Math 2150 Notes - July 20, 2022

To begin, I'm again going to restate our list of assumptions as they will continue to play a role in our work.

Assumptions

1. basic algebra[1]
2. the sum, difference, product, and negatives of integers are integers,
3. every integer is either even or odd,
4. no integer is both even and odd,
5. $1 > 0$,
6. for all real numbers a, b, and c: if $a < b$, then $a + c < b + c$,
7. for all real numbers a, b, and c: if $a < b$ and $c > 0$, then $ac < bc$,
8. for all real numbers a, b, and c: if $a < b$ and $b < c$, then $a < c$, and
9. for all real numbers a, b, and c: exactly one of $a = b$, $a < b$, $b < a$ holds.
10. every rational number can be expressed in reduced form.
11. the product of two nonzero real numbers is nonzero.
12. if x is an integer and $x|1$, then $x = \pm 1$.

We now move on to so-called "existence proofs". Almost all of the direct proofs and proofs by contraposition that we have done have involved proving statements which begin with universal quantifiers. This is why we have always let the actors in the proof be arbitrary, since the proof needs to work for *all* values in the universally quantified domains. What I am going to introduce now is how to prove statements which begin with existential quantifiers instead of universal ones. In general (but there are exceptions!), existence proofs are shorter and easier than proofs of universally quantified statements. Let's begin with a very simple example. We will prove it shortly, but I want to discuss it briefly before doing so.

Example 1. *Consider the statement, "There exists an integer x such that $x + x = x$."*

Using our formal language, this statement translates to $\exists x(x + x = x)$ in predicate logic, where the domain for x is the set of integers. Remember that, by definition, this statement is true as long as THERE IS AT LEAST ONE INTEGER x such that $x + x = x$. So to prove the above statement, you simply need to prove that there is at least one integer x such that $x + x = x$. In fact, there is exactly one such integer in this case, though this need not be the case when doing existence proofs. So your job is simply to find an explicit integer x such that $x + x = x$. In general, you will NOT use the word "arbitrary" when you do existence proofs. Your job is simply to find a concrete value for the variable(s) for which the statement is true. In this case, the proof is incredibly short.

Proof. Let $x = 0$. Note that $x + x = 0 + 0 = 0 = x$, and so we have demonstrated the existence of an integer x such that $x + x = x$. □

VERY IMPORTANT: many of you will want to do something like this, which is NOT logically valid in general:

Proof. Note that $x + x = x$ can be rewritten as $2x = x$. Subtracting x from both sides, $x = 0$. So $x = 0$ is an example of such an x. □

[1] When you do basic algebra, you need NOT say "by assumption 1."; you can just do it in the course of a proof.

The reason why this method is not valid is because it is circular: you have begun by ASSUMING that the equation has a solution (which is what you need to prove). Many equations simply do not have solutions within a certain set of numbers. Let me try to drive this point home (please pay careful attention to this). First, recall from class that if x is a positive number, then $\sqrt{x}$ is defined to be the *positive* square root of x. For example, $\sqrt{4} = 2$; it is NOT the case that $\sqrt{4} = \pm 2$. Now, suppose you wanted to prove that there exists a real number x such that $\sqrt{x} = -1$. If you simply solve this equation for x, you would square both sides and get that $x = 1$. But $x = 1$ is actually NOT a solution to this equation, since $\sqrt{1} = 1$, not -1! To summarize, if you want to prove, for example, that a solution to an equation exists, you solve it on scratch paper first; the actual proof is the demonstration that the solution is indeed a solution. Let's look at another example.

Example 2. *Prove that there exist real numbers x and y such that $x + y = \frac{1}{2}$.*

Proof. Let $x = 0$ and $y = \frac{1}{2}$. Then $x + y = 0 + \frac{1}{2} = \frac{1}{2}$. □

(yes, it really is that short and easy)
Now let's look at a more difficult problem which ties together many of the concepts we have been discussing.

Example 3. *Prove that every odd integer can be written as the difference of two perfect squares.*

I'm going to present the preformalization in words this time: "For all integers y: if y is odd, then there exist perfect squares a and b such that $y = a - b$." (please pause to try to put your head around this and convince yourself that this really is what is stated in the example above) This is globally a universally quantified statement, but inside it, you also have an existential "piece". Presenting a formal proof at this point is likely not going to be very enlightening, so I'd rather present an informal discussion instead. Note that the statement in quotes above begins with "for all integers y". If we were to ignore the existential part and proceed with a direct proof, how do we start (you should all know this by now)? Answer:

Let y be an arbitrary integer. Assume that y is odd.

