then S is a nonzero subring of R which is not reduced, and hence $S = R$. Moreover, every nontrivial additive subgroup of R is a subring of R which is not reduced, and hence is equal to R. We conclude that $(R, +)$ is a simple abelian group, hence isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Combining this result with the fact that R has zero multiplication, it follows from Lemma 1 that $R \cong X\mathbb{F}_p[X]/\langle X^2\rangle$, and this concludes the proof. □

The following corollary is immediate.

Corollary 1. *Let R be a nontrivial ring which is not a division ring. Then the following are equivalent.*

(1) *The only left ideals of R are $\{0\}$ and R.*
(2) *R is not unital, but every proper subring of R is unital.*
(3) *$R \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime number p.*

We conclude the paper with a final theorem which is a natural extension of our previous work. In particular, we classify the rings T for which a subring R of T has an identity if and only if R is "at least two rings from T" in the sense that there is a subring S of T such that $R \subsetneq S \subsetneq T$. The proof of this theorem makes essential use of the results of [2]. This paper classifies all rings of cardinality pq for primes p and q (not necessarily distinct). We recall the results of this paper below. In what follows, if G is an abelian group, then $G(0)$ denotes the ring with G as additive group and zero multiplication.

Lemma 4 ([2], Corollary 2 and Theorem 2)**.** *Let R be a ring. Then R has order pq for some primes p and q if and only if R is isomorphic to one of the following rings.*

(a) *$\mathbb{Z}/pq\mathbb{Z}$,*
(b) *$\mathbb{Z}/pq\mathbb{Z}(0)$,*
(c) *$\mathbb{Z}/p\mathbb{Z}(0) \times \mathbb{Z}/q\mathbb{Z}$,*
(d) *$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}(0)$,*
(e) *$\mathbb{Z}/p^2\mathbb{Z}$,*
(f) *$\langle a\colon p^2a = 0, a^2 = pa\rangle$,*
(g) *$\mathbb{Z}/p^2\mathbb{Z}(0)$,*
(h) *$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$,*
(i) *$\langle a, b\colon pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$,*
(j) *$\langle a, b\colon pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle$,*
(k) *$\langle a, b\colon pa = pb = 0, a^2 = 0, b^2 = b, ab = a, ba = a\rangle$,*
(l) *$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}(0)$,*
(m) *$\langle a, b\colon pa = pb = 0, a^2 = b, ab = 0\rangle$,*
(n) *$(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})(0)$, or*
(o) *$\mathbb{F}_{p^2}$, the field of order p^2.*

We conclude the paper with our final theorem.

Theorem 2. *Let T be a ring. Then for all subrings R of T, R has an identity if and only if there is a subring S of T such that $R \subsetneq S \subsetneq T$ exactly when T belongs to one of the following families, where p and q are primes:*

(b) *$\mathbb{Z}/pq\mathbb{Z}(0)$,*
(f) *$\langle a\colon p^2a = 0, a^2 = pa\rangle$,*
(g) *$\mathbb{Z}/p^2\mathbb{Z}(0)$,*
(m) *$\langle a, b\colon pa = pb = 0,\ a^2 = b,\ ab = 0\rangle$, or*
(n) *$(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})(0)$.*

Proof. Let us agree to call a ring T *special* if for all subrings R of T: R has an identity if and only if there is a subring S of T such that $R \subsetneq S \subsetneq T$. Let us first show that the rings in (b), (f), (g), (m), and (n) are special.

As for (b), note that any additive subgroup of $\mathbb{Z}/pq\mathbb{Z}(0)$ is also a subring as a result of the zero multiplication. For any subgroup G of $\mathbb{Z}/pq\mathbb{Z}$, G is trivial if and only if $G \subsetneq H \subsetneq \mathbb{Z}/pq\mathbb{Z}$ for some subgroup H of $\mathbb{Z}/pq\mathbb{Z}$. Since the trivial subring of $\mathbb{Z}/pq\mathbb{Z}(0)$ is the only subring with an identity, this verifies that the rings in (b) are special.