Now what do we need to show? We need to prove that there exist perfect squares a and b such that $y = a - b$. So now our job has shifted to simply giving an example of perfect squares a and b such that $y = a - b$. There is an added difficulty now though: we don't know what y is; it can be ANY odd integer. But we do want a and b to be perfect squares, so we want $a = c^2$ and $b = d^2$ for some integers c and d. Then the equation $y = a - b$ becomes $y = c^2 - d^2$. We can factor the right side to get $y = (c + d)(c - d)$. Can we find integers c and d such that $c + d = y$ and $c - d = 1$? If we can, then note that, indeed, $y = (c + d)(c - d)$ (process this in your mind; it is simply because in this case, the right side is $y \cdot 1$, which is y). So let's consider the equations $c + d = y$ and $c - d = 1$. Can we solve these equations and get integer values for c and d? Let's try: Adding the equations together, $2c = y + 1$. Solving For c, $c = \frac{y+1}{2}$. But is c an integer? YES!! c is an integer because y is odd (remember... look above), so $y + 1$ is even, and an even integer divided by 2 is still an integer. Now, the second equation above is $c - d = 1$. Plugging in $\frac{y+1}{2}$ for c, we get $\frac{y+1}{2} - d = 1$, and solving for d, we obtain $d = \frac{y+1}{2} - 1 = \frac{y+1}{2} - \frac{2}{2} = \frac{y+1-2}{2} = \frac{y-1}{2}$, which is also an integer for a similar reason: y is odd, so $y - 1$ is even, and hence $\frac{y-1}{2}$ is an integer. This is the "scratch work" we need to construct our proof. Remember: when you want to prove that a solution to an equation or a pair of equations exists, your proof DOES NOT CONSIST OF SOLVING THE EQUATION(S); IT CONSISTS OF VERIFYING THE SOLUTIONS ACTUALLY ARE SOLUTIONS!!! (the solving is the "scratch work"). Now let's give the proof.

Proof. Let y be an arbitrary integer. Assume that y is odd. We must prove the existence of perfect squares a and b such that $y = a - b$. Since y is odd, $y = 2e + 1$ for some integer e. Let $c = \frac{y+1}{2}$. Then we have $c = \frac{y+1}{2} = \frac{2e+1+1}{2} = \frac{2e+2}{2} = e + 1$. Since e and 1 are integers, so is $e + 1$ (assumption (2)). Further, let $d = \frac{y-1}{2}$. Then $d = \frac{y-1}{2} = \frac{2e+1-1}{2} = \frac{2e}{2} = e$, and hence d is also an integer. Finally, let $a = c^2$ and $b = d^2$. Then by definition, a and b are perfect squares. Finally, we have $a - b = c^2 - d^2 = \frac{(y+1)^2}{4} - \frac{(y-1)^2}{4} = \frac{y^2+2y+1-(y^2-2y+1)}{4} = \frac{y^2+2y+1-y^2+2y-1}{4} = \frac{2y+2y}{4} = \frac{4y}{4} = y$, which is what we needed to prove. □

Note that our "scratch work" above is actually more lengthy than the proof!
Next, briefly discussing the notion of "disproof".

Definition 1. *Let $\mathcal{S}$ be a statement. To* ***disprove*** *$\mathcal{S}$ simply means to prove that $\mathcal{S}$ is false, that is, it means to prove $\neg\mathcal{S}$.*

Example 4. *Disprove that there exist positive real numbers x and y such that $x + y < x$.*

The statement to be disproved is the statement $\exists x \exists y(x + y < x)$, where the domain for x and y is the set of POSITIVE real numbers. By definition, disproving this statement means proving the negation, that is, we want to prove that there do NOT exist positive real numbers x and y such that $x + y < x$. We can do this in several ways, but let's do this by contradiction.

Disproof Suppose by way of contradiction that there exist positive real numbers x and y such that $x+y < x$. By assumption (8), we may add $-x$ to both sides to obtain $y < 0$. But this contradicts that y is positive, and the proof is complete. □

Let's do an example of a disproof of a universally quantified statement instead of an existentially quantified one as in the previous example.

Example 5. *Disprove that for all real numbers x, there exists a real number y such that $xy = 1$.*

Formally, we must disprove the statement $[\forall x \exists y(xy = 1)]$, where the domain for x and y is the set of real numbers. Again, by definition, this means we want to prove the negation of the above statement. Using our negation rules, the negation of the statement is $\exists x \forall y(xy \neq 1)$. So we now need to prove a statement that begins with "$\exists x$". Our job is to find such an x. In this case, there is only one real number that x can be? Can you see which number? If $x \neq 0$, then there IS such a y; namely, let $y = \frac{1}{x}$. So the only possibility for x is that $x = 0$. Here is the proof.

Proof. Again, we must show that there exists a real number x such that for all real numbers y, $xy \neq 1$. Let $x = 0$. We are not done yet!! We have to demonstrate that $x = 0$ has the required property, namely, that for all real numbers y, $xy \neq 1$. In other words, we still have to show that for all real numbers y, $0 \cdot y \neq 1$. So what we are left with is proving a *universally* quantified statement, namely, that for all real numbers y, $0 \cdot y \neq 1$. Since we must prove a universally quantified statement, we let y be an arbitrary real number, and we must show that $0 \cdot y \neq 1$. This is just basic algebra: $0 \cdot y = 0 \neq 1$, so this concludes the proof (note that I'm being wordier than is required in a proof because I'm also trying to explain as we go through the argument). □

Now, let's do a final example of a disproof of a universally quantified statement.

Example 6. *Disprove that for all real numbers x, $x^2 \geq x$.*

disproof. We must prove the negation of "for all real numbers x, $x^2 \geq x$". The negation of this statement is "there exists a real number x such that $x^2 < x$" (this uses assumption (9) implicitly). Let $x = \frac{1}{2}$, and note that $x^2 = \frac{1}{4} < \frac{1}{2} = x$, and so the disproof is complete. □

We will finish up by simply doing more examples of proofs. Our first example is to prove a certain inequality, but this time, we will need to do more work than we did for previous such problems. I'm going to state the assertion to be proved, and then we will have an informal discussion before beginning the proof. Given positive real numbers x and y, their **arithmetic mean** is $\frac{x+y}{2}$ and their **geometric mean** is $\sqrt{xy}$.

Example 7. *Prove that if x and y are positive real numbers, then the arithmetic mean is greater than or equal to the geometric mean, that is, $\frac{x+y}{2} \geq \sqrt{xy}$.*

Preformalization: $\forall x \forall y(\frac{x+y}{2} \geq \sqrt{xy})$ (the domain for x and y is the set of positive real numbers).

Preliminary discussion (this is NOT the proof!): as I said, this proof is going to be a bit more complicated than the proofs we've seen before. Here's the general idea: we want to prove that $\frac{x+y}{2} \geq \sqrt{xy}$. What you want to do is the following: manipulate this inequality to reduce it to something that you know is true, and then run the argument backwards, providing justification. The reason for running the argument backwards is that you never want to start a proof by assuming what you want to prove: this is circular. OK, so let's get started. Begin with $\frac{x+y}{2} \geq \sqrt{xy}$ (again, in the actual proof, we will reverse these steps so as to avoid circularity). Square both sides (ultimately we will need to justify this, of course) to get $\frac{x^2+2xy+y^2}{4} \geq xy$. Now multiply both sides through by 4 to obtain $x^2 + 2xy + y^2 \geq 4xy$. Next, subtract $4xy$ from both sides to get $x^2 - 2xy + y^2 \geq 0$. Now factor the left-hand side to get $(x-y)^2 \geq 0$. Note where we are now: the square of some real number is greater than or equal to zero. We know that this is true: I proved this in Example 23 above. So the idea of the proof is to start here, and then run the above argument backwards to conclude that the statement we are trying to prove is true. As we go through the argument, there will be a couple of "holes" that we will have to plug, which I will do after the proof (I will offset these holes with asterisks). Again, remember that THE ABOVE DISCUSSION IS NOT THE PROOF AND IS NOT WORK THAT YOU TURN IN; IT'S SCRATCH WORK. THE PROOF IS BELOW.

Proof. Let x and y be arbitrary positive real numbers. We must prove that $\frac{x+y}{2} \geq \sqrt{xy}$. It follows from the proof presented in Example 23 that $(x-y)^2 \geq 0$. Multiplying out on the left, we get $x^2 - 2xy + y^2 \geq 0$. (*) Adding $4xy$ to both sides, we get $x^2 + 2xy + y^2 \geq 4xy$.(**) Now multiply both sides by $\frac{1}{4}$ to obtain $\frac{x^2+2xy+y^2}{4} \geq xy$. Factor the numerator on the left to get $\frac{(x+y)^2}{4} \geq xy$. Recall that both x and y are positive, and thus (***) $(x+y)^2$ and xy are both positive. So now (****) we may take the square root of both sides of $\frac{(x+y)^2}{4} \geq xy$ to get $\frac{x+y}{2} \geq \sqrt{xy}$, which is what needed to be shown. □

We will now pay the debts incurred in the previous proof by verifying (*)-(****). I will verify two of these and may leave two to you to do in the homework.