Next, let $T := \langle a\colon p^2a = 0, a^2 = pa\rangle$. It is clear from this presentation that $(T, +) \cong \mathbb{Z}/p^2\mathbb{Z}$, and is thus cyclic. Hence $(T, +)$ has a unique subgroup of order p. This implies that T has at most one subring of order p. One verifies at once that $S := \{mpa\colon m \in \mathbb{Z}\}$ is the unique such subring. It is clear that S has zero multiplication, hence cannot have an identity. We claim that T too has no identity. For suppose that ma is an identity for T, where $1 \leq m < p^2$. Then $ma \cdot a = a$. But this means that $mpa = a$ (using the above presentation). Because a has additive order p^2, we deduce that $p^2 \mid mp - 1$, which is absurd. Hence T does not have an identity. It now follows that the only subring of T which has an identity is $R := \{0\}$. Since $R \subsetneq S \subsetneq T$, we see that T is special.

It is trivial to check that the rings in (g) are special.

We now come to the rings in (m). Let $T := \langle a, b\colon pa = pb = 0,\ a^2 = b,\ ab = 0\rangle$. Note first that $b^2 = b \cdot b = a^2 b = a(ab) = a \cdot 0 = 0$. Thus every member of T can be expressed in the form $ma+nb$ for some integers m and n. It is clear that $\{mb\colon m \in \mathbb{Z}\}$ is a subring of T of size p with zero multiplication. We claim that every subring of T of size p has zero multiplication. Thus suppose that S is a subring of T of size p without zero multiplication; say $(m_1a+n_1b)(m_2a+n_2b) \neq 0$, where $m_ia + n_ib \in S$. Upon multiplying out and using the above presentation, we see that $m_1m_2a^2$ is a nonzero member of S, that is, m_1m_2b is a nonzero member of S. As b has order p and $m_1m_2b \neq 0$, it follows that p and m_1m_2 are relatively prime. By elementary number theory, there are integers x and y such that $xp + ym_1m_2 = 1$. Multiplying through by b, we get $b = xpb + ym_1m_2b = ym_1m_2b \in S$. Hence $S = \{mb\colon m \in \mathbb{Z}\}$ (since S has exactly p elements and $b \in S$ has order p). But this is a contradiction since $\{mb\colon m \in \mathbb{Z}\}$ has zero multiplication. Thus to show that T is special, it suffices to prove that T does not have an identity. This is easy; every member of T annihilates b, so T cannot have an identity.

Finally, it is clear that $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})(0)$ is special, and this concludes the proof that the rings in families (b), (f), (g), (m), and (n) are special.

Conversely, suppose that T is any special ring. Since $\{0\}$ is a subring of T with identity, it follows immediately from the definition of "special ring" that

(2.12) there exists a proper, nontrivial subring S of T.

Next, we prove that

(2.13) T has a maximal subring.

Suppose not, and let R be any proper subring of T. Then there is a subring S of T such that $R \subsetneq S \subsetneq T$. Hence R has an identity, by definition of "special". We have shown that every proper subring of T has an identity. By definition of "special", T does not have an identity. Theorem 1 implies that $T \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime p. But then T has no proper, nontrivial subring, a contradiction to (2.12). We now characterize all maximal subrings of T:

(2.14) A proper subring S of T is maximal if and only if $S \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime p.

To begin, let S be a maximal subring of T. Then by (2.12), S is nontrivial. Because S is maximal, there is no subring of T properly between S and T, and so S has no identity. Moreover, every proper subring of S has an identity since T is special. Hence by Theorem 1, $S \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime p. Conversely, suppose that S is a proper subring of T and $S \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime p. Then S must be proper due to (2.12). If S were not maximal, then S would have an identity, which we know is false. This proves (2.14).

We are now in position to prove that

(2.15) every proper, nontrivial subring of T is maximal.

Let R be a proper, nontrivial subring of T. Next, set $\mathcal{S} := \{S\colon S$ is a proper subring of T containing $R\}$. We claim that $\mathcal{S}$ is closed under unions of nonempty chains. For suppose there is a nonempty chain $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup \mathcal{C} \notin \mathcal{S}$. Then it follows that $R \subseteq \bigcup \mathcal{C} = T$. Let $t \in T$ be nonzero. Then $t \in S \in \mathcal{C}$ for some proper subring S of T containing R. Now, S cannot be a maximal element of $\mathcal{C}$; otherwise $\bigcup \mathcal{C} = S = T$, which is a contradiction. Thus $S \subsetneq S' \subsetneq T$ for some $S' \in \mathcal{C}$. Because T is special, we deduce that S has an identity, and hence so does every subring of S. Invoking Theorem 1, S is a finite direct product of fields, and hence $t \in S$ is not nilpotent. Recall that the nonzero $t \in T$ above was arbitrary. We deduce that T is reduced. However, by (2.13), T has a maximal subring M, and by (2.14), M is not reduced. This is a contradiction. Hence $\mathcal{S}$ is indeed closed under unions of nonempty chains. Invoking Zorn's Lemma, $R \subseteq R^*$ for some maximal subring R^* of T. By (2.14), $R^* \cong X\mathbb{F}_p[X]/\langle X^2\rangle$ for some prime p, and we see that R^* has order p. Because R is nontrivial and R is a subring of R^*, we deduce that $R = R^*$, proving (2.15).