Example 8. *Prove that for all real numbers a, b, and c: if $a \geq b$, then $a + c \geq b + c$.*

Preformalization: $\forall a \forall b \forall c(a \geq b \to a + c \geq b + c)$.

Proof. Let a, b, and c be arbitrary real numbers. Assume that $a \geq b$. We must show that $a + c \geq b + c$. By definition, since $a \geq b$, either $a = b$ or $a > b$. We consider two cases.

Case 1. $a = b$. Adding c to both sides, $a + c = b + c$, and thus $a + c \geq b + c$.

Case 2. $a > b$. Then by definition, $b < a$. By assumption (6), we may add c to both sides of this inequality to obtain $b + c < a + c$, which means that $a + c > b + c$. Hence in this case too, $a + c \geq b + c$, and the proof is complete. □

Example 9. *Prove that for all real numbers a, b, and c: if $a \geq b$ and $c > 0$, then $ac \geq bc$.*

Proof. Exercise. Try to emulate what I did in the previous example. □

Example 10. *Prove the following:*

1. *for all real numbers a and b: if $a > 0$ and $b > 0$, then $a + b > 0$. (hint: assumption 8 should be used in your proof)*

2. *for all real numbers a and b: if $a > 0$ and $b > 0$, then $ab > 0$.*

Proof. . Exercise. □

Example 11. *For all positive real numbers a and b: if $a \geq b$, then $\sqrt{a} \geq \sqrt{b}$.*

Preformalization: $\forall a \forall b(a \geq b \to \sqrt{a} \geq \sqrt{b})$. Negation: $\exists a \exists b(a \geq b \wedge \sqrt{a} < \sqrt{b})$.

Proof. Suppose by way of contradiction that there exist positive real numbers a and b such that $a \geq b$ and $\sqrt{a} < \sqrt{b}$. By assumption (7), we may multiply both sides of $\sqrt{a} < \sqrt{b}$ by $\sqrt{a}$ to obtain $\sqrt{a}\sqrt{a} < \sqrt{a}\sqrt{b}$, that is, $a < \sqrt{a}\sqrt{b}$. Recall above that $\sqrt{a} < \sqrt{b}$. By assumption (7) again, we may multiply both sides of $\sqrt{a} < \sqrt{b}$ by $\sqrt{b}$ to get $\sqrt{a}\sqrt{b} < \sqrt{b}\sqrt{b} = b$. Recall above that we have $a < \sqrt{a}\sqrt{b}$. Thus we now have $a < \sqrt{a}\sqrt{b}$ and $\sqrt{a}\sqrt{b} < b$. By assumption (8), it follows that $a < b$. But recall above that $a \geq b$, and so by assumption (9), it cannot be that $a < b$, and we have a contradiction. This completes the proof. □

Next, we will look at another set of problems where we may apply the technique of proof by contradiction. I am indirectly introducing the so-called "pigeon-hole principle" which says that if you have n birdhouses and you have $n + 1$ pigeons to place in the houses, then at least one house has to contain at least two pigeons (think about this for a minute).

Example 12. *Suppose that a sock drawer contains only black and blue socks. Prove that if you pick 3 socks out of the drawer, you must have at least two of the same color.*

Preformalization: for any choice of 3 socks from the drawer, you must get at least two of the same color. Negation: there exist 3 socks from the drawer such that no two are of the same color.

Proof. Suppose by way of contradiction that there is a sock drawer containing only blue and black socks and there exist three socks from the drawer such that no two are of the same color. Then the sock drawer must contain socks of at least three different colors, contradicting our assumption. □

Example 13. *Prove that of any 8 dates chosen, at least two must fall on the same day of the week.*

Preformalization: For all possible 8 (different) dates chosen, at least two must fall on the same day of the week.

We will also proceed by contradiction here. First a question: what is the negation of "at least two"? We are negating "greater than or equal to 2", so the negation is "less than two".

Negation: There exist 8 dates such that less than two fall on the same day of the week.

Proof. Suppose by way of contradiction that there exist 8 dates such that less than two fall on the same day of the week. But this means that at most one date falls on a given day of the week. Thus at most one date falls on Monday, at most one on Tuesday, at most one on Wednesday, at most one on Thursday, at most one on Friday, at most one on Saturday, and at most one on Sunday. But of course, every date falls on some day of the week, and so we deduce that there are at most seven dates total. But this is false; there are eight. This contradiction completes the proof. □