The next step in the proof is to show that T has cardinality pq for some (not necessarily distinct) primes p and q. Our first claim is that

(2.16) if T has a proper, nontrivial two-sided ideal, then $|T| = pq$ for some primes p and q.

Indeed, suppose that I is a proper, nonzero two-sided ideal of T. Then I is a proper, nonzero subring, hence a maximal subring by (2.15). Invoking (2.14), $|I|$ is prime. But then by maximality of I, the ring T/I has no proper, nonzero subrings, thus no proper, nonzero left ideals. Applying Lemma 1, either T/I has prime order or T/I is a division ring. In the former case, I has prime order and T/I has prime order. It follows that T has order pq for some primes p and q. In the latter case, T/I is a division ring with no proper, nontrivial subrings. It is clear from this fact that T/I is isomorphic to $\mathbb{F}_p$ for some prime p. Hence in this case as well, T has order pq for some primes p and q. We have now established (2.16). Our next claim is that

(2.17) $|T| = pq$ for some primes p and q.

First, suppose that T is generated by a single element as a ring. Then T is commutative. If T has no proper, nontrivial ideals, then by Lemma 1, T is a field or T has p elements for some prime p. But T is special, hence does not have an identity; this precludes T from being a field. Further, (2.12) precludes the latter. We deduce that T has a proper, nontrivial ideal, and so by (2.16), $|T| = pq$ for some primes p and q. Suppose now that T is not generated by a single element. Invoking Lemma 1, we see that T has a proper, nontrivial right ideal I and a proper, nontrivial left ideal J.[2] Then I and J are also maximal subrings of T, by (2.15). We conlude from (2.14) that $I = \{ax\colon m \in \mathbb{Z}\}$ and $J = \{by\colon n \in \mathbb{Z}\}$ for some $x \in I$ and $y \in J$ such that $x^2 = y^2 = 0$ and both x and y of prime additive order. It follows that either $I = J$ or $I \cap J = \{0\}$. In the former case, I is a nontrivial two-sided ideal of T, and we apply (2.16) to conclude that $|T| = pq$ for some primes p and q. Thus let us suppose that $I \cap J = \{0\}$. Then we see that $xy \in I \cap J$, so $xy = 0$. From these observations, we see that $L := \{ax + by + cyx\colon a, b, c \in \mathbb{Z}\}$ is a subring of T containing the maximal subrings I and J of T. Since $I \cap J = \{0\}$, we conclude by maximality

[2] It is easy to see that we may replace "left ideal" with "right ideal" in Lemma 1 and obtain the same conclusion.

of I and J that $T = \{ax + by + cyx : a, b, c \in \mathbb{Z}\}$. Now, x has additive order p_1 for some prime p_1, y has additive order p_2 for some prime p_2, and either $yx = 0$ or yx has additive order p_2. In any case, there is a surjective additive group homomorphism $\varphi : \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \to T$. If the kernel is nontrivial, we see that $|T|$ is the product of two primes. Thus let us suppose that the kernel of φ is trivial. Then we have $\{by : b \in \mathbb{Z}\} \subsetneq \{by + cyx : b, c \in \mathbb{Z}\} \subsetneq T$. But this contradicts that $\{by : b \in \mathbb{Z}\}$ is a maximal subring of T. This proves (2.17).

We may now bring Lemma 4 into play. We simply must argue that every ring in (a) - (o) with the exception of (b), (f), (g), (m), and (n) is *not* special. Observe that the rings in (a), (e), (h), and (o) have identities, hence cannot be special by definition. Applying (2.14) and (2.15), every proper subring of T has zero multiplication. It is easy to see that each of the rings outside of (b), (f), (g), (m), and (n) and the rings in (a), (e), (h), and (o) has at least one proper subring with nonzero multiplication. This concludes the proof. □

Acknowledgment The authors wish to thank the referee for useful comments which have improved the exposition of the paper.

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